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ALMOST MULTIPLICATIVE SETS

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Abstract. Let R be a commutative ring with identity and let S be a nonempty subset of R. We define S to be an almost multiplicative subset of R if for each $a, b \in S$, there exist integers $m, n \geq 1$ such that $a^m b^n \in S$. In this article, we study some utilization of almost multiplicative subsets.

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1. Introduction

Let R be a commutative ring with identity and let S be a nonempty subset of R. Recall that S is a *multiplicative subset* of R if for each $a, b \in S$, $ab \in S$; and a multiplicative subset S of R is *saturated* if whenever $a, b \in R$ with $ab \in S$, both a and b belong to S. In commutative algebra, multiplicative sets have been very useful to study many algebraic properties. Especially, multiplicative subsets are related to prime ideals. For example, multiplicative subsets are used to construct prime ideals and to express radical ideals by the intersection of prime ideals. The simplest fact is that an ideal P of R is a prime ideal of R if and only if $R \setminus P$ is a multiplicative subset of R. Motivated by this result, Krull showed that if P is an ideal of R maximal with respect to the exclusion of a multiplicative subset of R, then P is a prime ideal of R [6, Theorem 1]. Also, multiplicative subsets are very important tools to construct quotient rings. It is well known that if S is a multiplicative subset of R, then R_S becomes a commutative ring with identity which shares ideal structures with R. Another application of a multiplicative subset is the study of S-Noetherian rings as a generalization of Noetherian rings.

The purpose of this article is to define a concept of almost multiplicative subsets and to study some applications. (The definition of almost multiplicative subsets will be introduced in the next section.) While our new notion is a weaker version than multiplicative subsets, it plays similar roles as multiplicative

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subsets. In Section 2, we recover the Krull's result by using almost multiplicative subsets. We also construct quotient rings and compare ideal structures with the base ring. Finally, we study S-Noetherian rings in terms of almost multiplicative subset S.

2. Main results

Let R be a commutative ring with identity and let S be a nonempty subset of R. We say that S is an *almost multiplicative subset* of R if for each $a, b \in S$, there exist integers $m, n \geq 1$ such that $a^m b^n \in S$. If we can always take $m = n = 1$, then the concept of almost multiplicative subsets is precisely the same as that of multiplicative subsets. Also, it is clear that every multiplicative subset of R is an almost multiplicative subset of R but not vice versa. For example, if $S = \{2^{2n+1} \,|\, n \in \mathbb{N}_0\}$, then S is an almost multiplicative subset of $\mathbb{Z}$ which is not a multiplicative subset of $\mathbb{Z}$, where $\mathbb{N}_0$ is the set of nonnegative integers and $\mathbb{Z}$ is the ring of integers.

Our first result in this paper is a slight generalization of [6, Theorem 1].

Theorem 2.1. *Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. If P is an ideal of R maximal with respect to the exclusion of S, then P is a prime ideal of R.*

Proof. Suppose to the contrary that P is not a prime ideal of R. Then there exist $a, b \in R \setminus P$ such that $ab \in P$. Note that $P + (a)$ and $P + (b)$ are ideals of R properly containing P; so by the maximality of P, $(P + (a)) \cap S \neq \emptyset$ and $(P + (b)) \cap S \neq \emptyset$. Let $s_1 \in (P + (a)) \cap S$ and $s_2 \in (P + (b)) \cap S$. Then $s_1 = p_1 + ax$ and $s_2 = p_2 + by$ for some $p_1, p_2 \in P$ and $x, y \in R$. Since S is an almost multiplicative subset of R, there exist positive integers m and n such that $s_1^m s_2^n \in S$. Also, we have

$$\begin{aligned}
s_1^m s_2^n &= (p_1 + ax)^m (p_2 + by)^n \\
&= \left((ax)^m + \sum_{i=1}^{m} p_1^i (ax)^{m-i} \right) \left((by)^n + \sum_{j=1}^{n} p_2^j (by)^{n-j} \right) \\
&= a^m b^n x^m y^n + \sum_{j=1}^{n} (ax)^m p_2^j (by)^{n-j} \\
&\qquad + \sum_{i=1}^{m} (by)^n p_1^i (ax)^{m-i} + \sum_{i=1}^{m} \sum_{j=1}^{n} p_1^i p_2^j (ax)^{m-i} (by)^{n-j} \\
&\in P.
\end{aligned}$$

