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PRECONDITIONED SSOR METHODS FOR THE LINEAR COMPLEMENTARITY PROBLEM WITH M-MATRIX

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Abstract. In this paper, we consider the preconditioned iterative methods for solving linear complementarity problem associated with an M-matrix. Based on the generalized Gunawardena's preconditioner, two preconditioned SSOR methods for solving the linear complementarity problem are proposed. The convergence of the proposed methods are analyzed, and the comparison results are derived. The comparison results showed that preconditioned SSOR methods accelerate the convergent rate of the original SSOR method. Numerical examples are used to illustrate the theoretical results.

1. Introduction

For a given matrix $A \in \mathbb{R}^{n\times n}$ and a given vector $f \in \mathbb{R}^n$, the linear complementarity problem, abbreviated as LCP, consists of finding a vector $x \in \mathbb{R}^n$ such that

$$x \geq 0,\ r = Ax - f \geq 0,\ x^T r = 0. \tag{1.1}$$

Here, the notation " $\geq$ " denotes the componentwise defined partial ordering between two vectors, and the superscript T denotes the transpose of a given vector.

The LCP of the form (1.1) arising in many scientific computing and engineering applications, for example, contact problems with friction, free boundary value problems of fluid mechanics, the solution of optimization and behavioral models in biology and molecular biology, see [5,6,11]. The LCP (1.1) possesses a unique solution if and only if $A \in \mathbb{R}^{n\times n}$ is a P-matrix, namely, a matrix whose all principal submatrices have positive determinants, see [5, 6, 22]. A positive diagonal M-matrix (see Section 2) is a P-matrix, and the LCP (1.1) with an M-matrix has the unique solution [4].

Numerical methods for LCP (1.1) have attracted much attentions. There are three main classes of iterative methods for the solution of the LCP (1.1),

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that are the projected methods [14, 15, 21], the modulus algorithms [16] and the modulus-based matrix splitting iterative methods [3,8,27,28], see [15] for a survey of the solvers for LCP (1.1). We pay our attention in the present work to the SSOR method [9, 26], which is a special projected method, for solving the LCP (1.1) with an M-matrix.

Preconditioning techniques for solving the large sparse linear algebraic equations have been investigated in depth, a number of preconditioners for the iterative methods were proposed [10, 13, 19, 20, 25]. In [12], Gunawardena et al. proposed the preconditioner

$$P_1 = I + S = \begin{bmatrix} 1 & -a_{12} & 0 & \cdots & 0 \\ 0 & 1 & -a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

for solving the linear system with L-matrices. Toshiyuki Kohno et al. [17] generalized the Gunawardena preconditioner P_1 as

$$P = I + S(\alpha) = \begin{bmatrix} 1 & -\alpha_1 a_{12} & 0 & \cdots & 0 \\ 0 & 1 & -\alpha_2 a_{23} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\alpha_{n-1} a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}, \tag{1.2}$$

where $\alpha_1, \ldots, \alpha_{n-1}$ are real constants. The preconditioner P is used to accelerate the convergent rate of the Gauss-Seidel method for solving the linear system with an M-matrix.

In this paper, the preconditioner P in (1.2) is used to accelerate the convergent rate of the SSOR method for solving the LCP of the form (1.1). Two preconditioned SSOR (PSSOR) methods are proposed. In [7], Dai et al. also give the preconditioner P to solve the LCP (see [7] for details). In the part of numerical examples, SSOR method [9], Dai et al.'s method ($PS\tilde{S}OR$) [7] and this paper's method (PSSOR) are compared in the number of IT (iteration steps) (see Table 4). Numerical examples tested show the prominent efficiency of the proposed methods in some situations. The remainder of the paper are organized as follows.

In Section 2, some preliminaries are given. The projected method for solving LCP is recalled, and two preconditioned SSOR methods are presented. In Section 3, the convergence of the preconditioned SSOR methods are studied. The comparison results about the convergent rates between the proposed preconditioned SSOR methods with the original SSOR method for LCP (1.1) with an M-matrix are given in Section 4. Numerical examples are given to demonstrate our theoretical results in Section 5. Finally, a brief conclusion is drawn in Section 6.

2. Preliminaries

Let us first briefly summarise the notation. In reference to $\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$, the relation $\geq$ denotes partial ordering. In addition, for $x, y \in \mathbb{R}^n$ we write $x > y$ if $x_i > y_i$ hold for $i = 1, \ldots, n$. A nonsingular matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is termed an M-matrix if $a_{ij} \leq 0$ for $i \neq j$ and $A^{-1} \geq 0$. Its comparison matrix $\langle A\rangle = (\alpha_{ij})$ is defined by $\alpha_{ii} = |a_{ii}|$, $\alpha_{ij} = -|a_{ij}|(i \neq j)$ for $i, j = 1, \ldots, n$. A is said to be an H-matrix if $\langle A\rangle$ is an M-matrix. For simplicity, we may assume that $a_{ii} = 1$, $i = 1, \ldots, n$.

Definition. For $x \in \mathbb{R}^n$, vector x_+ is defined such that $(x_+)_j = \max\{0, x_j\}$, $j = 1, \ldots, n$. Then, for any $x, y \in \mathbb{R}^n$. The following facts hold:

(1) $(x + y)_+ \leq x_+ + y_+$;
(2) $x_+ - y_+ \leq (x - y)_+$;
(3) $|x| = x_+ + (-x)_+$; and
(4) $x \leq y$ implies that $x_+ \leq y_+$.

The linear complementarity problem (1.1), conveniently denoted by LCP (1.1), is equivalent to [1]

$$z = (z - \alpha E(Az + f))_+,$$

where α is a positive constant and the matrix E is positive diagonal. We begin with a lemma together with its appropriate reference, a practice we also continue elsewhere if no proof is provided.

