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TOPOLOGICAL STABILITY IN HYPERSPACE DYNAMICAL SYSTEMS

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Abstract. In this paper we extend the concept of topological stability from continuous maps to the corresponding induced maps and prove that a continuous map is topologically stable if and only if its induced map also is topologically stable.

1. Introduction

The notion of expansiveness for dynamical systems which is a property shared by a large class of dynamical systems exhibiting chaotic behavior was introduced in the middle of the twentieth century by Utz [17]. The shadowing theory is now an important and rapidly developing branch of the modern global theory of dynamical systems. The notion of pseudo orbit goes back to Birkhoff [5]. The real development of the shadowing theory started after the classical results of Anosov [1] and Bowen [6].

The theory of dynamical systems on hyperspaces has several variations and directions. Bauer and Sigmund [4] considered the relationship between a homeomorphism on a compact metric space and its induced action on the space of probability measures. Also, they proved that the induced homeomorphism on the space of probability measures is expansive if and only if the initial space is finite. In the similar way, Artigue [3] defined the strong concept of expansiveness on a hyperspace which is called *hyper-expansive* and proved that if a compact metric space X admits a hyper-expansive homeomorphism, then X is a countable set. Wu and Xue [19] showed that if a continuous map f is a positively expansive open map, then the induced map 2^f has the shadowing property.

The definitions of expansiveness and the shadowing property for homeomorphisms are usually playing an important role in the investigation of the stability theory. Walters [18] introduced the notion of topological stability for homeomorphisms in which continuous perturbations are allowed, and showed that

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every expansive homeomorphism with the shadowing property on a compact metric space is topologically stable. Thereafter, Lee and Morales [12] introduced the concepts of topological stability and shadowing property for Borel measures on a compact metric space, and showed that any expansive measure with the shadowing property is topologically stable. Afterwards, Chung and Lee [7] extended the notion of topological stability from homeomorphisms to group actions on compact metric spaces, and proved that if every finite generated group action is expansive and has the shadowing property, then it is topologically stable. Also, Lee *et al.* [11] extended the Walters's stability theorem to homeomorphisms on locally compact metric spaces. Recently, Koo *et al.* [10] introduced the some pointwise notions for homeomorphisms and studied the relationship between the pointwise topological stability and shadowableness for expansive homeomorphisms.

Inspired by many interesting results relating initial dynamics and their induced hyperspace dynamics, we introduce the notion of topological stability in hyperspace dynamics and obtain implications and relations related to the topological stability between continuous maps and their induced maps. Furthermore, we give some examples to illustrate our results.

2. Preliminaries

We recall the definitions and fix the notations which will be important in our work. For the basic notions and results concerning hyperspaces, we mainly refer to some books [8, 14].

Throughout this paper, X denotes a non-empty compact metric space with a metric d and $f : X \to X$ a continuous map on it. Let 2^X denote the collection of all nonempty compact subsets of X, i.e.,

$$2^X = \{A \subseteq X \mid A \text{ is non-empty and closed}\}$$

which will be referred to as a *hyperspace* of X. The map $2^f : 2^X \to 2^X$ given by $2^f(A) = f(A)$ is called the induced map on a hyperspace by a continuous map $f : X \to X$.

For any $A \in 2^X$ and any $r > 0$, the *open d-ball in X* about A of radius r is given by

$$B_d(A, r) = \{x \in X \mid d(x, A) < r\}.$$

Then 2^X is a compact metric space with Hausdorff metric d_H given by

$$d_H(A, B) = \inf\{\epsilon > 0 \mid A \subseteq B_d(B, \epsilon) \text{ and } B \subseteq B_d(A, \epsilon)\}$$

for any $A, B \in 2^X$.

The hyperspace 2^X has very nice properties. For example, it is known that 2^X inherits the compactness, connectedness and arc-wise connectedness of X, respectively.

