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Stability Criterion for Volterra Type Delay Difference Equations Including a Generalized Difference Operator

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Abstract. The stability of a class of Volterra-type difference equations that include a generalized difference operator Δ_a is investigated using Krasnoselskii's fixed point theorem and some results are obtained. In addition, some examples are given to illustrate our theoretical results.

1. Introduction

Difference equations are the discrete analogues of differential equations and they usually describe certain phenomena over the course of time. Difference equations have many applications in a wide variety of disciplines, such as economics, mathematical biology, social sciences and physics. We refer to [1, 2, 4, 6] for the basic theory and some applications of difference equations. Volterra difference equations are extensively used to model phenomena in engineering, economics, and in the natural and social sciences; their stability has been studied by many authors.

In [5], Khandaker and Raffoul considered a Volterra discrete system with nonlinear perturbation

$$x(n+1) = A(n)x(n) + \sum_{s=0}^{n} B(n,s)x(s) + g(n, x(n))$$

and obtained necessary and sufficient conditions for stability properties of the zero solution employing the resolvent equation coupled with a variation of parameters formula.

In [7], Migda et al. investigated the boundedness and asymptotic stability of

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the zero solution of the discrete Volterra equation

$$x(n+1) = a(n) + b(n)x(n) + \sum_{i=n_0}^{n} K(n,i)x(i)$$

using fixed point theory.

In [3], Islam and Yankson studied the stability and boundedness of the nonlinear difference equation

$$x(t+1) = a(t)x(t) + c(t)\Delta x(t-g(t)) + q(x(t), x(t-g(t)))$$

using fixed point theorems.

In [9], Yankson studied the asymptotic stability of the zero solution of the Volterra difference delay equation

$$x(n+1) = a(n)x(n) + c(n)\Delta x(n-g(n)) + \sum_{s=n-g(n)}^{n-1} k(n,s)h\left(x(s)\right)$$

using Krasnoselskii's fixed point theorem.

In this paper, motivated by [9], we investigate the asymptotic stability of the zero solution of neutral and Volterra type difference equations which include a generalized difference operator of the form

$$\Delta_a\left[x\left(n\right) - b\left(n\right)x\left(n-\sigma\right)\right] = c\left(n\right)x\left(n\right) + \sum_{u=n-\sigma}^{n-1} k(u,n)h\left(x\left(u\right), x\left(u-\tau\right)\right) \tag{1.1}$$

using Krasnoselskii's fixed point theorem. Here $b(n) : \mathbb{Z} \to \mathbb{R}$ and $c(n) : \mathbb{Z} \to \mathbb{R}$ are discrete bounded functions, $k(u,n) : \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}^+$, $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, σ and τ are non-negative integers with $\lim(n-\sigma) = \infty$ and $\lim(n-\tau) = \infty$.

The difference operator Δ and generalized difference operator Δ_a are defined as

$$\Delta x(n) = x(n+1) - x(n)$$

and

$$\Delta_a x(n) = x(n+1) - ax(n),\ a > 0 \tag{1.2}$$

respectively.

We assume that $h(0,0) = 0$ and

$$\left|h(x_1,y_1) - h(x_2,y_2)\right| \leq K \max\left\{\left|x_1 - x_2\right|, \left|y_1 - y_2\right|\right\} \tag{1.3}$$

for some positive constant K.

2. Basic Definitions, Theorems and Lemmas

For any integer $n_0 \geq 0$ we define Z_0 as the set of all integers in the interval $[-\sigma - \tau, n_0]$. Let $\omega : Z_0 \to \mathbb{R}$ be a discrete and bounded initial function.

Definition 2.1. $x(n) = x(n, n_0, \omega)$ is a solution of (1.1) if $x(n) = \omega(n)$ for $n \in Z_0$ and satisfies (1.1) for $n \geq n_0$.

Definition 2.2. The zero solution of (1.1) is stable if for any $\varepsilon > 0$ and any integer $n_0 \geq 0$ there exists a $\delta = \delta(\varepsilon)$ such that $|\omega(n)| < \delta$ for $n \in Z_0$ implies $|x(n, n_0, \omega)| < \varepsilon$ for $n \geq n_0$.

Definition 2.3. The zero solution of (1.1) is asymptotically stable if it is stable and for any integer $n_0 \geq 0$ there exists a $\delta = \delta(n_0)$ such that $|\omega(n)| < \delta$ for $n \in Z_0$ implies $\lim_{n\to\infty} x(n) = 0$.

