Bull. Korean Math. Soc. **58** (2021), No. 1, pp. 59–70
https://doi.org/10.4134/BKMS.b191089
pISSN: 1015-8634 / eISSN: 2234-3016

INEQUALITIES OF OPERATOR VALUED QUANTUM SKEW INFORMATION

Byoung Jin Choi and Mi Ra Lee

Abstract. In this paper, we study two operator-valued inequalities for quantum Wigner-Yanase-Dyson skew information related to module operators. These are extended results of the trace inequalities for Wigner-Yanase-Dyson skew information. Moreover, we study a sufficient condition to prove an uncertainty relation for operator-valued generalized quantum Wigner-Yanase-Dyson skew information related to module operators and a pair of functions (f, g). Also, we obtain several previous results of scalar-valued cases as a consequence of our main result.

1. Introduction

In quantum information theory, quantum skew information plays an important role in matrix algebras. Quantum skew information is a significant tool for understanding uncertainty relations. For example, in [9], Heisenberg first proved the following uncertainty relation for a quantum state (or density operator) ρ and a pair of self-adjoint matrices (or observables) A and B:

$$V_\rho(A)V_\rho(B) \geq \left|\frac{1}{2}\mathrm{Tr}\left(\rho[A,B]\right)\right|^2, \tag{1.1}$$

where $V_\rho(A) = \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho A)^2$ and $[A, B] = AB - BA$ is the commutator. Also Schrödinger in [14] proved that

$$V_\rho(A)V_\rho(B) - |\mathrm{Re}\,\mathrm{Cov}_\rho(A,B)|^2 \geq \left|\frac{1}{2}\mathrm{Tr}\left(\rho[A,B]\right)\right|^2, \tag{1.2}$$

where $\mathrm{Cov}_\rho(A, B) := \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho A)\mathrm{Tr}(\rho B)$ which is stronger result than Heisenberg's result (1.1). In [15], Yanagi et al. defined the new skew information

Received December 20, 2019; Revised August 27, 2020; Accepted October 16, 2020.
2010 *Mathematics Subject Classification.* Primary 94A17, 81P45.
Key words and phrases. Uncertainty relation, Wigner-Yanase-Dyson skew information, module operator.
The first author was supported by the National Research Foundation of Korea(NRF) grant funded by the Korea government(MSIT) (No. 2020R1C1C1A01003305).

(Wigner-Yanase-Dyson skew information) for an observable A, as the following:

$$\begin{aligned} I_\rho^\alpha(A) &:= \frac{1}{2}\mathrm{Tr}\left((i[\rho^\alpha, A_0])\left(i\left[\rho^{1-\alpha}, A_0\right]\right)\right) \\ &= \mathrm{Tr}(\rho A^2) - \mathrm{Tr}(\rho^\alpha A \rho^{1-\alpha} A), \quad \alpha \in (0,1), \end{aligned}$$

where $A_0 = A - \mathrm{Tr}(\rho A)I$. It can be considered as a kind of measurement for non-commutativity between the density operator ρ and A. Note that if ρ is a pure state, then $I_\rho^{1/2}(A) = V_\rho(A)$. But the Heisenberg type inequality for $I_\rho^{1/2}(A)$, i.e.,

$$I_\rho^{1/2}(A) I_\rho^{1/2}(B) \geq \frac{1}{4}\left|\mathrm{Tr}\left(\rho[A,B]\right)\right|^2,$$

fails in general (see e.g., [15]). On the other hand, for this information (indeed, more general form), Furuichi proved that the following inequality [5]: for any self-adjoint operators A and B,

$$\left|\mathrm{Re}\,\mathrm{Corr}_\rho^{(f,g)}(A,B)\right|^2 \leq I_\rho^{(f,g)}(A) I_\rho^{(f,g)}(B), \tag{1.3}$$

where (f,g) is a monotone pair of functions,

$$\mathrm{Corr}_\rho^{(f,g)}(A,B) = \mathrm{Tr}(f(\rho)g(\rho)AB) - \mathrm{Tr}(f(\rho)Ag(\rho)B)$$

and

$$I_\rho^{(f,g)}(A) := \mathrm{Corr}_\rho^{(f,g)}(A,A).$$

Note that if we take $f(x) = x^\alpha$ and $g(x) = x^{1-\alpha}$ with $\alpha \in (0,1)$, then one reduces $I_\rho^{(f,g)}$ to I_ρ^α. Indeed, in [15], Yanagi et al. proved that for any self-adjoint elements A and B,

$$\left|\mathrm{Re}\,\mathrm{Corr}_\rho^\alpha(A,B)\right|^2 \leq I_\rho^\alpha(A) I_\rho^\alpha(B),$$

where

$$\mathrm{Corr}_\rho^\alpha(A,B) = \mathrm{Tr}(\rho AB) - \mathrm{Tr}(\rho^\alpha A \rho^{1-\alpha} B).$$