To explain the extended topological stability on hyperspaces of a compact metric space we recall the classical C_0-distance between continuous maps f

and g. We denote by $C(X)$ the set of all continuous maps from X to itself and by $H(X)$ the set of all homeomorphisms of X. We define two metrics d_{C_0} on $C(X)$ and $\bar{d}_{C_0}$ on $H(X)$ by

$$d_{C_0}(f,g) = \sup\{d(f(x),g(x)) \mid x \in X\}$$

for $f,g \in C(X)$ and

$$\bar{d}_{C_0}(f,g) = \max\{d_{C_0}(f,g), d_{C_0}(f^{-1},g^{-1})\}$$

for $f,g \in H(X)$, respectively. Then we note that d_{C_0} and $\bar{d}_{C_0}$ are equivalent metrics on $H(X)$.

Next, we recall some classical concepts of dynamical systems which are used in this paper. For the basic notions and results concerning shadowing properties in dynamical systems, we mainly refer to some books [15, 16].

Let $\mathbb{Z}$ and $\mathbb{N}_0$ be the set of integers and nonnegative integers, respectively.

For $\delta > 0$, a sequence $\xi = \{x_i\}_{i\in\mathbb{N}_0}$ of points in X is a *δ-pseudo orbit* of f if $d(f(x_i), x_{i+1}) < \delta$ for every $i \in \mathbb{N}_0$. For given $\varepsilon > 0$, we say that a δ-pseudo orbit $\xi = \{x_i\}_{i\in\mathbb{N}_0}$ for f is ε-shadowed by a point $y \in X$ if $d(f^i(y), x_i) < \varepsilon$ for every $i \in \mathbb{N}_0$. We say that a continuous map $f : X \to X$ has the *shadowing property* if for all $\epsilon > 0$ there is $\delta > 0$ such that every δ-pseudo orbit is ε-shadowed by a point in X. We write $\mathrm{Orb}_f(x)$ for the orbit of f at x. We say that a point $p \in X$ is a *periodic* of f if there is the minimal integer $n > 0$ (called *period*) such that $f^n(p) = p$. Periodic points of period 1 are fixed points. We say that a point $p \in X$ is a *non-wandering point* if for each neighborhood U of p there exists $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. We say that a point $p \in X$ is *stable* if for all $\epsilon > 0$ there is $\delta > 0$ such that if $d(x,p) < \delta$, then $d(f^n(x), f^n(p)) < \epsilon$ for all $n \geq 0$. A point $p \in X$ is said to be *unstable* if it is stable for f^{-1}. We say that a point $p \in X$ is *asymptotically stable* if it is stable and there is $\gamma > 0$ such that if $d(x,p) < \gamma$, then $d(f^n(x), f^n(p)) \to 0$ as $n \to \infty$. If $Orb_f(p)$ is an asymptotically stable periodic orbit, then the orbit of p is said to be an *attractor*. A *repeller* is an attractor of f^{-1}.

We write $\mathrm{Fix}(f)$ for the set of fixed points of f, $\Omega(f)$ the set of non-wandering points of f, $\mathrm{Per}(f)$ the set of all periodic points of f, $\mathrm{Per}_a(f)$ the set of attractor periodic points of f and $\mathrm{Per}_r(f)$ the set of repeller periodic points of f.

We recall that a continuous surjection $f : X \to X$ is *positively expansive* if there is a constant $e > 0$ such that if $x \neq y$ in X, then $d(f^n(x), f^n(y)) > e$ for some non-negative integer n (e is called an *expansive constant* for f). For a compact metric space, this is independent on the choice of the compatible metrics used, although not the expansive constants.