Lemma 2.1. *Where the generalized difference operator Δ_a is as defined in* (1.2), *we have*

$$\Delta_a x(n) = a^{n+1} \Delta \left(\frac{x(n)}{a^n} \right).$$

Proof. It is obvious. □

Now below we state Krasnoselskii's theorem. For the proof we refer to [8].

Theorem 2.1. *Let M be a closed convex nonempty subset of a Banach space $(B, \|.\|)$. Suppose that A and Q map M into B such that*

(i) $x, y \in M$ *implies* $Ax + Qy \in M$,

(ii) A *is continuous and* AM *is contained in a compact set,*

(iii) Q *is a contraction mapping.*

Then, there exits $z \in M$ with $z = Az + Qz$.

Theorem 2.2.(Ascoli-Arzela Theorem) *Let (X, d) be a compact metric space and $C(X)$ be a vector space consisting of all continuous function $f : X \to \mathbb{R}$. A subset F of $C(X)$ is relatively compact if and only if F is equibounded and equicontinuous.*

3. Main Results

Lemma 3.1. *Assume that $(a + c(n)) \neq 0$ for all $n \in \mathbb{Z}$. Necessary and sufficient condition for $x(n)$ to be the solution of* (1.1) *are*

$$\begin{aligned} x(n) &= (x(n_0) - b(n_0)\, x(n_0 - \sigma)) \prod_{u=n_0}^{n-1} (a + c(u)) + b(n)\, x(n - \sigma) \\ &\quad + \sum_{r=n_0}^{n-1} \left[c(r)\, b(r)\, x(r - \sigma) + \sum_{u=r-\sigma}^{r-1} k(u, r)\, h(x(u), x(u - \tau)) \right] \prod_{s=r+1}^{n-1} (a + c(s)), \\ &\quad n \geq n_0. \end{aligned}$$

Proof. From (1.1) we can write

$$\Delta_a x(n) - c(n)\, x(n) = \Delta_a \left(b(n)\, x(n-\sigma)\right) + \sum_{u=n-\sigma}^{n-1} k(u,n) h\left(x(u), x(u-\tau)\right). \tag{3.1}$$

Using the definition of the operator Δ_a in the left-hand side of (3.1) and multiplying both sides of (3.1) with $\prod_{s=n_0}^{n} (a+c(s))^{-1}$ we have

$$\Delta\left(x(n) \prod_{s=n_0}^{n-1} (a+c(s))^{-1}\right) = \left[\Delta_a \left(b(n)\, x(n-\sigma)\right) + \sum_{u=n-\sigma}^{n-1} k(u,n) h\left(x(u), x(u-\tau)\right)\right] \prod_{s=n_0}^{n} (a+c(s))^{-1}. \tag{3.2}$$

By summing both sides of (3.2) from n_0 to $n-1$, we obtain

$$x(n) \prod_{s=n_0}^{n-1} (a+c(s))^{-1} = x(n_0) + \sum_{r=n_0}^{n-1} \left[\Delta_a \left(b(n)\, x(n-\sigma)\right) + \sum_{u=n-\sigma}^{n-1} k(u,n) h\left(x(u), x(u-\tau)\right)\right] \prod_{s=n_0}^{r} (a+c(s))^{-1}$$

from this last equality, we write

$$x(n) = x(n_0) \prod_{s=n_0}^{n-1} (a+c(s)) + \left\{ \sum_{r=n_0}^{n-1} \left[\Delta_a \left(b(r)\, x(r-\sigma)\right) + \sum_{u=r-\sigma}^{r-1} k(u,r) h\left(x(u), x(u-\tau)\right)\right] \prod_{s=n_0}^{r} (a+c(s))^{-1} \right\} \prod_{s=n_0}^{n-1} (a+c(s)).$$

Because

$$\prod_{s=n_0}^{r} (a+c(s))^{-1} \prod_{s=n_0}^{n-1} (a+c(s)) = \prod_{s=r+1}^{n-1} (a+c(s)),$$

we can write

$$x(n) = x(n_0) \prod_{s=n_0}^{n-1} (a+c(s)) + \sum_{r=n_0}^{n-1} \left[\Delta_a \left(b(r)\, x(r-\sigma)\right) + \sum_{u=r-\sigma}^{r-1} k(u,r) h\left(x(u), x(u-\tau)\right)\right] \prod_{s=r+1}^{n-1} (a+c(s))$$

or

$$x(n) = x(n_0)\prod_{s=n_0}^{n-1}(a+c(s)) + \sum_{r=n_0}^{n-1}\Delta_a(b(r)x(r-\sigma))\prod_{s=r+1}^{n-1}(a+c(s)) + \sum_{r=n_0}^{n-1}\left[\sum_{u=r-\sigma}^{r-1}k(u,r)h(x(u),x(u-\tau))\right]\prod_{s=r+1}^{n-1}(a+c(s)). \tag{3.3}$$