Hence $P \cap S \neq \emptyset$, which is a contradiction to the choice of P. Thus P is a prime ideal of R. $\square$

Let R be a commutative ring with identity. For an almost multiplicative subset S of R, $\langle S \rangle$ denotes the smallest multiplicative subset of R containing S.

Remark 2.1. Let R be a commutative ring with identity and let S be an almost multiplicative subset of R.

(1) Let P be a prime ideal of R which is maximal with respect to the exclusion of S. If $P\cap\langle S\rangle \neq \emptyset$, then there exists an element $a \in P\cap\langle S\rangle$. Write $a = s_1 \cdots s_m$ for some $s_1, \ldots, s_m \in S$. Since P is a prime ideal of R, $s_i \in P$ for some $i \in \{1, \ldots, m\}$. Therefore $P \cap S \neq \emptyset$. This is absurd. Hence $P \cap \langle S\rangle = \emptyset$. Also, let Q be a prime ideal of R such that $P \subseteq Q$ and $Q \cap \langle S\rangle = \emptyset$. Since $S \subseteq \langle S\rangle$, $Q \cap S = \emptyset$; so $P = Q$ by the maximality of P. Thus P is a prime ideal of R which is maximal with respect to the exclusion of $\langle S\rangle$.

(2) Let P be a prime ideal of R which is maximal with respect to the exclusion of $\langle S\rangle$. Then a similar argument as in (1) shows that P is a prime ideal of R which is maximal with respect to the exclusion of S.

Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. We say that S is *saturated* if whenever $a, b \in R$ with $ab \in S$, both a and b belong to S; and S is *almost saturated* if whenever $a, b \in R$ with $ab \in S$, there exist positive integers m and n (depending on a and b) such that $a^m \in S$ and $b^n \in S$. It is clear that every saturated almost multiplicative set is almost saturated. However, the converse is not generally true. For example, if $R = \mathbb{Z}_2 \times \mathbb{Z}_8$ and $S = \{(1, 0), (1, 1), (1, 2)\}$, then S is an almost saturated almost multiplicative subset of R which is not saturated. Also, it is easy to check that $\{2^{2n+1} \,|\, n \in \mathbb{N}_0\}$ is an almost multiplicative subset of $\mathbb{Z}$ which is not almost saturated.

Remark 2.2. Let R be a commutative ring with identity and let S be an almost multiplicative subset of R.

(1) It is easy to see that if S is saturated, then S is a (saturated) multiplicative subset of R. Hence S is a saturated almost multiplicative subset of R if and only if the complement of S in R is a union of prime ideals of R [6, Theorem 2].

(2) The condition 'saturated multiplicative' in [6, Theorem 2] cannot be replaced by 'almost saturated almost multiplicative'. For instance, let $R = \mathbb{Z}_2 \times \mathbb{Z}_8$ and $S = \{(1, 0), (1, 1), (1, 2)\}$. Then S is an almost saturated almost multiplicative subset of R. If $R \setminus S$ is a union of prime ideals of R, then there exists a prime ideal P of R such that $P \cap S = \emptyset$ and $(1, 4) \in P$; so $(1, 2) \in P \cap S$. This is a contradiction. Hence the complement of S in R cannot be a union of prime ideals of R.

Lemma 2.2. *Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. Then the relation $\sim$ defined on $R \times S$ by*

$$(r_1, s_1) \sim (r_2, s_2) \text{ if and only if } t(r_1s_2 - r_2s_1) = 0 \text{ for some } t \in S$$

is an equivalence relation.