Another generalization of Wigner-Yanase-Dyson skew information was studied by Furuichi et al. in [7]. More precisely, the authors defined the new skew information as

$$K_{\rho,\mathrm{Tr}}^\alpha(A) := \frac{1}{2}\mathrm{Tr}\left(\left(i\left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A_0\right]\right)^2\right)$$

and

$$L_{\rho,\mathrm{Tr}}^\alpha(A) := \frac{1}{2}\mathrm{Tr}\left(\left\{\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A_0\right\}^2\right),$$

where $A_0 = A - \mathrm{Tr}(\rho A)I$ and $\{A, B\} = AB + BA$ is the anti-commutator, and proved the uncertainty relation

$$W^{\alpha}_{\rho,\mathrm{Tr}}(A)W^{\alpha}_{\rho,\mathrm{Tr}}(B) \geq \frac{1}{4}\left|\mathrm{Tr}\left(\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2 [A,B]\right)\right|^2 \tag{1.4}$$

for $\alpha \in (0, 1)$, a quantum state ρ, an observable A and

$$W^{\alpha}_{\rho,\mathrm{Tr}}(A) = \sqrt{K^{\alpha}_{\rho,\mathrm{Tr}}(A)L^{\alpha}_{\rho,\mathrm{Tr}}(A)}.$$

For more results of uncertainty relations for quantum skew information, we refer to [6, 8, 10–13], and references cited therein.

In [2], the authors introduced the new notion of Φ-density operators, where Φ is a tracial positive linear operator on a C^*-algebra $\mathcal{A}$, and the authors proved the uncertainty relation for Wigner-Yanase-Dyson skew information valued in a C^*-algebra $\mathcal{A}$ for any self-adjoint elements A and B in $\mathcal{A}$ and Φ-density operator $\rho \in \mathcal{A}$ (see Corollary 2.6). Also, Heisenberg and Schrödinger's uncertainty relations for positive operators valued in C^* or von Neumman algebras were studied in [1–3].

The main purpose of this paper is to study the operator valued inequalities concerning generalized quantum Wigner-Yanase-Dyson skew information of the forms (1.3) and (1.4) (see Theorems 2.4 and 2.10). Also, we study a sufficient condition to prove an uncertainty relation for operator valued generalized quantum Wigner-Yanase-Dyson skew information of the form (1.3) (see Theorem 2.4). To do this, in Section 2, we define the generalized quantum (f, g)-skew information and the generalized quantum Wigner-Yanase skew information with the certain condition of a pair of functions (f, g) (see the property (**P**) in Section 2), and then we present main results in this paper. Also, we have several corollaries as a consequence of main results, which include previous results. In Section 3, we give the proofs of the inequalities with some lemmas.

2. Quantum skew information and main results

Let $\mathcal{A}$ and $\mathcal{B}$ be two unital C^*-algebras (or von Neumann algebras). A linear operator $T : \mathcal{A} \longrightarrow \mathcal{B}$ is called *tracial* if $T(XY) = T(YX)$ for all $X, Y \in \mathcal{A}$ and *positive* if $T(X) \geq 0$ for all $X \in \mathcal{A}_+$, where $\mathcal{A}_+$ is a set of positive elements in $\mathcal{A}$.

Definition ([3])**.** Let (f, g) be a pair of continuous functions on a domain D in $\mathbb{R}$. Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be a tracial positive operator and ρ be a T-density operator, i.e., ρ is positive and $T(\rho) = \mathbf{1}$. Then

$$\mathrm{Corr}^{(f,g)}_{\rho,T}(A, B) = T(f(\rho)g(\rho)A^*B) - T(f(\rho)A^*g(\rho)B), \quad A, B \in \mathcal{A},$$

is called the *generalized correlation* and

$$I^{(f,g)}_{\rho,T}(A) := \mathrm{Corr}^{(f,g)}_{\rho,T}(A, A), \quad A \in \mathcal{A},$$

is called the *generalized quantum* (f,g)*-skew information*.

Now, we say that a linear operator $T : \mathcal{A} \longrightarrow \mathcal{B}$ and a pair function (f,g) satisfy the property **(P)** if T and (f,g) satisfy the following property:

$$T(f(A)Bg(A)B) \leq T(f(A)g(A)B^2)$$

for any self-adjoint elements A and B in $\mathcal{A}$. Note that in the above definition, we always assume that (f,g) is defined on an interval containing the spectrums of A and B, respectively.