Now, using Lemma 2.1 in the second term on the right-hand side of (3.3), we have

$$\begin{aligned}
&\sum_{r=n_0}^{n-1}\Delta_a(b(r)x(r-\sigma))\prod_{s=r+1}^{n-1}(a+c(s)) \\
&= \sum_{r=n_0}^{n-1}a^{r+1}\Delta\left(\frac{b(r)x(r-\sigma)}{a^r}\right)\prod_{s=r+1}^{n-1}(a+c(s)) \\
&= \sum_{r=n_0}^{n-1}\left[\Delta\left(b(r)x(r-\sigma)\prod_{s=r}^{n-1}(a+c(s))\right) - \Delta\left(\prod_{s=r}^{n-1}(a+c(s))a^r\right)\frac{b(r)x(r-\sigma)}{a^r}\right] \\
&= \left|b(r)x(r-\sigma)\prod_{s=r}^{n-1}(a+c(s))\right|_{r=n_0}^{r=n} - \sum_{r=n_0}^{n-1}\left[\Delta\left(\prod_{s=r}^{n-1}(a+c(s))a^r\right)\frac{b(r)x(r-\sigma)}{a^r}\right] \\
&= b(n)x(n-\sigma) - b(n_0)x(n_0-\sigma)\prod_{s=n_0}^{n-1}(a+c(s)) \\
&\qquad - \sum_{r=n_0}^{n-1}\left[\Delta\left(\prod_{s=r}^{n-1}(a+c(s))a^r\right)\frac{b(r)x(r-\sigma)}{a^r}\right].
\end{aligned}$$

Hence, by putting this last equality in (3.3), we reach

$$\begin{aligned}
x(n) &= x(n_0)\prod_{s=n_0}^{n-1}(a+c(s)) \\
&\quad + \sum_{r=n_0}^{n-1}\left[\sum_{u=r-\sigma}^{r-1}k(u,r)h(x(u),x(u-\tau))\right]\prod_{s=r+1}^{n-1}(a+c(s)) \\
&\quad + b(n)x(n-\sigma) - b(n_0)x(n_0-\sigma)\prod_{s=n_0}^{n-1}(a+c(s)) \\
&\quad - \sum_{r=n_0}^{n-1}\left[\Delta\left(\prod_{s=r}^{n-1}(a+c(s))a^r\right)\frac{b(r)x(r-\sigma)}{a^r}\right].
\end{aligned} \tag{3.4}$$

Because in the last term on the right-hand side of (3.4)

$$\Delta\left(\prod_{s=r}^{n-1}(a+c(s))\,a^{r}\right)=\prod_{s=r+1}^{n-1}(a+c(s))\,a^{r+1}-\prod_{s=r}^{n-1}(a+c(s))\,a^{r}$$
$$=-c(r)\prod_{s=r+1}^{n-1}(a+c(s))\,a^{r},$$

from (3.4) we obtain

$$\begin{aligned}x(n)&=[x(n_0)-b(n_0)\,x(n_0-\sigma)]\prod_{s=n_0}^{n-1}(a+c(s))+b(n)\,x(n-\sigma)\\&+\sum_{r=n_0}^{n-1}\Bigg[c(r)b(r)x(r-\sigma)\\&\qquad+\sum_{u=r-\sigma}^{r-1}k(u,r)h\left(x(u),x(u-\tau)\right)\Bigg]\prod_{s=r+1}^{n-1}(a+c(s)),\ n\geq n_0.\end{aligned}$$