Proof. Let $(r_1, s_1), (r_2, s_2), (r_3, s_3) \in R \times S$. Then it is obvious that $(r_1, s_1) \sim (r_1, s_1)$. Also, it is easy to see that if $(r_1, s_1) \sim (r_2, s_2)$, then $(r_2, s_2) \sim (r_1, s_1)$. Suppose that $(r_1, s_1) \sim (r_2, s_2)$ and $(r_2, s_2) \sim (r_3, s_3)$. Then there exist $t_1, t_2 \in S$ such that

$$t_1(r_1s_2 - r_2s_1) = 0 \text{ and } t_2(r_2s_3 - r_3s_2) = 0;$$

so by a routine calculation, $t_1t_2s_2(r_1s_3 - r_3s_1) = 0$. Since S is an almost multiplicative subset of R, there exist positive integers ℓ, m, n such that $t_1^\ell t_2^m s_2^n \in S$. Therefore $t_1^\ell t_2^m s_2^n(r_1s_3 - r_3s_1) = 0$. Hence $(r_1, s_1) \sim (r_3, s_3)$. Thus the relation $\sim$ is an equivalence relation on $R \times S$. □

Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. Then the equivalence relation $\sim$ defined in Lemma 2.2 gives the partition of $R \times S$ into equivalence classes. For an element $(r, s) \in R \times S$, $\frac{r}{s}$ denotes the equivalence class of (r, s) under $\sim$; and R_S stands for the set of equivalence classes in $R \times S$ under $\sim$.

Remark 2.3. Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. If $\frac{r}{s} \in R_S$, then for any $t \in R$ with $st \in S$, $\frac{r}{s} = \frac{rt}{st}$.

Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. In order to make the set R_S be a ring, we need to define addition and multiplication on R_S.

Lemma 2.3. *Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. Define addition and multiplication on R_S by*

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1s_1^{m-1}s_2^n + r_2s_1^ms_2^{n-1}}{s_1^ms_2^n} \text{ and } \frac{r_1}{s_1}\frac{r_2}{s_2} = \frac{r_1r_2s_1^{m-1}s_2^{n-1}}{s_1^ms_2^n}$$

for all $\frac{r_1}{s_1}, \frac{r_2}{s_2} \in R_S$, where m and n are positive integers satisfying $s_1^ms_2^n \in S$. Then $+$ and $\cdot$ are binary operations on R_S.

Proof. Suppose that $\frac{a_1}{s_1} = \frac{b_1}{t_1}$ and $\frac{a_2}{s_2} = \frac{b_2}{t_2}$ in R_S. Then there exist $u_1, u_2 \in S$ such that

$$u_1(a_1t_1 - b_1s_1) = 0 \text{ and } u_2(a_2t_2 - b_2s_2) = 0.$$

Let ℓ_1 and ℓ_2 be positive integers such that $u_1^{\ell_1}u_2^{\ell_2} \in S$. Then we have

$$u_1^{\ell_1}u_2^{\ell_2}(a_1t_1 - b_1s_1) = 0 \text{ and } u_1^{\ell_1}u_2^{\ell_2}(a_2t_2 - b_2s_2) = 0.$$

Let m_1, m_2, n_1, n_2 be positive integers satisfying $s_1^{m_1}s_2^{m_2}, t_1^{n_1}t_2^{n_2} \in S$. Then we have

$$\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{a_1s_1^{m_1-1}s_2^{m_2} + a_2s_1^{m_1}s_2^{m_2-1}}{s_1^{m_1}s_2^{m_2}} \text{ and } \frac{b_1}{t_1} + \frac{b_2}{t_2} = \frac{b_1t_1^{n_1-1}t_2^{n_2} + b_2t_1^{n_1}t_2^{n_2-1}}{t_1^{n_1}t_2^{n_2}}.$$