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. A linear operator $T : \mathcal{A} \to \mathcal{B}$ is called a *module operator* ([1]) if T satisfies the following property:

$$T(AXB) = AT(X)B \tag{2.1}$$

for any $X \in \mathcal{A}$ and $A, B \in \mathcal{B}$. Note that if a module operator T is tracial, then by using module and tracial properties, we have $T(\mathcal{A}) \subseteq Z(\mathcal{B})$, where $Z(\mathcal{B})$ is the center of $\mathcal{B}$. A pair (f,g) is said to be a *monotone pair* of operator monotone functions on the domain D in $\mathbb{R}$ if

$$(f(a) - f(b))\,(g(a) - g(b)) \geq 0$$

for any $a, b \in D$. There are many examples of monotone pairs as following: for more examples, see [10].

Example 2.1. (i) For any operator monotone function f, the pair (f,f) is a monotone pair.
(ii) $\left(x^\alpha, x^\beta\right)$ with $\alpha, \beta \in (0,1)$ on $[0,1]$ is a monotone pair.
(iii) $\left(\frac{x^\alpha + x^{1-\alpha}}{2}, \frac{x^\alpha + x^{1-\alpha}}{2}\right)$ with $\alpha \in (0,1)$ on $[0,1]$ is a monotone pair.

Now, we give several examples of a pair of functions satisfying the property (**P**) as followings:

Example 2.2. (i) Any trace Tr on an $n \times n$-matrix algebra $\mathcal{M}_{n\times n}$ and any monotone pair (f,g) satisfy the property (**P**) for any self-adjoint matrices A and B (see [4, Theorem 2]).
(ii) Any tracial positive linear operator $T : \mathcal{A} \longrightarrow \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras, and a monotone pair (f,g) satisfy the property (**P**) for any positive element $A \in \mathcal{A}$ and any self-adjoint element $B \in \mathcal{A}$ (see [3, Theorem 4.1]).
(iii) Any tracial positive linear operator $T : \mathcal{A} \longrightarrow \mathcal{B}$, where $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, and the monotone pair $(x^{1-\alpha}, x^\alpha)$ with $\alpha \in (0,1)$ satisfy the property (**P**) for any positive element $A \in \mathcal{A}$ and any self-adjoint element $B \in \mathcal{A}$ (see [2, Lemma 3.1]).

Example 2.3. Any tracial positive module operator $T : \mathcal{A} \longrightarrow \mathcal{B} \subseteq \mathcal{A}$, where $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras, and a pair (f,f) of positive preserving function f (i.e., $f(x) \geq 0$ for $x \geq 0$) satisfy the property (**P**) for any positive elements A and B. Indeed, using the fact that $T(\mathcal{A}) \subseteq Z(\mathcal{B})$, and the inequality (2.13) (Araki-Lieb-Thirring inequality) in [1, Theorem 2.8], we have

$$T(f(A)Bf(A)B) = T((f(A)B)^2) \leq T(f(A)BBf(A)) = T(f(A)f(A)BB),$$

where A and B are positive elements in $\mathcal{A}$.

The following theorem is one of the main results in this paper which is an uncertainty relation for a generalized quantum (f, g)-skew information.

Theorem 2.4. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be a tracial positive module operator. If T and (f, g) satisfy property ($\boldsymbol{P}$), then for all self-adjoint elements $A, B \in \mathcal{A}$ and T-density operator $\rho \in \mathcal{A}$,*

$$\left|\operatorname{Re}\operatorname{Corr}_{\rho,T}^{(f,g)}(A, B)\right|^2 \leq I_{\rho,T}^{(f,g)}(A) I_{\rho,T}^{(f,g)}(B),$$

where

$$\operatorname{Re}\operatorname{Corr}_{\rho,T}^{(f,g)}(A, B) := \frac{1}{2}\left(\operatorname{Corr}_{\rho,T}^{(f,g)}(A, B) + \operatorname{Corr}_{\rho,T}^{(f,g)}(A, B)^*\right)$$

and (f, g) is a pair of functions which are defined on some interval containing the spectrum of ρ.

By (ii), (iii) in Example 2.2 and Theorem 2.4, we have the following corollaries.

Corollary 2.5 ([3])**.** *Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{A}$. Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be a tracial positive module operator. Then for all self-adjoint elements $A, B \in \mathcal{A}$ and T-density operator $\rho \in \mathcal{A}$,*

$$\left|\operatorname{Re}\operatorname{Corr}_{\rho,T}^{(f,g)}(A, B)\right|^2 \leq I_{\rho,T}^{(f,g)}(A) I_{\rho,T}^{(f,g)}(B),$$

where (f, g) is a monotone pair of functions.

If we take the monotone pair $(f, g) = (x^{1-\alpha}, x^{\alpha})$ with $\alpha \in (0, 1)$, we have the following result.