Definition ([24])**.** Let $A \in \mathbb{R}^{n\times n}$. The representation $A = M - N$ is called a splitting of A if M is nonsingular. Then $A = M - N$ is called

1. convergent if $\rho(M^{-1}N) < 1$;
2. regular if $M^{-1} \geq 0, N \geq 0$;
3. weak regular if $M^{-1} \geq 0, M^{-1}N \geq 0$;
4. an M-splitting of A if M is an M-matrix and $N \geq 0$.

Lemma 2.1 ([20])**.** *Let $A = M - N$ be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is an M-matrix.*

Lemma 2.2 ([2])**.** *A is monotone if and only if A is nonsingular with $A^{-1} \geq 0$.*

Lemma 2.3 ([18])**.** *Let A be an M-matrix, and x be a solution of LCP* (1.1)*. If $f_i > 0$, then $x_i > 0$ and therefore $\sum_{j=1}^{n} a_{ij}x_j - f_i = 0$. Moreover, if $f \leq 0$, then $x = 0$ is the solution of LCP* (1.1)*.*

Lemma 2.4 ([5])**.** *Let A be a Z-matrix. Then the following statements are equivalent*:

(1) *A is a nonsingular M-matrix.*
(2) *There exists a positive vector $v > 0$ such that $Av > 0$.*
(3) *Any weak regular splitting is convergent.*

Lemma 2.5 ([23])**.** *Suppose that $A_1 = M_1 - N_1$ and $A_2 = M_2 - N_2$ are weak regular splittings of the monotone matrices A_1 and A_2, respectively, such that*

$M_1^{-1} \le M_2^{-1}$. If there exists a positive vector x such that $0 \le A_1 x \le A_2 x$, then for the monotonic norm associated with x, $\| M_2^{-1} N_2 \|_x \le \| M_1^{-1} N_1 \|_x$. In particular, if $M_1^{-1} N_1$ has a positive Perron vector, then $\rho(M_2^{-1}N_2) \le \rho(M_1^{-1}N_1)$.

Then, we will give a general preconditioning transformation for the linear complementarity problem related with M-matrices and also recall SSOR methods for solving the LCP (1.1). The following result was proved in [4].

Lemma 2.6. *Let A be an H-matrix with positive diagonal entries. Then the LCP* (1.1) *has a unique solution $x^* \in \mathbb{R}^n$.*

For the problem (1.1) with $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ and $f = (f_i) \in \mathbb{R}^n$, we denote

$$\tilde{A} = PA = (\tilde{a}_{ij}),\ \tilde{f} = Pf = (\tilde{f}_i), \tag{2.1}$$

where P satisfies (1.2). So it follows that

$$\tilde{a}_{ij} = \begin{cases} a_{ij} - \alpha_i a_{i,i+1} a_{i+1,j},\ i \ne n,\ j = 1,2,\ldots,n, \\ a_{nj},\ i = n,\ j = 1,2,\ldots,n, \end{cases}$$

and

$$\tilde{f}_{ij} = \begin{cases} f_i - \alpha_i a_{i,i+1} f_{i+1},\ i \ne n, \\ f_n,\ i = n. \end{cases}$$

Lemma 2.7 ([25])**.** *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonsingular M-matrix. Then there exists $\epsilon_0 > 0$ such that, for any $0 < \epsilon \le \epsilon_0$, $A(\epsilon) = (a_{ij}(\epsilon))$ is a nonsingular M-matrix, where*

$$a_{ij}(\epsilon) = \begin{cases} a_{ij}, & \text{if } a_{ij} \ne 0, \\ -\epsilon, & \text{if } a_{ij} = 0. \end{cases}$$

Definition ([9])**.** Let $0 < w < 2$ and $A = D - L - U$, where D, L and U are diagonal, strictly lower and upper triangular matrices, respectively. (E, F) is called the SSOR splitting of A if (E, F) is a splitting of A,

$$E = 1/(w(2-w))(D - wL)D^{-1}(D - wU) \text{ and}$$
$$F = 1/(w(2-w))((1-w)D + wL)D^{-1}((1-w)D + wU).$$

By using SSOR splitting, two SSOR methods for the LCP(1.1), are defined as follows (see [9]):

Method 2.8. (*SSOR I*) :

(1) *Choose an initial vector $z^0 \in \mathbb{R}^n$, a parameter $w \in \mathbb{R}^+$ and set $k = 0$;*
(2) *Calculate $z^{k+1} = (z^k - D^{-1}[-wUz^{k+1} + (w(2-w)A + wU)z^k - w(2-w)f])_+$;*
(3) *If $z^{k+1} = z^k$, then stop, otherwise set $k := k+1$ and return to Step (2).*

Method 2.9. (*SSOR II*) :

(1) *Choose an initial vector $z^0 \in \mathbb{R}^n$, a parameter $w \in \mathbb{R}^+$ and set $k = 0$;*

(2) *Calculate* $z^{k+1} = (z^k - D^{-1}[-wLz^{k+1} + (w(2-w)A + wL)z^k - w(2-w)f])_+$;
(3) *If* $z^{k+1} = z^k$, *then stop, otherwise set* $k := k+1$ *and return to Step* (2).

Let

$$B_1 = I - wD^{-1}|L|,\ C_1 = |I - D^{-1}[w(2-w)A + wL]|, \tag{2.2}$$

and

$$B_2 = I - wD^{-1}|U|,\ C_2 = |I - D^{-1}[w(2-w)A + wU]|. \tag{2.3}$$

The following lemma follows from Theorem 2.1 and its proof in [9].