Let $k_1 = s_1^{m_1-1}s_2^{m_2}t_1^{n_1-1}t_2^{n_2}$ and $k_2 = s_1^{m_1}s_2^{m_2-1}t_1^{n_1}t_2^{n_2-1}$. Then we have

$$u_1^{\ell_1}u_2^{\ell_2}(k_1(a_1t_1 - b_1s_1) + k_2(a_2t_2 - b_2s_2)) = 0.$$

Hence $\frac{a_1}{s_1} + \frac{a_2}{s_2} = \frac{b_1}{t_1} + \frac{b_2}{t_2}$. Note that

$$\frac{a_1}{s_1}\frac{a_2}{s_2} = \frac{a_1a_2s_1^{m_1-1}s_2^{m_2-1}}{s_1^{m_1}s_2^{m_2}} \text{ and } \frac{b_1}{t_1}\frac{b_2}{t_2} = \frac{b_1b_2t_1^{n_1-1}t_2^{n_2-1}}{t_1^{n_1}t_2^{n_2}}.$$

Since $u_1^{\ell_1}u_2^{\ell_2}(a_1t_1 - b_1s_1) = 0$ and $u_1^{\ell_1}u_2^{\ell_2}(a_2t_2 - b_2s_2) = 0$, we have

$$u_1^{\ell_1}u_2^{\ell_2}(a_1a_2t_1t_2 - b_1b_2s_1s_2) = 0.$$

Let $k = s_1^{m_1-1}s_2^{m_2-1}t_1^{n_1-1}t_2^{n_2-1}$. Then we have

$$u_1^{\ell_1}u_2^{\ell_2}(k(a_1a_2t_1t_2 - b_1b_2s_1s_2)) = 0.$$

Hence $\frac{a_1}{s_1}\frac{a_2}{s_2} = \frac{b_1}{t_1}\frac{b_2}{t_2}$. Thus $+$ and $\cdot$ are binary operations on R_S. $\square$

Proposition 2.4. *Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. Then R_S is a commutative ring with identity under binary operations defined in Lemma 2.3.*

Proof. It is routine to check that R_S is a commutative ring with identity. $\square$

Theorem 2.5. *Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. Then R_S is isomorphic to $R_{\langle S\rangle}$.*

Proof. It is easy to show that the map $\phi : R_S \to R_{\langle S\rangle}$ given by $\frac{r}{s} \mapsto \frac{r}{s}$ is a ring isomorphism. $\square$

Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. For an ideal I of R, let $IR_S = \{\frac{r}{s} \,|\, r \in I \text{ and } s \in S\}$. Then it is easy to see that IR_S is an ideal of R_S. For an element $s \in S$, let $\psi_s : R \to R_S$ be the map defined by $r \mapsto \frac{rs}{s}$. Then it is routine to check that ψ_s is a ring homomorphism and $\psi_s = \psi_t$ for all $s, t \in S$. From now on, ψ_S represents the map ψ_s for a fixed $s \in S$.

Corollary 2.6. *Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. Then the following assertions hold.*

(1) *If A is an ideal of R_S, then $A = IR_S$ for some ideal I of R.*
(2) *If P is a prime ideal of R with $P \cap S = \emptyset$, then PR_S is a prime ideal of R_S.*
(3) *If Q is a prime ideal of R_S, then $Q = PR_S$ for some prime ideal P of R with $P \cap S = \emptyset$ and $\psi_S^{-1}(PR_S) = P$.*
(4) *There is a one-to-one order-preserving correspondence between the set of prime ideals of R which are disjoint from S and the set of prime ideals of R_S, given by $P \mapsto PR_S$.*

Proof. Let $\phi : R_S \to R_{\langle S\rangle}$ be the isomorphism given by $\phi(\frac{r}{s}) = \frac{r}{s}$ for all $\frac{r}{s} \in R_S$.