Corollary 2.6 ([2])**.** *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be a tracial positive module operator. Then for all self-adjoint elements $A, B \in \mathcal{A}$ and T-density operator $\rho \in \mathcal{A}$,*

$$\left|\operatorname{Re}\operatorname{Corr}_{\rho,T}^{\alpha}(A, B)\right|^2 \leq I_{\rho,T}^{\alpha}(A) I_{\rho,T}^{\alpha}(B), \quad (\alpha \in (0, 1)),$$

where $I_{\rho,T}^{\alpha}(A) := T(\rho A^2) - T(\rho^{\alpha} A \rho^{1-\alpha} A)$ and $\operatorname{Corr}_{\rho,T}^{\alpha}(A, B) := T(\rho A^* B) - T(\rho^{\alpha} A^* \rho^{1-\alpha} B)$.

The following result is an infinite dimensional version of the Furuichi's (f, g)-skew information result in [5].

Corollary 2.7 (c.f. [5]). *Let $\mathcal{B}(H)$ be the Banach space of all bounded linear operators on a Hilbert space H. Let $Tr : \mathcal{B}(H) \to \mathbb{C}$ be a usual trace and (f, g) be a pair of functions. If Tr and (f, g) satisfy property ($\boldsymbol{P}$), then for all self-adjoint elements $A, B \in \mathcal{B}(H)$ and Tr-density operator $\rho \in \mathcal{B}(H)$,*

$$\left|\mathrm{Re}\,\mathrm{Corr}^{(f,g)}_{\rho,Tr}(A,B)\right|^2 \leq I^{(f,g)}_{\rho,Tr}(A) I^{(f,g)}_{\rho,Tr}(B).$$

Remark 2.8. In Corollary 2.7, we take $H = \mathbb{C}^n$ with $n \in \mathbb{N}$ then by Example 2.2(i), Tr and any monotone pair (f, g) satisfy the property ($\mathbf{P}$) for any self-adjoint matrices A and B. Thus we have for any monotone pair (f, g),

$$\left|\mathrm{Re}\,\mathrm{Corr}^{(f,g)}_{\rho,\mathrm{Tr}}(A,B)\right|^2 \leq I^{(f,g)}_{\rho,\mathrm{Tr}}(A) I^{(f,g)}_{\rho,\mathrm{Tr}}(B),$$

(see [5]).

Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be a tracial positive module operator. For a T-density operator ρ and a self-adjoint element $A \in \mathcal{A}$, we define *the generalized quantum Wigner-Yanase skew information* by

$$K^f_{\rho,T}(A) := I^{(f,f)}_{\rho,T}(A) = \frac{1}{2}T\left(\left(i\big[f(\rho), A_0\big]\right)^2\right), \tag{2.2}$$

where $A_0 = A - T(\rho A)$. Indeed, for a tracial positive module operator T, since $f(\rho)g(\rho) = g(\rho)f(\rho)$, we have

$$I^{(f,g)}_{\rho,T}(A) = \frac{1}{2}T\left(\left(i\big[f(\rho), A_0\big]\right)\left(i\big[g(\rho), A_0\big]\right)\right).$$

It is a generalization of the quantum Wigner-Yanase skew information $I^{1/2}_{\rho,T}(A)$. We also define

$$L^f_{\rho,T}(A) = \frac{1}{2}T\left(\big\{f(\rho), A_0\big\}^2\right). \tag{2.3}$$

Note that for any self-adjoint element A,

$$\left(i\big[f(\rho), A_0\big]\right)^* = i\big[f(\rho), A_0\big]$$

induces that $K^f_{\rho,T}(A)$ is positive. Similarly, $L^f_{\rho,T}(A)$ is positive with self-adjoint element $A \in \mathcal{A}$.

By Example 2.2 and Theorem 2.4, the following result holds.

Corollary 2.9. *Let $\mathcal{A}$ be a von Neumann algebra and $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{A}$. Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be a tracial positive module operator. Then for any positive elements $A, B \in \mathcal{A}$ and T-density operator $\rho \in \mathcal{A}$,*

$$\left|\mathrm{Re}\,\mathrm{Corr}^{(f,f)}_{\rho,T}(A,B)\right|^2 \leq K^f_{\rho,T}(A) K^f_{\rho,T}(B),$$

where f is an operator monotone function or a positive preserving function.

Based on the above setting, we have the following inequality which is the other main result in this paper.

Theorem 2.10. *Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. If $T : \mathcal{A} \to \mathcal{B}$ is a tracial positive module operator and ρ is a density operator with respect to T, then it holds that*

$$W^f_{\rho,T}(A)W^f_{\rho,T}(B) \geq \frac{1}{4}\left|T\left(f(\rho)^2[A,B]\right)\right|^2$$

with self-adjoint $A, B \in \mathcal{A}$, where

$$W^f_{\rho,T}(A) = \sqrt{K^f_{\rho,T}(A)L^f_{\rho,T}(A)},$$

and f is a function which is defined on an interval containing the spectrum of T-density operator ρ.