This completes the proof. □

Now let $\phi(n)$ be a real sequence defined on $\mathbb{Z}$ and define the set S as

$$S=\{\phi:\mathbb{Z}\to\mathbb{R}\mid\|\phi\|\to 0,\ n\to\infty\}$$

where

$$\|\phi\|=\max|\phi(n)|\,,n\in\mathbb{Z}.$$

Then, we can see that $(S,\|.\|)$ is a Banach space. We then define the mapping $H:S\to S$ on Z_0 by

$$(H\phi)(n)=\omega(n)$$

and for $n\geq n_0$ by

$$\begin{aligned}(H\phi)(n)&=[\omega(n_0)-b(n_0)\,\omega(n_0-\sigma)]\prod_{s=n_0}^{n-1}(a+c(s))\\&+b(n)\,\phi(n-\sigma)+\sum_{r=n_0}^{n-1}\Bigg[c(r)b(r)\phi(r-\sigma)\\&\qquad+\sum_{u=r-\sigma}^{r-1}k(u,r)h\left(\phi(u),\phi(u-\tau)\right)\Bigg]\prod_{s=r+1}^{n-1}(a+c(s)).\end{aligned}\tag{3.5}$$

Lemma 3.2. *Let* (1.3) *hold. Suppose that*

$$\prod_{s=n_0}^{n-1}(a+c(s))\to 0\ \text{as}\ n\to\infty\tag{3.6}$$

and there exists $\alpha \in (0,1)$ *such that for* $n \geq n_0$

$$\sum_{r=n_0}^{n-1}\left[|c(r)b(r)| + K\sum_{u=r-\sigma}^{r-1} k(u,r)\right]\left|\prod_{s=r+1}^{n-1}(a+c(s))\right| \leq \alpha. \tag{3.7}$$

The mapping H *defined by* (3.5) *approaches 0 as* $n \to \infty$ *.*

Proof. Due to the condition (3.6) the first term of right-hand side of equation (3.5) approaches to zero as $n \to \infty$. Because $b(n)$ is bounded and $\phi \in S$ is also the second term of right-hand side of equation (3.5) approaches to zero as $n \to \infty$. Now, we show that the last term on the right-hand side of equation (3.5) approaches to zero as $n \to \infty$.

Given $\varepsilon_1 > 0$ and let n_1 be a positive integer such that for $n > n_1$ and $\phi \in S$, $|\phi(n-\sigma)| < \varepsilon_1$. Because $\phi(n-\sigma) \to 0$, for given $\varepsilon_2 > 0$ we can find a $n_2 > n_1$ such that for $n > n_2$ $|\phi(n-\sigma)| < \varepsilon_2$. Furthermore, because of condition (3.6) we can find a $n_3 > n_2$ such that for $n > n_3$ $\left|\prod_{s=n_2}^{n-1}(a+c(s))\right| < \frac{\varepsilon_2}{\alpha\varepsilon_1}$.

Hence, for $n > n_3$ from the last term of right-hand side of (3.5) we have

$$\begin{aligned}
&\left|\sum_{r=n_0}^{n-1}\left[c(r)b(r)\phi(r-\sigma) + \sum_{u=r-\sigma}^{r-1} k(u,r)h\left(\phi\left(u\right),\phi\left(u-\tau\right)\right)\right]\prod_{s=r+1}^{n-1}(a+c(s))\right| \\
&\leq \sum_{r=n_0}^{n-1}\left|\left[c(r)b(r)\phi(r-\sigma) + \sum_{u=r-\sigma}^{r-1} k(u,r)h\left(\phi\left(u\right),\phi\left(u-\tau\right)\right)\right]\prod_{s=r+1}^{n-1}(a+c(s))\right| \\
&\leq \sum_{r=n_0}^{n_2-1}\left|\left[c(r)b(r)\phi(r-\sigma) + \sum_{u=r-\sigma}^{r-1} k(u,r)h\left(\phi\left(u\right),\phi\left(u-\tau\right)\right)\right]\prod_{s=r+1}^{n-1}(a+c(s))\right| \\
&\quad + \sum_{r=n_2}^{n-1}\left|\left[c(r)b(r)\phi(r-\sigma) + \sum_{u=r-\sigma}^{r-1} k(u,r)h\left(\phi\left(u\right),\phi\left(u-\tau\right)\right)\right]\prod_{s=r+1}^{n-1}(a+c(s))\right| \\
&\leq \varepsilon_1\sum_{r=n_0}^{n_2-1}\left[|c(r)b(r)| + K\sum_{u=r-\sigma}^{r-1} k(u,r)\right]\left|\prod_{s=r+1}^{n-1}(a+c(s))\right| \\
&\quad + \varepsilon_2\sum_{r=n_0}^{n-1}\left[|c(r)b(r)| + K\sum_{u=r-\sigma}^{r-1} k(u,r)\right]\left|\prod_{s=r+1}^{n-1}(a+c(s))\right| \\
&= \varepsilon_1\sum_{r=n_0}^{n_2-1}\left[|c(r)b(r)| + K\sum_{u=r-\sigma}^{r-1} k(u,r)\right]\left|\prod_{s=r+1}^{n-1}(a+c(s))\right| + \varepsilon_2\alpha \\
&= \varepsilon_1\sum_{r=n_0}^{n_2-1}\left[|c(r)b(r)| + K\sum_{u=r-\sigma}^{r-1} k(u,r)\right]\left|\prod_{s=r+1}^{n_2-1}(a+c(s))\right|\left|\prod_{s=n_2}^{n-1}(a+c(s))\right| + \varepsilon_2\alpha
\end{aligned}$$