Lemma 2.10. *Let* $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ *be an* H*-matrix with positive diagonal elements. If* $0 < w < 2$, *then for any initial vector* $z^o \in \mathbb{R}^n$, *the iterative sequences* z^k *generated by the SSOR methods I and II converge to the unique solution* z^* *of the LCP* (1.1) *and it holds that* $\rho(B_1^{-1}C_1) < 1$ *and* $\rho(B_2^{-1}C_2) < 1$.

Then, let us present two preconditioned SSOR methods for the LCP (1.1) with an M-matrix based on the general preconditioner P given in (1.2).

By (2.1), let us recall that $\tilde{A} = PA$, $\tilde{f} = Pf$, and denote $\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U}$. $\tilde{D}$, $\tilde{L}$ and $\tilde{U}$ are diagonal, strictly lower and upper triangular matrices, respectively. In the following, two preconditioned SSOR methods (PSSOR) for linear complementarity problems are presented as follows:

Method 2.11. (*PSSOR I*) :

(1) *Choose an initial vector* $z^0 \in \mathbb{R}^n$, *a parameter* $w \in \mathbb{R}^+$ *and set* $k = 0$;
(2) *Calculate* $z^{k+1} = (z^k - \tilde{D}^{-1}[-w\tilde{U}z^{k+1} + (w(2-w)\tilde{A} + w\tilde{U})z^k - w(2-w)\tilde{f}])_+$;
(3) *If* $z^{k+1} = z^k$, *then stop, otherwise set* $k := k+1$ *and return to Step* (2).

Method 2.12. (*PSSOR II*) :

(1) *Choose an initial vector* $z^0 \in \mathbb{R}^n$, *a parameter* $w \in \mathbb{R}^+$ *and set* $k = 0$;
(2) *Calculate* $z^{k+1} = (z^k - \tilde{D}^{-1}[-w\tilde{L}z^{k+1} + (w(2-w)\tilde{A} + w\tilde{L})z^k - w(2-w)\tilde{f}])_+$;
(3) *If* $z^{k+1} = z^k$, *then stop, otherwise set* $k := k+1$ *and return to Step* (2).

3. Convergence analysis for PSSOR method

In this section, we will consider the convergence of the preconditioned SSOR method for solving LCP (1.1). From Lemma 2.3, if the problem LCP (1.1) has a nonzero solution, there is at least one index i such that $f_i > 0$. Let us assume that $f_{i+1} > 0$. From Lemma 2.6, we obtain:

Theorem 3.1. *Let* $\tilde{A} = PA \equiv [\tilde{a}_{ij}]$, $\tilde{f} = Pf \equiv \tilde{f}_i$. *If* $f_{i+1} > 0$, *then LCP* (1.1) *is equivalent to the linear complementarity problem*

$$x \geq 0,\ \tilde{r} = \tilde{A}x - \tilde{f} \geq 0,\ x^T\tilde{r} = 0. \tag{3.1}$$

Proof. Suppose that x is the solution to LCP (1.1). Because $f_{i+1} > 0$, from Lemma 2.3 we have that $x_{i+1} > 0$ and $\sum_{j=1}^{n} a_{i+1,j}x_j - f_{i+1} = 0$.

Thus if $i = n$,

$$\sum_{j=1}^{n} \tilde{a}_{ij}x_j - \tilde{f}_i = \sum_{j=1}^{n} a_{nj}x_j - f_n = \sum_{j=1}^{n} a_{ij}x_j - f_i, \tag{3.2}$$

on the other hand if $i \neq n$, then

$$\begin{aligned}\sum_{j=1}^{n} \tilde{a}_{ij}x_j - \tilde{f}_i &= \sum_{j=1}^{n}(a_{ij} - (\alpha_i a_{i,i+1}a_{i+1,j}))x_j - (f_i - \alpha_i a_{i,i+1}f_{i+1}) \\ &= \sum_{j=1}^{n}(a_{ij}x_j - f_i) - (\alpha_i a_{i,i+1})(\sum_{j=1}^{n} a_{i+1,j}x_j - f_{i+1}) \\ &= \sum_{j=1}^{n} a_{ij}x_j - f_i,\end{aligned} \tag{3.3}$$

thus, x is the solution of the LCP (3.1).

Conversely suppose that x is the solution to LCP (3.1). Then by Theorem 3.1 we get that $x_{i+1} > 0$ and $\sum_{j=1}^{n} \tilde{a}_{i+1,j}x_j - \tilde{f}_{i+1} = 0$. This is together with (3.2) and (3.3) give $\sum_{j=1}^{n} a_{i+1,j}x_j - f_{i+1} = 0$. Thus for $i = n$ we have

$$\sum_{j=1}^{n}(a_{ij}x_j - f_i) = \sum_{j=1}^{n} a_{nj}x_j - f_n = \sum_{j=1}^{n} \tilde{a}_{ij} - \tilde{f}_i.$$

While for $i \neq n$, we can deduce that

$$\begin{aligned}\sum_{j=1}^{n}(a_{ij}x_j - f_i) &= \sum_{j=1}^{n}(\tilde{a}_{ij} + \alpha_i a_{i,i+1}a_{i+1,j})x_j - \sum_{j=1}^{n}(\tilde{f}_i + \alpha_i a_{i,i+1}f_{i+1}) \\ &= \sum_{j=1}^{n}(\tilde{a}_{ij}x_j - \tilde{f}_i) + \alpha_i a_{i,i+1}(\sum_{j=1}^{n} a_{i+1,j}x_j - f_{i+1}) \\ &= \sum_{j=1}^{n} \tilde{a}_{ij} - \tilde{f}_i.\end{aligned}$$

Hence, x is the solution of the LCP (1.1). □

In what follows, we make the following assumptions:

(H1) $0 \leq \alpha_i \leq 1$ for $i = 1, 2, \ldots, n$;

(H2) $0 \leq \alpha_i a_{i,i+1}a_{i+1,j}$ for $i = 1, 2, 3, \ldots, n$.