Remark 2.11. (i) If we take $f(x) = \frac{x^\alpha + x^{1-\alpha}}{2}$ for $\alpha \in [0,1]$ in (2.2) and (2.3) respectively, then we obtain that for a self adjoint element $A \in \mathcal{A}$

$$K^\alpha_{\rho,T}(A) := K^f_{\rho,T}(A) = \frac{1}{2}T\left(\left(i\left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A_0\right]\right)^2\right)$$

and

$$L^\alpha_{\rho,T}(A) := L^f_{\rho,T}(A) = \frac{1}{2}T\left(\left\{\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, A_0\right\}^2\right),$$

where $A_0 = A - T(\rho A)$. Then Theorem 2.10 induces the inequality

$$W^\alpha_{\rho,T}(A)W^\alpha_{\rho,T}(B) \geq \frac{1}{4}\left|T\left(\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2}\right)^2[A,B]\right)\right|^2,$$

where $W^\alpha_{\rho,T}(A) = \sqrt{K^\alpha_{\rho,T}(A)L^\alpha_{\rho,T}(A)}$.

(ii) If we take $f(x) = x^{1/2}$ in (2.2) and (2.3) respectively, then we obtain that for a self-adjoint element $A \in \mathcal{A}$

$$I^{1/2}_{\rho,T}(A) = K^f_{\rho,T}(A) = \frac{1}{2}T\left(\left(i\left[\rho^{1/2}, A_0\right]\right)^2\right)$$

and

$$J^{1/2}_{\rho,T}(A) = L^f_{\rho,T}(A) = \frac{1}{2}T\left(\left\{\rho^{1/2}, A_0\right\}^2\right),$$

where $A_0 = A - T(\rho A)$. Also, Theorem 2.10 induces the inequality

$$U_{\rho,T}(A)U_{\rho,T}(B) \geq \frac{1}{4}\left|T\left(\rho[A,B]\right)\right|^2,$$

where $U_{\rho,T}(A) = \sqrt{I^{1/2}_{\rho,T}(A)J^{1/2}_{\rho,T}(A)}$ (see [2]).

Using Remark 2.11, we have the following corollaries which are related to the Wigner-Yanase skew information and Lao's information in [7, 12].

Corollary 2.12 ([7])**.** *Let $\mathcal{B}(H)$ be the Banach space of all bounded linear operators on a Hilbert space H. Let $Tr : \mathcal{B}(H) \to \mathbb{C}$ be a usual trace functional. Then for any self-adjoint elements $A, B \in \mathcal{B}(H)$ and Tr-density operator $\rho \in \mathcal{B}(H)$,*

$$W^{\alpha}_{\rho,Tr}(A)W^{\alpha}_{\rho,Tr}(B) \geq \frac{1}{4}\left| Tr\left(\left(\frac{\rho^{\alpha}+\rho^{1-\alpha}}{2}\right)^2 [A,B]\right)\right|^2,$$

where $W^{\alpha}_{\rho,Tr}(A) = \sqrt{K^{\alpha}_{\rho,Tr}(A)L^{\alpha}_{\rho,Tr}(A)}$.

Corollary 2.13 ([12])**.** *Let $Tr : \mathcal{B}(H) \to \mathbb{C}$ be a usual trace functional. Then for any self-adjoint elements $A, B \in \mathcal{B}(H)$ and Tr-density operator $\rho \in \mathcal{B}(H)$,*

$$U_{\rho,Tr}(A)U_{\rho,Tr}(B) \geq \frac{1}{4}\left| Tr(\rho[A,B])\right|^2,$$

where $U_{\rho,Tr}(A) = \sqrt{I^{1/2}_{\rho,Tr}(A)J^{1/2}_{\rho,Tr}(A)}$.

3. Proofs

3.1. Proof of Theorem 2.4

Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{B}$ be a C^*-subalgebra of $\mathcal{A}$. To prove Theorem 2.4, we need some lemmas.

Lemma 3.1. *Let $T : \mathcal{A} \longrightarrow \mathcal{B}$ be a tracial positive linear operator and (f, g) be a pair of functions. If T and (f, g) satisfy the property (**P**), then for any $A \in \mathcal{A}$*

$$I^{(f,g)}_{\rho,T}(A) \geq 0.$$

Proof. The proof is clear owing to the property (**P**). □

Now, for any $A, B \in \mathcal{A}$, we define

$$\widetilde{\mathrm{Corr}}^{(f,g)}_{\rho,T}(A,B) := \frac{1}{2}\left(\mathrm{Corr}^{(f,g)}_{\rho,T}(A,B) + \mathrm{Corr}^{(f,g)}_{\rho,T}(B^*,A^*)\right)$$

and

$$\widetilde{I}^{(f,g)}_{\rho,T}(A) := \widetilde{\mathrm{Corr}}^{(f,g)}_{\rho,T}(A,A).$$