$$
\begin{aligned}
&\leq \varepsilon_1 \alpha \left| \prod_{s=n_2}^{n-1} (a+c(s)) \right| + \varepsilon_2 \alpha \\
&\leq \varepsilon_2 (1+\alpha).
\end{aligned}
$$

This completes the proof. □

To use Krasnoselskii's theorem, we construct two mappings Q and A expressing (3.5) as

$$
(H\phi)(n) = (Q\phi)(n) + (A\phi)(n)
$$

where Q, $A : S \to S$ are mappings with

$$
(Q\phi)(n) = \left[\omega(n_0) - b(n_0)\,\omega(n_0-\sigma)\right] \prod_{s=n_0}^{n-1} (a+c(s)) + b(n)\,\phi(n-\sigma) \tag{3.8}
$$

and

$$
\begin{aligned}
(A\phi)(n) = \sum_{r=n_0}^{n-1} \Bigg[& c(r)b(r)\phi(r-\sigma) \\
& + \sum_{u=r-\sigma}^{r-1} k(u,r) h\left(\phi(u), \phi(u-\tau)\right) \Bigg] \prod_{s=r+1}^{n-1} (a+c(s))
\end{aligned} \tag{3.9}
$$

respectively.

Lemma 3.3. *Assume that* (1.3), (3.6) *and* (3.7) *hold and suppose that there exists a positive constant* ξ *such that*

$$
a + c(n) \leq 1 \text{ and } \max_{n\in\mathbb{Z}} |a+c(n)| = \xi \tag{3.10}
$$

Then, the mapping A *defined by* (3.9) *is continuous and compact.*

Proof. First, we show that the mapping A defined by (3.9) is continuous. Let ϕ, $\bar{\phi} \in S$. For a given $\varepsilon > 0$ choose $\delta = \frac{\varepsilon}{\alpha}$ such that $\left\|\phi - \bar{\phi}\right\| < \delta$ holds. Then, we have

$$
\begin{aligned}
&\left\|(A\phi) - (A\overline{\phi})\right\| \\
&= \max_{n\in\mathbb{Z}} \Bigg| \left\{ \sum_{r=n_0}^{n-1} \left[c(r)b(r)\phi(r-\sigma) + \sum_{u=r-\sigma}^{r-1} k(u,r) h\left(\phi(u), \phi(u-\tau)\right) \right] \prod_{s=r+1}^{n-1} (a+c(s)) \right\} \\
&\quad - \left\{ \sum_{r=n_0}^{n-1} \left[c(r)b(r)\overline{\phi}(r-\sigma) + \sum_{u=r-\sigma}^{r-1} k(u,r) h\left(\overline{\phi}(u), \overline{\phi}(u-\tau)\right) \right] \prod_{s=r+1}^{n-1} (a+c(s)) \right\} \Bigg| \\
&\leq \sum_{r=n_0}^{n-1} \left[|c(r)b(r)| \left|\phi(r-\sigma) - \overline{\phi}(r-\sigma)\right| \right] \left| \prod_{s=r+1}^{n-1} (a+c(s)) \right| \\
&\quad + \sum_{r=n_0}^{n-1} \left| \sum_{u=r-\sigma}^{r-1} k(u,r) h\left(\phi(u), \phi(u-\tau)\right) \right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \sum_{u=r-\sigma}^{r-1} k(u,r)h\left(\overline{\phi}\,(u)\,,\overline{\phi}\,(u-\tau)\right)\Bigg|\left|\prod_{s=r+1}^{n-1}(a+c(s))\right| \\
&\leq \sum_{r=n_0}^{n-1}|c(r)b(r)|\left|\prod_{s=r+1}^{n-1}(a+c(s))\right|\left\|\phi-\bar{\phi}\right\| \\
&\quad + \sum_{r=n_0}^{n-1}\left|\sum_{u=r-\sigma}^{r-1} k(u,r)\left[h\left(\phi\,(u)\,,\phi\,(u-\tau)\right)-h\left(\overline{\phi}\,(u)\,,\overline{\phi}\,(u-\tau)\right)\right]\right|\left|\prod_{s=r+1}^{n-1}(a+c(s))\right| \\
&\leq \sum_{r=n_0}^{n-1}\left[|c(r)b(r)|+K\sum_{u=r-\sigma}^{r-1}k(u,r)\right]\left|\prod_{s=r+1}^{n-1}(a+c(s))\right|\left\|\phi-\bar{\phi}\right\| \\
&\leq \alpha\left\|\phi-\bar{\phi}\right\| \leq \varepsilon
\end{aligned}
$$