3. Main results

In this section we introduce the notion of topological stability for the induced map $2^f : 2^X \to 2^X$ on a hyperspace 2^X by a continuous map $f : X \to X$ on a compact metric space X and study the connection on topological stabilities for a continuous map f and its induced map 2^f via an imbedding of $C(X)$

into $C(2^X)$ given by the induced maps. Furthermore, in case $f : X \to X$ is a homeomorphism on a compact metric space, we show that if the map 2^f induced by a homeomorphism $f : X \to X$ is expansive, then f has the shadowing property, so it is topologically stable. Firstly, we introduce the notion of topological stability for the induced maps on a subspace $IC(2^X)$ of $C(2^X)$ which is similar to the standard notion given by Walters [18]. Here $IC(2^X)$ is given in below Lemma 3.1.

Definition. Let $2^f : 2^X \to 2^X$ be the induced map of a continuous map $f : X \to X$ on a compact metric space X. We say that an induced map 2^f is *topologically stable* in $IC(2^X)$ (or topological stable in induced sense) if given $\epsilon > 0$ there exists $\delta > 0$ such that for each $2^g \in C(2^X)$ with $d_H^{C_0}(2^f, 2^g) < \delta$ there is a continuous map $h \in C(X)$ satisfying $2^f \circ 2^h = 2^h \circ 2^g$ and $d_H^{C_0}(2^h, \mathrm{Id}) < \epsilon$, where $\mathrm{Id} : 2^X \to 2^X$ is the identity map and $d_H^{C_0}$ is given by

$$d_H^{C_0}(2^f, 2^g) = \sup\{d_H(2^f(A), 2^g(A)) \mid A \in 2^X\}$$

for $2^f, 2^g \in C(2^X)$.

For the proof of Theorem 3.2, we need the following result.

Lemma 3.1. *The map $\phi : C(X) \to C(2^X)$ given by $\phi(f) = 2^f$ is an imbedding of $C(X)$ into $C(2^X)$. Moreover, $IC(2^X)$ is a closed subspace of $C(2^X)$. Here $\phi(C(X)) := IC(2^X)$.*

Proof. The map ϕ is well defined since the induced map 2^f given by a continuous map f is also continuous (see [14, Theorem 0.52]). Suppose that $\phi(f) = \phi(g)$. Taking $A = \{x\}$ for each $x \in X$, we have $\{f(x)\} = 2^f(A) = 2^g(A) = \{g(x)\}$ for each $x \in X$. Thus ϕ is injective.

To show that ϕ and ϕ^{-1} are continuous, first, we prove that if a sequence $\{f_n\}$ in $(C(X), d_{C_0})$ converges to f, then $\{2^{f_n}\}$ converges to 2^f in $(C(2^X), d_H^{C_0})$. Assume that for any $\epsilon > 0$ and each $A \in 2^X$, there exists $N \in \mathbb{N}$ such that $d_{C_0}(f_n, f) < \epsilon$ for all $n \geq N$. For all $a \in X$ and $n \geq N$, we have $d(f_n(a), f(a)) < \epsilon$. This we see that $f_n(A) \subset B_d(f(A), \epsilon)$, $n \geq N$.

Similarly, for all $A \in 2^X$ we get $f(A) \subset B_d(f_n(A), \epsilon)$ for every $n \geq N$. Thus, there exists $N \in \mathbb{N}$ such that $d_H^{C_0}(2^{f_n}(A), 2^f(A)) < \epsilon$ for all $n \geq N$ and each $A \in 2^X$.

On the other hand, similarly we show that if $\{2^{f_n}\}$ converges to 2^f in $C(2^X)$, then $\{f_n\}$ in $C(X)$ converges to f. Therefore, the map ϕ is an embedding of $C(X)$ into $C(2^X)$.