Note that if A is a self-adjoint element, then $\widetilde{I}^{(f,g)}_{\rho,T}(A) = I^{(f,g)}_{\rho,T}(A)$. Also, it is easy to show that the following properties hold:

(i) $\widetilde{\mathrm{Corr}}^{(f,g)}_{\rho,T}(A,A) \geq 0$ for $A \in \mathcal{A}$ (using Lemma 3.1);

(ii) $\widetilde{\mathrm{Corr}}^{(f,g)}_{\rho,T}(A,B+\alpha C) = \widetilde{\mathrm{Corr}}^{(f,g)}_{\rho,T}(A,B)+\alpha\widetilde{\mathrm{Corr}}^{(f,g)}_{\rho,T}(A,C)$ for $A, B, C \in \mathcal{A}$ and $\alpha \in \mathbb{C}$;

(iii) $\widetilde{\mathrm{Corr}}^{(f,g)}_{\rho,T}(A,B)^* = \widetilde{\mathrm{Corr}}^{(f,g)}_{\rho,T}(B,A)$ for $A, B \in \mathcal{A}$ (using the fact that $f(\rho)g(\rho) = g(\rho)f(\rho)$).

Using the module property (2.1), we can prove the following lemma.

Lemma 3.2. *If $T : \mathcal{A} \longrightarrow \mathcal{B}$ is a tracial positive module operator, then we have*

$$\widetilde{\mathrm{Corr}}_{\rho,T}^{(f,g)}(A, BC) = \widetilde{\mathrm{Corr}}_{\rho,T}^{(f,g)}(A, B)C$$

for all $A, B \in \mathcal{A}$ and $C \in \mathcal{B}$ and T-density operator ρ.

Proof. For any $A, B \in \mathcal{A}$ and any $C \in \mathcal{B}$ we obtain that

$$\begin{aligned}
\widetilde{\mathrm{Corr}}_{\rho,T}^{(f,g)}(A, BC) &= \frac{1}{2}\left(\mathrm{Corr}_{\rho,T}^{(f,g)}(A, BC) + \mathrm{Corr}_{\rho,T}^{(f,g)}((BC)^*, A^*)\right) \\
&= \frac{1}{2}\left(T(f(\rho)g(\rho)A^*BC) - T(f(\rho)A^*g(\rho)BC)\right. \\
&\quad \left.+T(f(\rho)g(\rho)BCA^*) - T(f(\rho)BCg(\rho)A^*)\right) \\
&= \frac{1}{2}\left(T(f(\rho)g(\rho)A^*B)C - T(f(\rho)A^*g(\rho)B)C\right. \\
&\quad \left.+T(f(\rho)g(\rho)BA^*)C - T(f(\rho)Bg(\rho)A^*)C\right) \\
&= \frac{1}{2}\left(T(f(\rho)g(\rho)A^*B) - T(f(\rho)A^*g(\rho)B)\right. \\
&\quad \left.+T(f(\rho)g(\rho)BA^*) - T(f(\rho)Bg(\rho)A^*)\right) C \\
&= \widetilde{\mathrm{Corr}}_{\rho,T}^{(f,g)}(A, B)C,
\end{aligned}$$

by using module and tracial properties of T. □

Lemma 3.3 (Cauchy-Schwarz inequality, see [1,2])**.** *Let $T : \mathcal{A} \to \mathcal{B}$ be a tracial positive module operator. Then we have*

$$|T(x^*y)|^2 \le T(x^*x)T(y^*y), \qquad x, y \in \mathcal{A}.$$

Now we prove the following theorem by applying the above lemmas.

Proof of Theorem 2.4. Define the map $\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{B}$ as $\langle A, B \rangle = \widetilde{\mathrm{Corr}}_{\rho,T}^{(f,g)}(A, B)$. Then due to Lemma 3.2, it is clear that $\langle \cdot, \cdot \rangle$ is a $\mathcal{B}$-valued semi-inner product and $\mathcal{A}$ is a $\mathcal{B}$-module. Using Lemma 3.3, we obtain for all self-adjoint elements $A, B \in \mathcal{A}$ that

$$\begin{aligned}
\left|\mathrm{Re}\,\mathrm{Corr}_{\rho,T}^{(f,g)}(A, B)\right|^2 &= \left|\frac{1}{2}\left(\mathrm{Corr}_{\rho,T}^{(f,g)}(A, B) + \mathrm{Corr}_{\rho,T}^{(f,g)}(B, A)\right)\right|^2 \\
&= \left|\widetilde{\mathrm{Corr}}_{\rho,T}^{(f,g)}(A, B)\right|^2 \\
&\le \langle A, A \rangle \langle B, B \rangle \\
&= I_{\rho,T}^{(f,g)}(A) I_{\rho,T}^{(f,g)}(B),
\end{aligned}$$