which shows that the mapping A is continuous. Now we show that A is compact. For this we use Arzela-Ascoli theorem. Let $\{\phi_n\} \subset S$ be a sequence of uniformly bounded functions where $\|\phi_n\| \leq m$ for $m > 0$ and n is a positive integer. Then using (1.3) we have

$$
\begin{aligned}
&\quad \|A\phi_n\| \\
&= \max_{n\in\mathbb{Z}}\left|\sum_{r=n_0}^{n-1}\left[c(r)b(r)\phi(r-\sigma)+\sum_{u=r-\sigma}^{r-1}k(u,r)h\left(\phi\,(u)\,,\phi\,(u-\tau)\right)\right]\prod_{s=r+1}^{n-1}(a+c(s))\right| \\
&\leq \sum_{r=n_0}^{n-1}\left|\left[c(r)b(r)\phi(r-\sigma)+\sum_{u=r-\sigma}^{r-1}k(u,r)h\left(\phi\,(u)\,,\phi\,(u-\tau)\right)\right]\prod_{s=r+1}^{n-1}(a+c(s))\right| \\
&\leq \sum_{r=n_0}^{n-1}\left|\left[|c(r)b(r)|+L\sum_{u=r-\sigma}^{r-1}k(u,r)\right]\right|\left|\prod_{s=r+1}^{n-1}(a+c(s))\right|\|\phi\| \\
&\leq \alpha\,\|\phi\| \leq \alpha m
\end{aligned}
$$

which shows that $(A\phi_n)$ is uniformly bounded. Furthermore,

$$
\begin{aligned}
&\quad \|\Delta\,(A\phi)\| \\
&= \max_{n\in\mathbb{Z}}|(A\phi)\,(n+1)-(A\phi)\,(n)| \\
&\leq |a+c(n)|\left|c(n)b(n)\phi(n-\sigma)+\sum_{u=n-\sigma}^{n-1}k(u,r)h\left(\phi\,(u)\,,\phi\,(u-\tau)\right)\right|\left|\prod_{s=r+1}^{n-1}(a+c(s))\right| \\
&\leq \xi\left(|c(n)b(n)|+K\sum_{u=n-\sigma}^{n-1}k(u,r)\right)\|\phi\| \\
&\leq \xi\alpha m \leq \gamma
\end{aligned}
$$

for some positive constant γ. This shows that $(A\phi_n)$ is equi-continuous. Hence, by Arzela-Ascoli's theorem, the mapping A is compact. □

Lemma 3.4. *Consider the mapping Q defined by* (3.8) *and assume that*

$$|b(n)| \le \mu < 1 \tag{3.11}$$

holds for some positive constant μ. Then, Q is a contraction.

Proof. Take any two functions $\phi, \overline{\phi} \in S$. We then have

$$\begin{aligned}\left\|(Q\phi) - (Q\overline{\phi})\right\| = \max_{n\in\mathbb{Z}} &\Bigg| [\omega(n_0) - b(n_0)\omega(n_0-\sigma)] \prod_{s=n_0}^{n-1}(a+c(s)) + b(n)\phi(n-\sigma) \\ &- [\omega(n_0) - b(n_0)\omega(n_0-\sigma)] \prod_{s=n_0}^{n-1}(a+c(s)) - b(n)\phi(n-\sigma)\Bigg| \\ \le\ & |b(n)| \left\|\phi - \overline{\phi}\right\| \le \mu \left\|\phi - \overline{\phi}\right\|\end{aligned}$$

which shows that Q is a contraction mapping. □

Theorem 3.1. *Suppose that* (1.3), (3.6), (3.7), (3.10) *and* (3.11) *hold. Also suppose that there exists positive constants c and $\beta \in (0,1)$ such that*