For our purpose, we need the following equivalent conditions of nonsingular M-matrices.

When A is an M-matrix, we assume that PA in (2.1) preserves the Z-matrix character, i.e.,

$$\tilde{a}_{ij} \leq 0 \quad \text{for all} \quad i \neq j,$$

or equivalently,

$$a_{ij} - \alpha_i a_{i,i+1} a_{i+1,j} \leq 0, \; i \neq j. \tag{3.4}$$

The following result holds for the matrix PA in (2.1) satisfying (3.4).

Theorem 3.2. *Suppose that $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is a nonsingular M-matrix. If P is given by* (1.2) *such that* (3.4) *follow, then $\tilde{A} = PA$ is a nonsingular M-matrix.*

Proof. Since A is a nonsingular M-matrix, from Lemma 2.4, there exists a positive vector $v > 0$ so that $Av > 0$. Due to the fact that $P \geq 0$ and the diagonal entries of P satisfy $p_{ii} = 1$, then $PAv > 0$. Note that PA is a Z-matrix. The result is directly derived from Lemma 2.4. □

By using the previous Theorem 3.1 and Theorem 3.2, we can establish the following convergence theorem for the PSSOR methods for solving the LCP (1.1).

Theorem 3.3. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be a nonsingular M-matrix. If P given in* (1.2) *satisfies the conditions of Theorem 3.2, then for $0 < w < 2$, the iterative sequences of the PSSOR methods I and II converge to the unique solution x^* of the LCP* (1.1)*, where for the given vector f, its components $f_{i+1} > 0$.*

Proof. Since A is a nonsingular M-matrix, by Theorem 3.2 $\tilde{A}$ is still an M-matrix, then $\tilde{A}$ is also an H-matrix with positive diagonals. Hence, according to Lemma 2.10, the iterative sequences of the PSSOR methods I and II converge to the unique solution x^* of the LCP (3.1), namely, by Theorem 3.1, the unique solution x^* of the LCP (1.1). □

4. Comparison results

In this subsection, we will consider comparison theorems, which show that the PSSOR methods can increase the convergence of corresponding SSOR methods for the LCPs of M-matrices. Let us consider the problem (1.1) with the splitting

$$A = D - L - U, \tag{4.1}$$

where D, L and U are diagonal, strictly lower and strictly upper triangular parts of A, respectively. By (2.1) we assume that

$$\tilde{A} = PA = (\tilde{a}_{ij}), \; \tilde{f} = Pf, \tag{4.2}$$

where P satisfies Theorem 3.3 and

$$\tilde{a}_{ij} = \begin{cases} a_{ij} - \alpha_i a_{i,i+1} a_{i+1,j}, \ i \neq n, \ j = 1, 2, \dots n, \\ a_{nj} \ , i = n, \ j = 1, 2, \dots, n. \end{cases}$$

We split $\tilde{A}$ in (4.2) as

$$\tilde{A} = \tilde{D} - \tilde{L} - \tilde{U}, \tag{4.3}$$

where $\tilde{D}$, $\tilde{L}$, and $\tilde{U}$ are diagonal, strictly lower and strictly upper triangular parts of $\tilde{A}$, respectively. Apparently, it follows that $\tilde{D} =: (d_{ii})$ with

$$d_{ii} = \begin{cases} a_{ii} - \alpha_i a_{i,i+1} a_{i+1,i}, \ i \neq n, \\ a_{nn}, \ i = n. \end{cases}$$

$\tilde{L} =: (l_{ij})$ with

$$l_{ij} = \begin{cases} a_{ij} - \alpha_i a_{i,i+1} a_{i+1,j}, \ i \neq n, \ i > j, \\ a_{nj}, \ i = n, \end{cases}$$

and $\tilde{U} =: (u_{ij})$ with

$$u_{ij} = a_{ij} - \alpha_i a_{i,i+1} a_{i+1,j}, \ i < j,$$

respectively.

In what follows, we give some useful auxiliary results that are important for us to provide comparison theorems.

Lemma 4.1. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an M-matrix. Assume that A is written as the splitting* (4.1) *and D, L, U, $\tilde{D}$, $\tilde{L}$, $\tilde{U}$ are given by* (4.1)–(4.3)*. Then $D^{-1}|L| \leq \tilde{D}^{-1}|\tilde{L}|, D^{-1}|U| \leq \tilde{D}^{-1}|\tilde{U}|$.*

Proof. Since $\tilde{A}$ is an M-matrix, naturally, an H-matrix with positive diagonals,

$$\begin{cases} a_{ii} - \alpha_i a_{i,i+1} a_{i+1,i} > 0, \ i \neq n, \\ a_{nn} > 0, \ i = n. \end{cases} \tag{4.4}$$

Let us denote $D^{-1}|L| =: (\bar{l}_{ij}), \tilde{D}^{-1}|\tilde{L}| =: (\tilde{l}_{ij})$. Then we have

$$\bar{l}_{ij} = \begin{cases} \frac{1}{a_{ii}} |a_{ij}|, \ i > j, \\ 0, \ \text{otherwise}, \end{cases}$$

and

$$\tilde{l}_{ij} = \begin{cases} \frac{1}{a_{ii} - \alpha_i a_{i,i+1} a_{i+1,i}} (|a_{ij}| + \alpha_i a_{i,i+1} a_{i+1,j}), \ i > j, \ i \neq n, \\ \frac{1}{a_{nn}} |a_{nj}|, \ i = n. \end{cases}$$