which gives the proof. □

3.2. Proof of Theorem 2.10

Proof of Theorem 2.10. Let $Z \in \mathcal{B}$ be a self-adjoint element and put

$$M = i\big[f(\rho), A_0\big]Z + \big\{f(\rho), B_0\big\}, \tag{3.1}$$

where $A_0 = A - T(\rho A)$ and $B_0 = B - T(\rho B)$. Then by the positivity of T we obtain that

$$\begin{aligned}
0 &\le T(M^*M) \\
&= T\left(\left(iZ\big[f(\rho), A_0\big] + \big\{f(\rho), B_0\big\}\right)\left(i\big[f(\rho), A_0\big]Z + \big\{f(\rho), B_0\big\}\right)\right) \\
&= T\left(-Z[f(\rho), A_0]^2 Z + iZ[f(\rho), A_0]\{f(\rho), B_0\}\right. \\
&\quad \left. + i\{f(\rho), B_0\}[f(\rho), A_0]Z + \{f(\rho), B_0\}^2\right) \\
&= ZT\left((i[f(\rho), A_0])^2\right) Z + 2iT\left([f(\rho), A_0]\{f(\rho), B_0\}\right) Z + T\left(\{f(\rho), B_0\}^2\right) \\
&= 2K^f_{\rho,T}(A)Z^2 + 2iT\left([f(\rho), A_0]\{f(\rho), B_0\}\right) Z + 2L^f_{\rho,T}(B), \qquad (3.2)
\end{aligned}$$

where for the last equality, we use the fact that $T(\mathcal{A}) \subseteq Z(\mathcal{B})$. On the other hand, since T is a tracial module operator, we obtain that

$$\begin{aligned}
&T\left([f(\rho), A_0]\{f(\rho), B_0\}\right) \\
&= T\left(f(\rho)A_0 f(\rho)B_0 + f(\rho)A_0 B_0 f(\rho) - A_0 f(\rho)^2 B_0 - f(\rho)A_0 f(\rho)B_0\right) \\
&= T\left(f(\rho)^2(A_0B_0 - B_0A_0)\right) \\
&= T\left(f(\rho)^2[A, B]\right). \qquad (3.3)
\end{aligned}$$

Hence, combining (3.2) with (3.3), we have

$$0 \le 2K^f_{\rho,T}(A)Z^2 + 2iT\left(f(\rho)^2[A, B]\right) Z + 2L^f_{\rho,T}(B). \tag{3.4}$$

Now, without loss of the generality, we assume that $K^f_{\rho,T}(A) > 0$. Put

$$Z := -\frac{i}{2}K^f_{\rho,T}(A)^{-1}T\left(f(\rho)^2[A, B]\right).$$

Then we can see, using the fact that $T(\mathcal{A}) \subseteq Z(\mathcal{B})$,

$$-\frac{1}{2}K^f_{\rho,T}(A)^{-1}T\left(f(\rho)^2[A, B]\right)^2 + K^f_{\rho,T}(A)^{-1}T\left(f(\rho)^2[A, B]\right)^2 + 2L^f_{\rho,T}(B) \ge 0,$$

equivalently,

$$K^f_{\rho,T}(A)L^f_{\rho,T}(B) \ge -\frac{1}{4}T\left(f(\rho)^2[A, B]\right)^2.$$

Since $T\left(f(\rho)[A, B]\right)^* = -T\left(f(\rho)[A, B]\right)$, we have

$$K^f_{\rho,T}(A)L^f_{\rho,T}(B) \ge \frac{1}{4}\Big|T\left(f(\rho)^2[A, B]\right)\Big|^2. \tag{3.5}$$

Finally, since $T(\mathcal{A}) \subseteq Z(\mathcal{B})$,

$$\begin{aligned}
W^f_{\rho,T}(A)W^f_{\rho,T}(B) &= \sqrt{K^f_{\rho,T}(A)L^f_{\rho,T}(A)}\sqrt{K^f_{\rho,T}(B)L^f_{\rho,T}(B)} \\
&= (K^f_{\rho,T}(A)^{\frac{1}{2}}L^f_{\rho,T}(A)^{\frac{1}{2}})(K^f_{\rho,T}(B)^{\frac{1}{2}}L^f_{\rho,T}(B)^{\frac{1}{2}})
\end{aligned}$$

$$= \left(K^f_{\rho,T}(A)L^f_{\rho,T}(B)\right)^{\frac{1}{2}}\left(K^f_{\rho,T}(B)L^f_{\rho,T}(A)\right)^{\frac{1}{2}}$$
$$\geq \frac{1}{4}\Big|T\left(f(\rho)^2[A,B]\right)\Big|^2$$