$$\left|\prod_{s=n_0}^{n-1}(a+c(s))\right| \le c \tag{3.12}$$

and

$$|b(n)| + \sum_{r=n_0}^{n-1}\left[|c(r)b(r)| + K\sum_{u=r-\sigma}^{r-1}k(u,r)\right]\left|\prod_{s=r+1}^{n-1}(a+c(s))\right| \le \beta,\ n \ge n_0 \tag{3.13}$$

hold. Then, the zero solution of (1.1) *is asymptotically stable.*

Proof. Given $\varepsilon > 0$. Choose δ such that

$$|1 - b(n_0)|\,\delta c < \varepsilon(1-\beta)$$

Let ω be a given initial function such that $|\omega(n)| < \delta$. Let us define the set M as

$$M = \{\phi \in S : \|\phi\| < \varepsilon\}$$

and take any $\phi, \varphi \in M$. Then, we have

$$\begin{aligned}&\|(Q\varphi) + (A\phi)\| \\ &= \max_{n\in\mathbb{Z}}\Bigg|[\omega(n_0) - b(n_0)\omega(n_0-\sigma)]\prod_{u=n_0}^{n-1}(a+c(u)) + b(n)\varphi(n-\sigma) \\ &\quad + \sum_{r=n_0}^{n-1}\left[c(r)b(r)\phi(r-\sigma) + \sum_{u=r-\sigma}^{r-1}k(u,r)h(\phi(u),\phi(u-\tau))\right]\prod_{s=r+1}^{n-1}(a+c(s))\Bigg|\end{aligned}$$

$$
\begin{aligned}
&\leq \left| \left[\omega\left(n_0\right) - b\left(n_0\right)\omega\left(n_0-\sigma\right)\right] \prod_{u=n_0}^{n-1}\left(a+c\left(u\right)\right)\right| + \left|b\left(n\right)\varphi\left(n-\sigma\right)\right| \\
&\quad + \sum_{r=n_0}^{n-1}\left|\left[c\left(r\right)b\left(r\right)\phi\left(r-\sigma\right) + \sum_{u=r-\sigma}^{r-1} k\left(u,r\right)h\left(\phi\left(u\right),\phi\left(u-\tau\right)\right)\right]\prod_{s=r+1}^{n-1}\left(a+c\left(s\right)\right)\right| \\
&\leq \left|1-b(n_0)\right|\delta c + \left|b(n)\right|\varepsilon + \varepsilon\sum_{r=n_0}^{n-1}\left[\left|c\left(r\right)b\left(r\right)\right| + K\sum_{u=r-\sigma}^{r-1} k\left(u,r\right)\right]\left|\prod_{s=r+1}^{n-1}\left(a+c\left(s\right)\right)\right| \\
&\leq \left|1-b(n_0)\right|\delta c + \left\{\left|b(n)\right| + \sum_{r=n_0}^{n-1}\left[\left|c\left(r\right)b\left(r\right)\right| + K\sum_{u=r-\sigma}^{r-1} k\left(u,r\right)\right]\left|\prod_{s=r+1}^{n-1}\left(a+c\left(s\right)\right)\right|\right\}\varepsilon \\
&\leq \left|1-b(n_0)\right|\delta c + \beta\varepsilon \\
&< \varepsilon
\end{aligned}
$$

which shows that $(Q\varphi) + (A\phi) \in M$.

By the last result, Lemma 4 and Lemma 5 all conditions of Theorem 1 are satisfied on M. Consequently, there exits a fixed point $x \in M$ such that $x = Qx + Ax$ holds. Lemma 2 implies that this fixed point $x(n)$ is a solution of (1.1). Furthermore the solution $x(n)$ is stable because $\|x\| < \varepsilon$ for a given $\varepsilon > 0$. By Lemma 3 the solution $x(n)$ is asymptotically stable. □

Example 3.1. Consider the difference equation

$$
\begin{aligned}
\text{(3.14)}\qquad &\Delta_2\left[x\left(n\right) - \frac{1}{32\left(n+1\right)!}x\left(n-2\right)\right] \\
&= -\frac{2n}{n+1}x\left(n\right) + \sum_{u=n-2}^{n-1}\frac{2^n}{16\left(n+1\right)!\left(u^2+4\right)}h\left(x\left(u\right),x\left(u-3\right)\right), n \geq 1
\end{aligned}
$$