On the one hand, from (4.4), $p_{ii} \geq 0$ and the fact that A is an M-matrix, we have

$$\frac{1}{a_{ii}} \leq \frac{1}{a_{ii} - \alpha_i a_{i,i+1} a_{i+1,i}} \quad \text{and} \quad |a_{ij}| \leq (|a_{ij}| + \alpha_i a_{i,i+1} a_{i+1,j}).$$

Therefore, we obtain that $\bar{l}_{ij} \leq \tilde{l}_{ij}$, $i, j \in \mathbb{N}$. In other words, $D^{-1}|L| \leq \tilde{D}^{-1}|\tilde{L}|$.

Similarly, one can achieve that $D^{-1}|U| \leq \tilde{D}^{-1}|\tilde{U}|$. $\square$

Let

$$\tilde{B}_1 = I - w\tilde{D}^{-1}|\tilde{L}|, \tilde{C}_1 = |I - \tilde{D}^{-1}[w(2-w)\tilde{A} + w\tilde{L}]|, \quad (4.5)$$

$$\tilde{B}_2 = I - w\tilde{D}^{-1}|\tilde{U}|, \tilde{C}_2 = |I - \tilde{D}^{-1}[w(2-w)\tilde{A} + w\tilde{U}]|. \quad (4.6)$$

Lemma 4.2. *Let $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ be an M-matrix. Suppose that $\tilde{A}$ and $\tilde{f}$ are given by* (4.2) *and $\tilde{B}_1, \tilde{C}_1$ and $\tilde{B}_2, \tilde{C}_2$ are defined by* (4.5) *and* (4.6)*, respectively. If $0 < w < 2$, then for any initial vector $x_0 \in \mathbb{R}^n$, the iterative sequences x^kgenerated by the PSSOR methods I and II converge to the unique solution x^* of the LCP* (1.1) *and it follows that $\rho(\tilde{B}_1^{-1}\tilde{C}_1) < 1$ and $\rho(\tilde{B}_2^{-1}\tilde{C}_2) < 1$.*

Proof. By (4.2), $\tilde{A}$ is an H-matrix with positive diagonals. Hence, by Theorem 3.3, for any initial vector $x_0 \in \mathbb{R}^n$, the iterative sequences x^k of the PSSOR methods I and II converge to the unique solution of the LCP(1.1), and from Lemma 2.10 and the fact that $\tilde{A}$ is an H-matrix with positive diagonal entries, it follows that $\tilde{B}_1^{-1}\tilde{C}_1 < 1$ and $\tilde{B}_2^{-1}\tilde{C}_2 < 1$. □

Theorem 4.3. *Assume that A is a nonsingular M-matrix and A and $\tilde{A}$ have the splitting* (4.1) *and* (4.3)*, respectively. Let B_1, C_1 and $\tilde{B}_1, \tilde{C}_1$ be given as in* (2.2) *and* (4.5)*, respectively. Then for the matrices $B_1^{-1}C_1$ for SSOR I and $\tilde{B}_1^{-1}\tilde{C}_1$ for PSSOR I with respect to the LCPs, we have*

$$\rho(\tilde{B}_1^{-1}\tilde{C}_1) \le \rho(B_1^{-1}C_1) < 1.$$

Proof. By Lemma 2.10 and the fact that A is an M-matrix, for any initial vector $x^0 \in \mathbb{R}^n$, the iterative sequence x^k generated by SSOR I converges to the unique solution x^* of the LCP (1.1) and

$$\rho(B_1^{-1}C_1) < 1. \quad (4.7)$$

Analogously, by Lemma 4.2 and the fact that $\tilde{A}$ is an H-matrix with positive diagonals, for any initial vector $y^0 \in \mathbb{R}^n$, the iterative sequence y^k generated by PSSOR I converges to the unique solution x^* of the LCP (1.1) and

$$\rho(\tilde{B}_1^{-1}\tilde{C}_1) < 1. \quad (4.8)$$

Let us now consider the result $\rho(\tilde{B}_1^{-1}\tilde{C}_1) \le \rho(B_1^{-1}C_1)$. In terms of Lemma 4.1, we have that $D^{-1}|L| \le \tilde{D}^{-1}|\tilde{L}$, which is equivalent to

$$I - w\tilde{D}^{-1}|\tilde{L}| \le I - wD^{-1}|L|,$$

that is, $\tilde{B}_1 \le B_1$. Notice that B_1 and $\tilde{B}_1$ are M-matrices, this implies that $0 \le B_1^{-1} \le \tilde{B}_1^{-1}$. Let us denote $Q_1 := B_1 - C_1$ and $Q_2 := \tilde{B}_1 - \tilde{C}_1$. Observe that B_1 and $\tilde{B}_1$ are M-matrices and C_1 and $\tilde{C}_1$ are nonnegative, by Definition 2, it holds that $B_1 - C_1$ and $\tilde{B}_1 - \tilde{C}_1$ are M-splittings of Q_1 and Q_2, respectively. It means from (4.7), (4.8) and Lemma 2.2 that Q_1 and Q_2 are M-matrices. Therefore, $Q_1^{-1} \ge 0$ and $Q_2^{-1} \ge 0$, which show by Lemma 2.2 that Q_1 and Q_2 are monotone. From Definition 2 and the fact that an M-splitting is an regular splitting, it can be derived that $B_1 - C_1$ and $\tilde{B}_1 - \tilde{C}_1$ are regular splittings of the monotone matrices Q_1 and Q_2, respectively.