(1) Let A be an ideal of R_S. Then $\phi(A)$ is an ideal of $R_{\langle S\rangle}$; so $\phi(A) = IR_{\langle S\rangle}$ for some ideal I of R [5, Chapter III, Lemma 4.9(ii)]. Hence $A = \phi^{-1}(IR_{\langle S\rangle})$. We now claim that $IR_S = \phi^{-1}(IR_{\langle S\rangle})$. Note that $\phi(IR_S) \subseteq IR_{\langle S\rangle}$; so $IR_S \subseteq \phi^{-1}(IR_{\langle S\rangle})$. For the reverse containment, let $\frac{a}{s} \in \phi^{-1}(IR_{\langle S\rangle})$. Then $\phi(\frac{a}{s}) = \frac{i}{t}$ for some $\frac{i}{t} \in IR_{\langle S\rangle}$. Since S is an almost multiplicative subset of R, we can take an element $x \in R$ such that $tx \in S$; so $\frac{a}{s} = \frac{ix}{tx}$ in $R_{\langle S\rangle}$. Therefore there exists an element $u \in \langle S\rangle$ such that $u(atx - ixs) = 0$. Since S is an almost multiplicative subset of R, there exists an element $y \in R$ such that $uy \in S$; so

$uy(atx - ixs) = 0$. Hence $\frac{a}{s} = \frac{ix}{tx}$ in R_S, which implies that $\frac{a}{s} \in IR_S$. Thus $\phi^{-1}(IR_{\langle S \rangle}) \subseteq IR_S$. Consequently, $A = IR_S$.

(2) Let P be a prime ideal of R with $P \cap S = \emptyset$. Then $P \cap \langle S \rangle = \emptyset$ by an argument in Remark 2.1(1); so $PR_{\langle S \rangle}$ is a prime ideal of $R_{\langle S \rangle}$ [5, Chapter III, Lemma 4.9(iii)]. Note that $\phi^{-1}(PR_{\langle S \rangle}) = PR_S$ by the proof of (1). Thus PR_S is a prime ideal of R_S.

(3) Let Q be a prime ideal of R_S. Then $\phi(Q)$ is a prime ideal of $R_{\langle S \rangle}$; so $\phi(Q) = PR_{\langle S \rangle}$ for some prime ideal P of R with $P \cap \langle S \rangle = \emptyset$ [5, Chapter III, Theorem 4.10]. Hence by the proof of (1), $Q = \phi^{-1}(PR_{\langle S \rangle}) = PR_S$. Obviously, $P \cap S = \emptyset$ because $S \subseteq \langle S \rangle$.

Note that $\phi \circ \psi_S : R \to R_{\langle S \rangle}$ is a ring homomorphism and by the proof of (1), $\phi(PR_S) = PR_{\langle S \rangle}$; so $\psi_S^{-1}(PR_S) = (\phi \circ \psi_S)^{-1}(PR_{\langle S \rangle}) = P$ [5, Chapter III, Lemma 4.9(iii)].

(4) The result follows directly from (2) and (3). □

Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. We say that an ideal I of R is *S-finite* if there exist an element $s \in S$ and a finitely generated ideal J of R such that $sI \subseteq J \subseteq I$; and R is an *S-Noetherian ring* if every ideal of R is S-finite. These concepts generalize those of S-finiteness and S-Noetherian rings for multiplicative sets S. For more on S-finiteness and S-Noetherian rings for multiplicative sets, the readers can refer to [3, 7, 10, 11, 12, 13, 14].