Here,

$$
\begin{aligned}
&a = \sigma = 2, \qquad \tau = 3, \qquad n_0 = 1, \\
&c(n) = -\frac{2n}{n+1}, \qquad b(n) = \frac{1}{32\left(n+1\right)!}, \\
&K(u,n) = \frac{2^n}{16\left(n+1\right)!\left(u^2+4\right)}.
\end{aligned}
$$

We see that

$$
\prod_{s=1}^{n-1}\left(2-\frac{2n}{n+1}\right) = \frac{2^{n-1}}{n!} \to 0 \ as \ n \to \infty,
$$

so (3.6) holds. Because

$$
\sum_{r=1}^{n-1}\left[\frac{2r}{r+1}\frac{1}{32\left(n+1\right)!} + \sum_{u=r-2}^{r-1}\frac{2^r}{16\left(r+1\right)!\left(u^2+4\right)}\right]\left|\prod_{s=r+1}^{n-1}\left(\frac{2}{s+1}\right)\right| \leq \frac{3}{16} < 1,
$$

(3.7) holds. Because

$$a+c(n)=2-\frac{2n}{n+1}\leq 1 \text{ and } \max_{n\in\mathbb{Z}}|a+c(n)|=\max_{n\in\mathbb{Z}}\left|2-\frac{2n}{n+1}\right|=1$$

(3.10) holds. Because

$$\left|\frac{1}{32\,(n+1)!}\right|\leq\frac{1}{32}<1,$$

(3.11) holds. Because

$$\left|\prod_{s=1}^{n-1}\left(2-\frac{2s}{s+1})\right)\right|=\left|\prod_{s=1}^{n-1}\left(\frac{2}{s+1})\right)\right|\leq 1,$$

(3.12) holds. Also, because

$$\left|\frac{1}{32\,(n+1)!}\right|+\sum_{r=1}^{n-1}\left[\frac{2r}{r+1}\frac{1}{32\,(n+1)!}\right.\\ \left.+\sum_{u=r-2}^{r-1}\frac{2^r}{16\,(r+1)!\,(u^2+4)}\right]\left|\prod_{s=r+1}^{n-1}\left(\frac{2}{s+1}\right)\right|\leq\frac{13}{64}<1,$$

(3.13) holds. So, by Theorem 3 the zero solution of (3.14) is asymptotically stable.
The solution is of the form

$$x\,(n)=\left(x\,(1)-\frac{1}{64}x\,(-1)\right)\prod_{u=1}^{n-1}\left(2-\frac{2u}{u+1}\right)+\frac{1}{32\,(n+1)!}x\,(n-2)\\ +\sum_{r=1}^{n-1}\left[-\frac{2r}{r+1}\frac{1}{32\,(r+1)!}x\,(r-2)\right.\\ \left.+\sum_{u=r-2}^{r-1}\frac{2^r}{16\,(r+1)!\,(u^2+4)}h\,(x\,(u)\,,x\,(u-3))\right]\prod_{s=r+1}^{n-1}\left(2-\frac{2s}{s+1}\right),\\ n\geq 1.$$

References

[1] R. P. Agarwal, *Difference equations and inequalities: theory, methods and applications*, (2nd ed.), Marcel Dekker, Inc., New York, NY, 2000.

[2] S. Elaydi, *An introduction to difference equations*, (3rd ed.), Springer, New York, 2005.

[3] M. N. Islam and E. Yankson, *Boundedness and stability in nonlinear delay difference equations employing fixed point theory*, Electron. J. Qual. Theory Differ. Equ., **26**(2005), 18 pp.

[4] W. G. Kelley and A. C. Peterson, *Difference equations: an introduction with applications*, Academic Press, Boston, MA, 1991.

[5] T. M. Khandaker and Y. N. Raffoul, *Stability properties of linear Volterra discrete systems with nonlinear perturbation*, J. Difference Equ. Appl., **8**(2002), 857–874.

[6] R. E. Mickens, *Difference equations: theory and applications*, New York, Chapman and Hall, 1991.

[7] M. Migda, M. Ruzickova and E. Schmeidel, *Boundedness and stability of discrete Volterra equations*, Adv. Difference Equ., (2015), 2015:47, 11 pp.

[8] D. R. Smart, *Fixed point theorems*, Cambridge University Press, Cambridge, 1980.

[9] E. Yankson, *Stability of Volterra difference delay equations*, Electron. J. Qual. Theory Differ. Equ., **20**(2006), 14 pp.