Next, we prove that $IC(2^X)$ is closed in $C(2^X)$. We claim that there exists a sequence $\{2^{f_n}\}$ in $IC(2^X)$ which converges to F such that there is $f \in C(X)$ satisfying $2^f = F$. Consider $\phi^{-1}(2^{f_n}) = f_n$. For every $\epsilon > 0$ take $N > 0$ such that $d_H^{C_0}(2^{f_n}, 2^{f_m}) < \epsilon$ for all $n, m \geq N$. This means that any compact set A satisfies $d_H(2^{f_n}(A), 2^{f_m}(A)) < \epsilon$ for all $n, m \geq N$. For any compact subset A of X, we choose $A_x = \{x\}$ for all $x \in X$. Then we see that $d_{C_0}(f_n, f_m) < \epsilon$ for all $n, m \geq N$ and all $x \in X$. Since the sequence $\{f_n\}$ is Cauchy in complete metric space $(C(X), d_{C_0})$, there exists $f \in C(X)$ such that $\{f_n\}$ converge to f. Also, it follows from the continuity of ϕ that the sequence $\{\phi(f_n)\} = \{2^{f_n}\}$ converges to $\phi(f) = 2^f$ with metric $d_H^{C_0}$. In view of uniqueness of limit, we obtain $F = 2^f$. Thus we have $F \in IC(2^X)$. Hence $IC(2^X)$ is closed in $C(2^X)$ with metric $d_H^{C_0}$. The proof is complete. □

Using Lemma 3.1, we obtain an equivalent relation between the classical topological stability of a continuous map and the topological stability of its induced map.

Theorem 3.2. *A continuous map $f : X \to X$ is topologically stable in $C(X)$ if and only if the induced map $2^f : 2^X \to 2^X$ is topologically stable in $IC(2^X)$.*

Proof. Suppose that a continuous map $f : X \to X$ is topologically stable and $\epsilon > 0$ is given. Then there are $\delta > 0$ and $h \in C(X)$ satisfying the definition of topological stability for f. Since ϕ^{-1} is continuous at $2^f \in IC(2^X)$, then for $\delta > 0$ there exists a $\delta_1 > 0$ such that if $2^g \in IC(2^X)$ with $d_H^{C_0}(2^f, 2^g) < \delta_1$, then we have

$$d_{C_0}(\phi^{-1}(2^f), \phi^{-1}(2^g)) = d_{C_0}(f, g) < \delta.$$

From $f \circ h = h \circ g$, we obtain

$$\begin{aligned}
\phi(f \circ h)(A) &= (2^{f \circ h})(A) \\
&= (f \circ h)(A) \\
&= f(h(A)) \\
&= 2^f(h(A)) \\
&= 2^f(2^h(A)) \\
&= (2^f \circ 2^h)(A), \quad A \in 2^X,
\end{aligned}$$

and

$$\phi(h \circ g)(A) = (2^h \circ 2^g)(A), \quad A \in 2^X.$$

From above conditions, we have $2^f \circ 2^h = 2^h \circ 2^g$.

Fixed any $A \in 2^X$. From continuity of h and compactness of A, we can see that

$$h(A) \subset \bigcup_{a \in A} B(a, \epsilon) \quad \text{and} \quad A \subset \bigcup_{a \in A} B(h(a), \epsilon)$$

if and only if

$$h(A) \subset B_d(A, \epsilon) \quad \text{and} \quad A \subset B_d(h(A), \epsilon),$$

i.e., $h(A) \subset B_d(A, \epsilon)$ and $A \subset B_d(h(A), \epsilon)$ for any $A \in 2^X$. This means that $d_H^{C_0}(2^h, Id_{2^X}) < \epsilon$. Hence the induced map 2^f is topologically stable in $IC(2^X)$.

Conversely, suppose that $2^f \in IC(2^X)$ is topologically stable. For any $\epsilon > 0$, there exist $\delta > 0$ and $2^h \in IC(2^X)$ satisfying topological stability of 2^f. Since ϕ is continuous at $f \in C(X)$, there is $\delta_2 > 0$ such that for any $g \in C(X)$ with $d_{C_0}(f,g) < \delta_2$, we have

$$d_H^{C_0}(\phi(f), \phi(g)) = d_H^{C_0}(2^f, 2^g) < \delta.$$

Take $A = \{x\}$ for each $x \in X$, it follows from $2^f \circ 2^h = 2^h \circ 2^g$ that

$$\begin{aligned}(2^f \circ 2^h)(A) &= \{f(h(x))\}\\ &= \{h(g(x))\} = (2^h \circ 2^g)(A),\ x \in X.\end{aligned}$$

This implies that $f \circ h = h \circ g$.