If A is an irreducible matrix, taking into account that

$$B_1^{-1}C_1 = (I - wD^{-1}|L|)^{-1}|I - D^{-1}[w(2-w)A + wL]|,$$

this implies that the matrix $B_1^{-1}C_1$ is a nonnegative irreducible matrix. Thus, by means of Perron-Frobenius theorem (see Theorem 2.7 of [4]), $B_1^{-1}C_1$ has a positive Perron vector. By Lemma 2.5, as a result, we have $\rho(\tilde{B}_1^{-1}\tilde{C}_1) \le \rho(B_1^{-1}C_1)$.

If A is a reducible matrix, then by Lemma 2.7, for sufficiently small $\epsilon > 0$ the matrix $A(\epsilon)$ is an irreducible M-matrix and by the above proof one can see that

$$\rho(\tilde{B}_1^{-1}\tilde{C}_1) = \lim_{\epsilon\to 0^+} \rho(\tilde{B}_1^{-1}\tilde{C}_1)(\epsilon) \le \lim_{\epsilon\to 0^+} \rho(B_1^{-1}C_1)(\epsilon) = \rho(B_1^{-1}C_1).$$

This completes the proof. □

Similarly, we can obtain the following corollary.

Corollary 4.4. *Assume that A is a nonsingular M-matrix and A and $\tilde{A}$ have the splitting* (4.1) *and* (4.2)*, respectively. Let B_2, C_2 and $\tilde{B}_2, \tilde{C}_2$ be given as in* (2.3) *and* (4.6)*, respectively. Then for the matrices $B_2^{-1}C_2$ for SSORII and $\tilde{B}_2^{-1}\tilde{C}_2$ for PSSORII with respect to the LCP* (1.1)*, it holds that $\rho(\tilde{B}_2^{-1}\tilde{C}_2) \le \rho(B_2^{-1}C_2) < 1$.*

5. Numerical examples

In this section, two examples are given for verifying the theoretical result.

Example 5.1. The coefficient matrix A in Equation (1.1) is given by

$$A = \begin{pmatrix} I - Q & U \\ L & I - R \end{pmatrix},$$

where A is a irreducible M-matrix, $Q = (q_{ij}) \in \mathbb{R}^{p\times p}$, $R = (r_{ij}) \in \mathbb{R}^{q\times q}$, $L = (l_{ij}) \in \mathbb{R}^{q\times p}$, and $U = (u_{ij}) \in \mathbb{R}^{p\times q}$ with

$$\begin{aligned}
q_{ii} &= \frac{1}{10(i+1)},\ 1 \le i \le p,\\
q_{ij} &= \frac{1}{30} - \frac{1}{30j+i},\ 1 \le i < j \le p,\\
q_{ij} &= \frac{1}{30} - \frac{1}{30(i-j+1)+i},\ 1 \le j < i \le p,\\
r_{ii} &= \frac{1}{10(p+i+1)},\ 1 \le i \le q,\\
r_{ij} &= \frac{1}{30} - \frac{1}{30(p+j)+p+i},\ 1 \le i < j \le q,
\end{aligned}$$

$$r_{ij} = \frac{1}{30} - \frac{1}{30(i-j+1)+p+i}, \ 1 \le j < i \le q,$$
$$l_{ij} = \frac{1}{30(p+i-j+1)+p+i} - \frac{1}{30}, \ 1 \le i \le q, \ 1 \le j \le p,$$
$$u_{ij} = \frac{1}{30(p+j)+i} - \frac{1}{30}, \ 1 \le i \le p, \ 1 \le j \le q.$$

Table 1 and Table 2 list $\rho(B^{-1}C)$ and $\rho(\widetilde{B}^{-1}\widetilde{C})$ with different α and ω for Example 5.1.

TABLE 1. $\rho(B^{-1}C)$ and $\rho(\widetilde{B}^{-1}\widetilde{C})$ with $\alpha = 2/3$ and $\omega = 1.1$ for Example 5.1

n	*SSOR I*	*PSSOR I*	*SSOR II*	*PSSOR II*
5	0.0526	**0.0435**	0.0502	**0.0477**
10	0.1460	**0.1307**	0.1377	**0.1343**
15	0.2774	**0.2602**	0.2654	**0.2612**
20	0.4514	**0.4358**	0.4399	**0.4347**
25	0.6753	**0.6648**	0.6700	**0.6637**
30	0.9584	**0.9567**	0.9666	**0.9589**

TABLE 2. $\rho(B^{-1}C)$ and $\rho(\widetilde{B}^{-1}\widetilde{C})$ with $\alpha = 2/3$ and $\omega = 0.5$ for Example 5.1

n	*SSOR I*	*PSSOR I*	*SSOR II*	*PSSOR II*
5	0.3030	**0.2953**	0.3020	**0.2970**
10	0.3937	**0.3838**	0.3907	**0.3851**
15	0.4981	**0.4885**	0.4940	**0.4888**
20	0.6152	**0.6070**	0.6110	**0.6065**
25	0.7453	**0.7394**	0.7418	**0.7387**
30	0.8893	**0.8865**	0.8875	**0.8860**

Example 5.2. Consider the LCP(1.1) with the system matrix $A \in \mathbb{R}^{n\times n}$ and the vector $f \in \mathbb{R}^n$,

$$A = \begin{bmatrix} S & -I & -I & & \\ & S & -I & \ddots & \\ & & S & \ddots & -I \\ & & & \ddots & -I \\ & & & & S \end{bmatrix} \in \mathbb{R}^{n\times n}, \ f = \begin{bmatrix} -1 \\ 1 \\ -1 \\ \vdots \\ (-1)^{n-1} \\ (-1)^n \end{bmatrix} \in \mathbb{R}^n,$$

where $S = tridiag(-1, 8, -1) \in \mathbb{R}^{\bar{n}\times\bar{n}}$ and $I \in \mathbb{R}^{\bar{n}\times\bar{n}}$ is the identity matrix and $\bar{n}^2 = n$. It is easy to check that A is an M-matrix. So, the LCP(1.1) has a unique solution. Taking into account that $f_2 > 0, f_4 > 0, \ldots$, hence $k_m \in \{2, 4, 6, \ldots\}$. For this problem, we take the initial vector $z^0 = (5, 5, \ldots, 5)^T$. Let the termination criterion for the presented methods be $\epsilon(z^t) := \| \min(Az^t + q, z^t)\|_\infty < 10^{-6}$.