Proposition 2.7. *Let R be a commutative ring with identity, S an almost multiplicative subset of R and I an ideal of R. Then I is S-finite if and only if I is $\langle S \rangle$-finite.*

Proof. The "only if" part is clear, because $S \subseteq \langle S \rangle$. For the converse, suppose that I is an $\langle S \rangle$-finite ideal of R. Then there exist an element $t \in \langle S \rangle$ and a finitely generated ideal J of R such that $tI \subseteq J \subseteq I$. Note that $t = s_1 \cdots s_m$ for some $s_1, \dots, s_m \in S$. Since S is an almost multiplicative subset of R, there exist positive integers $n_1, \dots, n_m$ such that $s_1^{n_1} \cdots s_m^{n_m} \in S$. Thus $s_1^{n_1} \cdots s_m^{n_m} I \subseteq J \subseteq I$, which means that I is an S-finite ideal of R. □

By Proposition 2.7, we have

Corollary 2.8. *Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. Then R is an S-Noetherian ring if and only if R is an $\langle S \rangle$-Noetherian ring.*

Let R be a commutative ring with identity and let S be an (almost) multiplicative subset of R. We say that S is an *anti-Archimedean subset* of R if $\bigcap_{n \geq 1} s^n R \cap S \neq \emptyset$ for all $s \in S$.

Proposition 2.9. *Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. Then S is an anti-Archimedean subset of R if and only if $\langle S \rangle$ is an anti-Archimedean subset of R.*

Proof. ($\Rightarrow$) Let $t \in \langle S\rangle$. Then $t = s_1 \cdots s_m$ for some $s_1, \dots, s_m \in S$. Since S is an anti-Archimedean subset of R, $\bigcap_{n\geq 1} s_i^n R \cap S \neq \emptyset$ for all $i \in \{1, \dots, m\}$. For each $i \in \{1, \dots, m\}$, choose any element $\alpha_i \in \bigcap_{n\geq 1} s_i^n R \cap S$. Then $\alpha_1 \cdots \alpha_m \in t^n R \cap \langle S\rangle$ for all $n \geq 1$. Hence $\bigcap_{n\geq 1} t^n R \cap \langle S\rangle \neq \emptyset$. Thus $\langle S\rangle$ is an anti-Archimedean subset of R.

($\Leftarrow$) Suppose that $\langle S\rangle$ is an anti-Archimedean subset of R and let $s \in S$. Then there exists an element $t \in \bigcap_{n\geq 1} s^n R \cap \langle S\rangle$. Write $t = s_1 \cdots s_m$ for some $s_1, \dots, s_m \in S$. Since S is an almost multiplicative subset of R, there exist positive integers $n_1, \dots, n_m$ such that $s_1^{n_1} \cdots s_m^{n_m} \in S$. Note that $s_1^{n_1} \cdots s_m^{n_m} = ts_1^{n_1-1} \cdots s_m^{n_m-1} \in \bigcap_{n\geq 1} s^n R$. Hence $\bigcap_{n\geq 1} s^n R \cap S \neq \emptyset$. Thus S is an anti-Archimedean subset of R. $\square$

Let R be a commutative ring with identity and let $R[X]$ be the polynomial ring over R. For an element $f \in R[X]$, $c(f)$ denotes the *content ideal* of f, *i.e.*, the ideal of R generated by the coefficients of f. Let $U = \{f \in R[X] \mid f \text{ is monic}\}$ and let $N = \{f \in R[X] \mid c(f) = R\}$. Then U is a multiplicative subset of $R[X]$ and N is a saturated multiplicative subset of $R[X]$. Also, the quotient ring $R[X]_U$ is called the *Serre's conjecture ring* of R and the quotient ring $R[X]_N$ is called the *Nagata ring* of R. The readers can refer to [2, 8, 9] for the Serre's conjecture ring and to [1, 2, 4] for the Nagata ring.

Corollary 2.10. (cf. [11, Theorem 3]) *Let R be a commutative ring with identity and let S be an almost multiplicative subset of R. If S is an anti-Archimedean subset of R, then the following conditions are equivalent.*

(1) *R is an S-Noetherian ring.*
(2) *$R[X]$ is an S-Noetherian ring.*
(3) *$R[X]_U$ is an S-Noetherian ring.*
(4) *$R[X]_N$ is an S-Noetherian ring.*

Proof. These equivalences come directly from Corollary 2.8 and Proposition 2.9. $\square$

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