From $d_H^{C_0}(2^h, Id_{2^X}) < \epsilon$, we have

$$2^h(A) \subset B_d(A, \epsilon) \quad \text{and} \quad A \subset B_d(2^h(A), \epsilon),\ A \in 2^X. \tag{1}$$

Taking $A = \{a\}$ for each $a \in X$, (1) implies

$$\{h(a)\} \subset B(a, \epsilon) \quad \text{and} \quad \{a\} \subset B(h(a), \epsilon).$$

Since $a \in X$ is arbitrary, we obtain $d_{C_0}(h, Id_X) < \epsilon$. Hence, $f \in C(X)$ is topologically stable. This completes the proof. □

From Theorem 3.2, we can obtain a hyperspace version of the topological stability for induced maps on a hyperspace.

Corollary 3.3. *If a continuous map $f : X \to X$ is positively expansive with the shadowing property, then the induced map $2^f : 2^X \to 2^X$ is topologically stable in $IC(2^X)$.*

Lastly, we obtain some results concerning topological stability for the induced homeomorphisms on a hyperspace. For the rest part of our results, hereafter, let $f : X \to X$ be a homeomorphism on a compact metric space X. Recall that a homeomorphism $f : X \to X$ is *expansive* if there exists a positive constant e such that $d(f^n(x), f^n(y)) < e$ for all $n \in \mathbb{Z}$ implies $x = y$ (see [17]). Walters [18] showed that every expansive homeomorphism $f : X \to X$ on a compact metric space X with the shadowing property is topologically stable.

Remark 3.4. Artigue [3, Theorem 2.2] introduced the notion of hyper-expansiveness of a homeomorphism $f : X \to X$ on a compact metric space, i.e., every pair of different compact sets in X are separated by the homeomorphism f in the Hausdorff metric if and only if the induced map $2^f : 2^X \to 2^X$ is expansive. Also, he proved that a homeomorphism $f : X \to X$ is hyper-expansive if and only if f has a finite number of orbits and $\Omega(f) = \mathrm{Per}_a(f) \cup \mathrm{Per}_r(f)$. This implies that a compact metric space X admitting a hyper-expansive homeomorphism is a countable set and f has finitely many periodic points. Then we have $\Omega(f^k) = \mathrm{Fix}_a(f^k) \cup \mathrm{Fix}_r(f^k)$ for the least common multiple k of any periods of all periodic points in the finite set $\mathrm{Per}(f)$.

For topological stability of induced homeomorphisms, let $IH(2^X)$ be a subspace of $H(2^X)$ denoted by

$$IH(2^X) = \{2^f \in H(2^X) \mid f \in H(X)\}.$$

We recall that an induced homeomorphism $2^f : 2^X \to 2^X$ is *topologically stable* if given $\epsilon > 0$ there exists $\delta > 0$ such that for any $g \in H(X)$ with $\bar{d}_H^{C_0}(2^f, 2^g) < \delta$ there is a continuous map $2^h \in IC(2^X)$ satisfying $2^f \circ 2^h = 2^h \circ 2^g$ and $d_H^{C_0}(2^h, Id) < \epsilon$. Here $Id : 2^X \to 2^X$ is the identity map and $\bar{d}_H^{C_0}$ is a metric on $IH(2^X)$ given by

$$\bar{d}_H^{C_0}(2^f, 2^g) = \max\{d_H^{C_0}(2^f, 2^g), d_H^{C_0}(2^{f^{-1}}, 2^{g^{-1}})\}.$$

Note that it is not yet known whether the concept of induced topological stability is true in $H(2^X)$ since it guess that there is a big gab between two topological stabilities in $IH(2^X)$ and $H(2^X)$, respectively.