is obtained by (3.5). □

Remark 3.4. In the proof of Theorem 2.10, if we take $M := i\left(\rho^{1/2}A_0\right)Z + \rho^{1/2}B_0$ in (3.1), then we have

$$V_{\rho,T}(A)V_{\rho,T}(B) \geq \left|\frac{1}{2}T\left(\rho[A,B]\right)\right|^2, \tag{3.6}$$

where $V_{\rho,T}(A) = \mathrm{T}(\rho A^2) - \mathrm{T}(\rho A)^2$ which is the Heisenberg uncertainty relation for T (see [2]). Indeed, if we take $Z = -\frac{i}{2}T\left(\rho A_0^2\right)^{-1}T\left(\rho[A_0,B_0]\right)$, where $A_0 = A - T(\rho A)$ and $B_0 = B - T(\rho B)$, then we have (3.6) by using same method in the proof.

References

[1] B. J. Choi, U. C. Ji, and Y. Lim, *Inequalities for positive module operators on von Neumann algebras*, J. Math. Phys. **59** (2018), no. 6, 063513, 11 pp. https://doi.org/10.1063/1.5009615

[2] A. Dadkhah and M. S. Moslehian, *Quantum information inequalities via tracial positive linear maps*, J. Math. Anal. Appl. **447** (2017), no. 1, 666–680. https://doi.org/10.1016/j.jmaa.2016.10.027

[3] A. Dadkhah, M. S. Moslehian, and K. Yanagi, *Noncommutative versions of inequalities in quantum information theory*, Anal. Math. Phys. **9** (2019), no. 4, 2151–2169. https://doi.org/10.1007/s13324-019-00309-7

[4] J. I. Fujii, *A trace inequality arising from quantum information theory*, Linear Algebra Appl. **400** (2005), 141–146. https://doi.org/10.1016/j.laa.2004.11.009

[5] S. Furuichi, *Inequalities for Tsallis relative entropy and generalized skew information*, Linear Multilinear Algebra **59** (2011), no. 10, 1143–1158. https://doi.org/10.1080/03081087.2011.574624

[6] S. Furuichi and K. Yanagi, *Schrödinger uncertainty relation, Wigner-Yanase-Dyson skew information and metric adjusted correlation measure*, J. Math. Anal. Appl. **388** (2012), no. 2, 1147–1156. https://doi.org/10.1016/j.jmaa.2011.10.061

[7] S. Furuichi, K. Yanagi, and K. Kuriyama, *Trace inequalities on a generalized Wigner-Yanase skew information*, J. Math. Anal. Appl. **356** (2009), no. 1, 179–185. https://doi.org/10.1016/j.jmaa.2009.02.043

[8] P. Gibilisco and T. Isola, *Uncertainty principle and quantum Fisher information*, Ann. Inst. Statist. Math. **59** (2007), no. 1, 147–159. https://doi.org/10.1007/s10463-006-0103-3

[9] W. Heisenberg, *Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik*, Z. Phys. **43** (1927), 172–198.

[10] C. K. Ko and H. J. Yoo, *Uncertainty relation associated with a monotone pair skew information*, J. Math. Anal. Appl. **383** (2011), no. 1, 208–214. https://doi.org/10.1016/j.jmaa.2011.05.014

[11] ———, *Schrödinger uncertainty relation and convexity for the monotone pair skew information*, Tohoku Math. J. (2) **66** (2014), no. 1, 107–117. https://doi.org/10.2748/tmj/1396875665

[12] S. Luo, *Heisenberg uncertainty relation for mixed states*, Phys. Rev. A **72** (2005), 042110.

[13] Y. M. Park, *Improvement of uncertainty relations for mixed states*, J. Math. Phys. **46** (2005), no. 4, 042109, 13 pp. https://doi.org/10.1063/1.1876874
[14] E. Schrödinger, *About Heisenberg uncertainty relation*, Translation of Proc. Prussian Acad. Sci. Phys. Math. Sect. **19** (1930), 296–303, Bulgar. J. Phys. **26** (1999), 193–203.
[15] K. Yanagi, S. Furuichi, and K. Kuriyama, *A generalized skew information and uncertainty relation*, IEEE Trans. Inform. Theory **51** (2005), no. 12, 4401–4404. https://doi.org/10.1109/TIT.2005.858971

Byoung Jin Choi
Department of Mathematics Education
Jeju National University
Jeju 63243, Korea
Email address: choibj@jejunu.ac.kr

Mi Ra Lee
Department of Mathematics
Chungbuk National University
Cheongju 28644, Korea
Email address: mrlee1012@chungbuk.ac.kr