Table 3 list IT and CPU with different n and ω for Example 5.2.

Table 3. IT, CPU and ρ with different n and ω for Example 5.2

	PSSOR method I			SSOR method I			PSSOR method II			SSOR method II		
	IT	CPU	ρ	IT	CPU	ρ	IT	CPU	ρ	IT	CPU	ρ
$n = 100$												
$\omega = 0.01$	1190	0.3432	0.9829	1204	0.4056	0.9849	1147	0.3120	0.9829	1162	0.3432	0.9849
$\omega = 0.1$	117	0.0624	0.8359	118	0.1092	0.8538	107	0.0624	0.8355	107	0.0624	0.8538
$\omega = 0.5$	22	0.0312	0.3334	22	0.0468	0.3885	58	0.0624	0.3253	56	0.0312	0.3885
$\omega = 1.0$	12	0.0312	0.0399	12	0.0468	0.0575	−−	−−	0.0130	−−	−−	0.0575
$\omega = 1.2$	13	0.0624	0.1352	14	0.0468	0.2036	−−	−−	0.1204	−−	−−	0.2036
$n = 400$												
$\omega = 0.01$	1229	4.1184	0.9830	1241	4.9920	0.9850	1154	3.8064	0.9830	1165	3.8688	0.9850
$\omega = 0.1$	121	0.5772	0.8371	123	0.7176	0.8551	126	0.4992	0.8367	125	0.5304	0.8551
$\omega = 0.5$	24	0.2184	0.3375	24	0.3120	0.3931	−−	−−	0.3287	−−	−−	0.3931
$\omega = 1.0$	15	0.1872	0.0433	15	0.2184	0.0611	−−	−−	0.0140	−−	−−	0.0611
$\omega = 1.2$	16	0.2184	0.1403	16	0.2652	0.2099	−−	−−	0.1244	−−	−−	0.2099
$n = 900$												
$\omega = 0.01$	1229	27.8929	0.9831	1242	31.7774	0.9850	1157	24.6794	0.9831	1165	25.1317	0.9850
$\omega = 0.1$	121	3.8532	0.8374	123	4.3836	0.8554	164	4.6644	0.8369	162	4.5396	0.8554
$\omega = 0.5$	24	1.8252	0.3383	24	1.8720	0.3940	−−	−−	0.3294	−−	−−	0.3940
$\omega = 1.0$	15	1.5912	0.0444	15	1.7004	0.0620	−−	−−	0.0142	−−	−−	0.0619
$\omega = 1.2$	16	1.7628	0.1414	17	1.7784	0.2112	−−	−−	0.1252	−−	−−	0.2112

Table 4 list IT with different n and ω for Example 5.2.

Table 4

n	$PSSOR\ methodI$	$PS\tilde{S}OR\ methodI$	$SSOR\ methodI$
$n = 100$			
$\omega = 0.01$	1190	1120	1204
$\omega = 0.1$	117	109	118
$\omega = 0.5$	22	20	22
$\omega = 1.0$	12	10	12
$\omega = 1.2$	13	12	14
$n = 400$			
$\omega = 0.01$	1229	1214	1241
$\omega = 0.1$	121	119	123
$\omega = 0.5$	24	22	24
$\omega = 1.0$	15	12	15
$\omega = 1.2$	16	14	16
$n = 900$			
$\omega = 0.01$	**1229**	1244	1242
$\omega = 0.1$	**121**	123	123
$\omega = 0.5$	24	24	24
$\omega = 1.0$	15	14	15
$\omega = 1.2$	16	16	17
$n = 1600$			
$\omega = 0.01$	**1229**	1250	1242
$\omega = 0.1$	**121**	124	123
$\omega = 0.5$	24	24	24
$\omega = 1.0$	15	15	15
$\omega = 1.2$	16	16	17

In Table 1, 2, we compare the spectral radii for two PSSOR methods and the SSOR method with several values of n, α and ω. In Table 3, we compare the number of iterations (IT), the CPU time (CPU) and ρ (Spectral radius) for two PSSOR methods and the SSOR method [9] with several values of n and ω, when the coefficient matrix is a large sparse matrix ('$--$' indicates that there is no corresponding result when the number of iterations is less than 300 in matlab calculation). It is easy to see that two PSSOR method has faster convergence rate and fewer iteration steps than two SSOR method for the LCP(1.1), which confirm our theoretical results. In addition, we also compare the number of iterations (IT) for the SSOR method [9], Dai et al.'s method ($PS\tilde{S}OR$) [7] and PSSOR methods with several values of n and ω in Table 4, numerical examples demonstrate that the proposed method has fewer iterations and convergence speed is faster when the size of the matrix is larger and w is smaller.