We give two examples to illustrate the characterization of hyper-expansive homeomorphisms.

Example 3.5 ([9, Remark 3.4])**.** Let $X = \{0, 2\} \cup \{a_i = 2^{-i}, a_{-i} = 2 - 2^{-i}\}_{i \in \mathbb{N}_0}$ be a subspace of $\mathbb{R}$. The map $f : X \to X$ defined by

$$f(x) = \begin{cases} 0, & \text{if } x = 0, \\ 2, & \text{if } x = 2, \\ a_{i+1}, & \text{if } x = a_i,\ i \in \mathbb{Z}, \end{cases}$$

is a hyper-expansive homeomorphism.

Example 3.6. Let $\Sigma_2 = \{x = (x_i) \mid x_i \in \{0, 1\},\ i \in \mathbb{Z}\}$ be the compact metric space equipped with the metric $d(x, y) = \sum_{i=-\infty}^{\infty} \frac{|x_i - y_i|}{2^i}$ for each pair $x = (x_i), y = (y_i) \in \Sigma_2$. The shift map $\sigma : \Sigma_2 \to \Sigma_2$ defined by $\sigma(x_i) = x_{i+1}$ is expansive but not hyper-expansive. Since the shift map σ has infinitely many periodic points, the induced homeomorphism $2^\sigma : 2^{\Sigma_2} \to 2^{\Sigma_2}$ admits infinitely many fixed points.

Remark 3.7. Morales [13] introduced a shadowable point of a homeomorphism which defined to be a point such that the shadowing lemma holds for pseudo orbits through the point. Recall that a point $x \in X$ is *shadowable* of a homeomorphism $f : X \to X$ if for any $\epsilon > 0$ there exists $\delta > 0$ corresponding to ϵ and x such that every δ-pseudo orbit $\xi = \{x_i\}_{i \in \mathbb{Z}}$ with $x_0 = x$ (or through x) is ϵ-shadowed by some point in X. We denote $Sh(f)$ the set of all shadowable points of f. Note that $f \in H(X)$ has the shadowing property if and only if $Sh(f) = X$.

From Theorem 2.2 in [3] and Theorem 2.3.3 in [2], we can obtain the following result.

Theorem 3.8. *Let $f : X \to X$ be a homeomorphism on a compact metric space. If its induced map $2^f : 2^X \to 2^X$ is expansive, then $f : X \to X$ has the shadowing property.*

Proof. It follows from [2, Theorems 2.2.4 and 2.3.3] that a homeomorphism $f : X \to X$ is expansive with the shadowing property if and only if f^k is expansive with the shadowing property for nonzero integer k. From Remarks 3.4 and 3.7, it is sufficient to show that if 2^f is expansive with $\mathrm{Per}_s(f) = \mathrm{Fix}_s(f)$ for $s = a, r$, respectively, then $X \subset \mathrm{Sh}(f)$. We consider two cases for the proof.
Case 1. $X \setminus \Omega(f) \subset \mathrm{Sh}(f)$. Let $x \notin \Omega(f)$. Then there are $q \in \mathrm{Fix}_a(f)$ and $p \in \mathrm{Fix}_r(f)$ satisfying $f^i(x) \to q$ as $i \to \infty$ and $f^i(x) \to p$ as $i \to -\infty$. Thus there exists $\varepsilon_0 > 0$ such that $f^{-1}(B(p, \epsilon_0)) \subset B(p, \epsilon_0)$ and $f(B(q, \epsilon_0)) \subset B(q, \epsilon_0)$. For any $0 < \epsilon < \epsilon_0$ satisfying $B(p, \epsilon) \cap B(q, \epsilon) = \emptyset$, there is $N \in \mathbb{N}$ such that $f^i(x) \in B(q, \epsilon)$ for every $i \geq N$ and $f^i(x) \in B(p, \epsilon)$ for every $i \leq -N$. Also, there exists $0 < \delta < \epsilon$ such that $B(f^i(x), \delta) = \{f^i(x)\}$ for every $i \in [-N, N]$. Let $\xi = \{x_i\}_{i\in\mathbb{Z}}$ be any δ-pseudo orbit of f through x. From $B(f^i(x), \delta) = \{f^i(x)\}$ for every $i \in [-N, N]$, we obtain $x_i = f^i(x)$ and $d(f^i(x), x_i) = 0$ for each $i \in [-N, N]$. Furthermore, we have $f^i(x), x_i \in B(q, \varepsilon)$ for each $i = N + j$ with $j \in \mathbb{N}$. Similarly, we have $f^i(x), x_i \in B(p, \epsilon)$ for each $i \leq -N$ by same argument. Thus we get

$$d(f^i(x), x_i) \begin{cases} \leq d(f^i(x), q) + d(q, x_i) < 2\varepsilon, \ i \geq N + 1, \\ = 0, \ i \in [-N, N], \\ \leq d(f^i(x), p) + d(p, x_i) < 2\varepsilon, \ i \leq -N - 1. \end{cases}$$

Hence $\xi = \{x_i\}_{i\in\mathbb{Z}}$ can be 2ε-shadowing by the point $x \in X$. So $x \in \mathrm{Sh}(f)$.

Case 2. $\Omega(f) \subset \mathrm{Sh}(f)$. Let $x \in \Omega(f) = \mathrm{Fix}_a(f) \cup \mathrm{Fix}_r(f)$. First, we may assume that $x \in \mathrm{Fix}_a(f)$ and there exists $\{y^1, \dots, y^l\}$ such that $y \in X$ with $f^i(y) \to x$ as $i \to \infty$, then $y \in \bigcup_{i=1}^{l} \mathrm{Orb}_f(y^i)$. Given $\epsilon > 0$, we take $\delta_i > 0$ corresponding to ϵ by applying above proof for y^i for each $i = 1, 2, \dots, l$. Let $\delta = \min\{\delta_i \mid i = 1, \dots, l\}$. Clearly, we see that δ-pseudo orbit $\xi = \{\dots, x, x, x, \dots\}$ is ϵ-shadowed by x. Let $\xi = \{x_i\}_{i\in\mathbb{Z}}$ be a δ-pseudo orbit of f with $x_0 \neq x$. Then $x_0 \in \mathrm{Orb}_f(y^j)$ for some $j \in \{1, \dots, l\}$. We see that ξ can be considered as a δ_j-pseudo orbit with $x_0 = x$. From the similar argument of Case 1, then ξ is ϵ-shadowed by some point, so $x \in Sh(f)$. For the case of $x \in \mathrm{Fix}_r(f)$, we arrive the proof by the same argument for f^{-1}. This completes the proof. □

Notice that if a homeomorphism $f : X \to X$ is hyper-expansive, then it is expansive. From Theorem 3.8 and Theorem 4 in [18], we can obtain the following result.

Corollary 3.9. *Let $f : X \to X$ be a homeomorphism on a compact metric space. If its induced homeomorphism $2^f : 2^X \to 2^X$ is expansive, then $f : X \to X$ is topologically stable.*

Remark 3.10. From our results on hyper-dynamics obtained in this section, we describe implications and relations between three notions of expansiveness (Exp), shadowing property (SP) and topological stability (TS) for continuous maps (or homeomorphisms) and their induced maps via the below table.

$$\begin{array}{cccc} (2^X, 2^f) & \text{Exp} & \text{Exp} & \text{TS} \\ & {\scriptstyle\text{Example 3.6}}\not\Uparrow \Downarrow {\scriptstyle[2]} & \Downarrow {\scriptstyle\text{Theorem 3.8}} & \Updownarrow {\scriptstyle\text{Theorem 3.2}} \\ (X, f) & \text{Exp} & \text{SP} & \text{TS} \end{array}$$

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References

[1] D. V. Anosov, *On a class of invariant sets of smooth dynamical systems*, in Proc. 5th Int. Conf. Nonl. Oscill. Kiev. **2** (1970), 39–45.