6. Concluding remarks

In this paper, for the LCPs with an M-matrix A and the vector f, we first present a preconditioner P by using the number of positive sign of the components in f, and prove that the original LCP (1.1) is equivalent to the LCP (3.1). Then, on the basis of the preconditioner P, two PSSOR methods for linear complementarity problem are proposed and the convergence analysis is provided. Also we achieve comparison theorems on the PSSOR methods for the linear complementarity problem, which show that the PSSOR methods improve considerably the convergence rate and fewer iteration steps of the original SSOR methods for solving the LCP (1.1). Numerical examples tested show the prominent efficiency of the proposed methods. How to extend this technique to other methods for solving the LCPs is the content of future research.

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References

[1] B. H. Ahn, *Solution of nonsymmetric linear complementarity problems by iterative methods*, J. Optim. Theory Appl. **33** (1981), no. 2, 175–185.
[2] O. Axelsson, *Iterative Solution Methods*, Cambridge University Press, Cambridge, 1994.
[3] Z.-Z. Bai, *Modulus-based matrix splitting iteration methods for linear complementarity problems*, Numer. Linear Algebra Appl. **17** (2010), no. 6, 917–933.
[4] Z.-Z. Bai and D. J. Evans, *Matrix multisplitting relaxation methods for linear complementarity problems*, Int. J. Comput. Math. **63** (1997), no. 3-4, 309–326.
[5] A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, New York, 1979.
[6] R. W. Cottle, J.-S. Pang, and R. E. Stone, *The Linear Complementarity Problem*, Computer Science and Scientific Computing, Academic Press, Inc., Boston, MA, 1992.
[7] P.-F. Dai, J.-C. Lia, Y.-T. Lic, and J. Baia, *A general preconditioner for linear complementarity problem with an M-matrix*, J. Comput. Appl. Math. **317** (2017), 100–112.
[8] J.-L. Dong and M.-Q. Jiang, *A modified modulus method for symmetric positive-definite linear complementarity problems*, Numer. Linear Algebra Appl. **16** (2009), no. 2, 129–143.

[9] M. Dehghan and M. Hajarian, *Convergence of SSOR methods for linear complementarity problems*, Oper. Res. Lett. **37** (2009), no. 3, 219–223.

[10] ———, *Improving preconditioned SOR-type iterative methods for L-matrices*, Int. J. Numer. Methods Biomed. Eng. **27** (2011), no. 5, 774–784.

[11] M. C. Ferris and Y. Zhang, *Foreword: Special issue on mathematical programming in biology and medicine*, Math. Program. **101** (2004), no. 2, Ser. B, 297–299.

[12] A. D. Gunawardena, S. K. Jain, and L. Snyder, *Modified iterative methods for consistent linear systems*, Linear Algebra Appl. **154/156** (1991), 123–143.

[13] A. Hadjidimos, D. Noutsos, and M. Tzoumas, *More on modifications and improvements of classical iterative schemes for M-matrices*, Linear Algebra Appl. **364** (2003), 253–279.

[14] A. Hadjidimos and M. Tzoumas, *On the solution of the linear complementarity problem by the generalized accelerated overrelaxation iterative method*, J. Optim. Theory Appl. **165** (2015), no. 2, 545–562.

[15] ———, *The solution of the linear complementarity problem by the matrix analogue of the accelerated overrelaxation iterative method*, Numer. Algorithms **73** (2016), no. 3, 665–684.

[16] N. W. Kappel and L. T. Watson, *Iterative algorithms for the linear complementarity problems*, I, Int. J.Comput. Math. **19** (1986), no. 3–4, 273–297.

[17] T. Kohno, H. Kotakemori, and H. Niki, *Improving the modified Gauss-Seidel method for Z-matrices*, Linear Algebra Appl. **267** (1997), 113–123.

[18] D. Li, J. Zeng, and Z. Zhang, *Gaussian pivoting method for solving linear complementarity problem*, Appl. Math. J. Chinese Univ. Ser. B **12** (1997), no. 4, 419–426.

[19] W. Li, *The convergence of the modified Gauss-Seidel methods for consistent linear systems*, J. Comput. Appl. Math. **154** (2003), no. 1, 97–105.

[20] W. Li and W. Sun, *Modified Gauss-Seidel type methods and Jacobi type methods for Z-matrices*, Linear Algebra Appl. **317** (2000), no. 1-3, 227–240.

[21] Y. Li and P. Dai, *Generalized AOR methods for linear complementarity problem*, Appl. Math. Comput. **188** (2007), no. 1, 7–18.

[22] K. G. Murty, *Linear Complementarity, Linear and Nonlinear Programming*, Sigma Series in Applied Mathematics, **3**, Heldermann Verlag, Berlin, 1988.

[23] M. Neumann and R. J. Plemmons, *Convergence of parallel multisplitting iterative methods for M-matrices*, Linear Algebra Appl. **88/89** (1987), 559–573.

[24] R. S. Varga, *Matrix Iterative Analysis*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1962.

[25] L. Wang and Y. Song, *Preconditioned AOR iterative methods for M-matrices*, J. Comput. Appl. Math. **226** (2009), no. 1, 114–124.

[26] M. T. Yahyapour and S. A. Edalatpanah, *Modified SSOR Modelling for Linear Complementarity Problems*, Turkish Journal of Analysis and Number Theory. **2** (2014), no. 2, 47–52.

[27] L.-L. Zhang and Z.-R. Ren, *Improved convergence theorems of modulus-based matrix splitting iteration methods for linear complementarity problems*, Appl. Math. Lett. **26** (2013), no. 6, 638–642.

[28] N. Zheng and J.-F. Yin, *Accelerated modulus-based matrix splitting iteration methods for linear complementarity problem*, Numer. Algorithms **64** (2013), no. 2, 245–262.

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