[2] N. Aoki and K. Hiraide, *Topological theory of dynamical systems*, North-Holland Mathematical Library, 52, North-Holland Publishing Co., Amsterdam, 1994.

[3] A. Artigue, *Hyper-expansive homeomorphisms*, Publ. Mat. Urug. **14** (2013), 72–77.

[4] W. Bauer and K. Sigmund, *Topological dynamics of transformations induced on the space of probability measures*, Monatsh. Math. **79** (1975), 81–92. https://doi.org/10.1007/BF01585664

[5] G. D. Birkhoff, *An extension of Poincaré's last geometric theorem*, Acta Math. **47** (1926), no. 4, 297–311. https://doi.org/10.1007/BF02559515

[6] R. Bowen, *ω-limit sets for axiom* A *diffeomorphisms*, J. Differential Equations **18** (1975), no. 2, 333–339. https://doi.org/10.1016/0022-0396(75)90065-0

[7] N.-P. Chung and K. Lee, *Topological stability and pseudo-orbit tracing property of group actions*, Proc. Amer. Math. Soc. **146** (2018), no. 3, 1047–1057. https://doi.org/10.1090/proc/13654

[8] A. Illanes and S. B. Nadler, Jr., *Hyperspaces*, Monographs and Textbooks in Pure and Applied Mathematics, 216, Marcel Dekker, Inc., New York, 1999.

[9] I. S. Kim, H. Kato, and J. J. Park, *On the countable compacta and expansive homeomorphisms*, Bull. Korean Math. Soc. **36** (1999), no. 2, 403–409.

[10] N. Koo, K. Lee, and C. A. Morales, *Pointwise topological stability*, Proc. Edinb. Math. Soc. (2) **61** (2018), no. 4, 1179–1191. https://doi.org/10.1017/s0013091518000263

[11] K. Lee, N.-T. Nguyen, and Y. Yang, *Topological stability and spectral decomposition for homeomorphisms on noncompact spaces*, Discrete Contin. Dyn. Syst. **38** (2018), no. 5, 2487–2503. https://doi.org/10.3934/dcds.2018103

[12] K. Lee and C. A. Morales, *Topological stability and pseudo-orbit tracing property for expansive measures*, J. Differential Equations **262** (2017), no. 6, 3467–3487. https://doi.org/10.1016/j.jde.2016.04.029

[13] C. A. Morales, *Shadowable points*, Dyn. Syst. **31** (2016), no. 3, 347–356. https://doi.org/10.1080/14689367.2015.1131813

[14] S. B. Nadler, Jr., *Hyperspaces of Sets*, Marcel Dekker, Inc., New York, 1978.

[15] S. Yu. Pilyugin, *Shadowing in dynamical systems*, Lecture Notes in Mathematics, 1706, Springer-Verlag, Berlin, 1999.

[16] S. Yu. Pilyugin and K. Sakai, *Shadowing and hyperbolicity*, Lecture Notes in Mathematics, 2193, Springer, Cham, 2017. https://doi.org/10.1007/978-3-319-65184-2

[17] W. R. Utz, *Unstable homeomorphisms*, Proc. Amer. Math. Soc. **1** (1950), 769–774. https://doi.org/10.2307/2031982

[18] P. Walters, *On the pseudo-orbit tracing property and its relationship to stability*, in The structure of attractors in dynamical systems (Proc. Conf., North Dakota State Univ., Fargo, N.D., 1977), 231–244, Lecture Notes in Math., 668, Springer, Berlin, 1978.

[19] Y. Wu and X. Xue, *Shadowing property for induced set-valued dynamical systems of some expansive maps*, Dynam. Systems Appl. **19** (2010), no. 3-4, 405–414.

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