$([r, s], [t, u])$-INTERVAL-VALUED INTUITIONISTIC FUZZY ALPHA GENERALIZED CONTINUOUS MAPPINGS

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Abstract. In this paper, we introduce the concepts of $([r, s], [t, u])$-interval-valued intuitionistic fuzzy alpha generalized closed and open sets in the interval-valued intuitionistic smooth topological space and $([r, s], [t, u])$-interval-valued intuitionistic fuzzy alpha generalized continuous mappings and then investigate some of their properties.

1. Introduction

which is a generalization of both interval-valued fuzzy sets and intuitionistic fuzzy sets. Mondal and Samanta [9,15] introduced the concept of intuitionistic gradation of openness and defined an intuitionistic fuzzy topological space. Jeon, Jun and Park [8] introduced the concepts of intuitionistic fuzzy alpha closed sets and intuitionistic fuzzy alpha continuous mappings. Sakthivel [14] introduced the concepts of intuitionistic fuzzy alpha generalized closed sets and intuitionistic fuzzy alpha generalized continuous mappings.

In this paper, we introduce the concepts of \([r,s], [t,u]\)-interval-valued intuitionistic fuzzy alpha generalized closed and open sets in the interval-valued intuitionistic smooth topological space and \([r,s], [t,u]\)-interval-valued intuitionistic fuzzy alpha generalized continuous mappings and then investigate some of their properties.

2. Preliminaries

Throughout this paper, let \(X\) be a nonempty set, \(I = [0,1], I_0 = (0,1]\) and \(I_1 = [0,1)\). The family of all fuzzy sets of \(X\) will be denoted by \(I^X\). By \(0_X\) and \(1_X\) we denote the characteristic functions of \(\phi\) and \(X\), respectively. For any \(A \in I^X\), \(A^c\) denotes the complement of \(A\), i.e., \(A^c = 1_X - A\).

**Definition 2.1.** [3,5,13]. A gradation of openness (for short, GO) on \(X\), which is also called a smooth topology on \(X\), is a mapping \(\tau : I^X \rightarrow I\) satisfying the following conditions:

1. \((GO1)\) \(\tau(0_X) = \tau(1_X) = 1\),
2. \((GO2)\) \(\tau(A \cap B) \geq \tau(A) \land \tau(B)\) for each \(A, B \in I^X\),
3. \((GO3)\) \(\tau(\bigcup_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \tau(A_i)\) for each subfamily \(\{A_i : i \in \Gamma\} \subseteq I^X\).

The pair \((X, \tau)\) is called a smooth topological space (for short, STS).

**Definition 2.2.** [9]. An intuitionistic gradation of openness (for short, IGO) on \(X\), which is also called an intuitionistic smooth topology on \(X\), is an ordered pair \((\tau, \tau^*)\) of mappings from \(I^X\) to \(I\) satisfying the following conditions:

1. \((IGO1)\) \(\tau(A) + \tau^*(A) \leq 1\) for each \(A \in I^X\),
2. \((IGO2)\) \(\tau(0_X) = \tau(1_X) = 1\) and \(\tau^*(0_X) = \tau^*(1_X) = 0\),
3. \((IGO3)\) \(\tau(A \cap B) \geq \tau(A) \land \tau(B)\) and \(\tau^*(A \cap B) \leq \tau^*(A) \lor \tau^*(B)\) for each \(A, B \in I^X\),
(IGO4) \( \tau(\bigcup_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \tau(A_i) \) and \( \tau^*(\bigcup_{i \in \Gamma} A_i) \leq \bigvee_{i \in \Gamma} \tau^*(A_i) \) for each subfamily \( \{A_i : i \in \Gamma\} \subseteq I^X \).

The triple \((X, \tau, \tau^*)\) is called an intuitionistic smooth topological space (for short, ISTS). \( \tau \) and \( \tau^* \) may be interpreted as gradation of openness and gradation of nonopenness, respectively.

Let \( D(I) \) be the set of all closed subintervals of the unit interval \( I \). The elements of \( D(I) \) are generally denoted by capital letters \( M, N, \cdots \) and \( M = [M^L, M^U] \), where \( M^L \) and \( M^U \) are respectively the lower and the upper end points. Especially, we denote \( r = [r, r] \) for each \( r \in I \).

The complement of \( M \), denoted by \( M^c \), is defined by \( M^c = 1 - M = [1 - M^U, 1 - M^L] \). Note that \( M = N \) iff \( M^L = N^L \) and \( M^U = N^U \) and that \( M \leq N \) iff \( M^L \leq N^L \) and \( M^U \leq N^U \).

**Definition 2.3.** [17]. A mapping \( A = [A^L, A^U] : X \rightarrow D(I) \) is called an interval-valued fuzzy set (for short, IVFS) on \( X \), where \( A(x) = [A^L(x), A^U(x)] \) for each \( x \in X \). \( A^L(x) \) and \( A^U(x) \) are called the lower and upper end points of \( A(x) \), respectively.

**Definition 2.4.** [10]. Let \( A \) and \( B \) be IVFSs on \( X \).

(i) \( A = B \) iff \( A^L(x) = B^L(x) \) and \( A^U(x) = B^U(x) \) for all \( x \in X \).

(ii) \( A \subseteq B \) iff \( A^L(x) \leq B^L(x) \) and \( A^U(x) \leq B^U(x) \) for all \( x \in X \).

(iii) The complement \( A^c \) of \( A \) is defined by \( A^c(x) = [1 - A^U(x), 1 - A^L(x)] \) for all \( x \in X \).

(iv) For a family of IVFSs \( \{A_i : i \in \Gamma\} \), the union \( \bigcup_{i \in \Gamma} A_i \) and the intersection \( \bigcap_{i \in \Gamma} A_i \) are respectively defined by

\[
\bigcup_{i \in \Gamma} A_i(x) = [\bigvee_{i \in \Gamma} A_i^L(x), \bigvee_{i \in \Gamma} A_i^U(x)],
\]

\[
\bigcap_{i \in \Gamma} A_i(x) = [\bigwedge_{i \in \Gamma} A_i^L(x), \bigwedge_{i \in \Gamma} A_i^U(x)]
\]

for all \( x \in X \).

**Definition 2.5.** [2]. A mapping \( A = (\mu_A, \nu_A) : X \rightarrow D(I) \times D(I) \) is called an interval-valued intuitionistic fuzzy set (for short, IVIFS) on \( X \), where \( \mu_A : X \rightarrow D(I) \) and \( \nu_A : X \rightarrow D(I) \) are interval-valued fuzzy sets on \( X \) with the condition \( \sup_{x \in X} \mu_A^U(x) + \sup_{x \in X} \nu_A^U(x) \leq 1 \). The intervals \( \mu_A(x) = [\mu_A^L(x), \mu_A^U(x)] \) and \( \nu_A(x) = [\nu_A^L(x), \nu_A^U(x)] \) denote the degree of belongingness and the degree of nonbelongingness of the element \( x \) to the set \( A \), respectively.

**Definition 2.6.** [11]. Let \( A = (\mu_A, \nu_A) \) and \( B = (\mu_B, \nu_B) \) be IVIFSs on \( X \).
(i) $A \subseteq B$ iff $\mu_A^R(x) \leq \mu_B^R(x)$, $\mu_A^L(x) \leq \mu_B^L(x)$ and $\nu_A^R(x) \geq \nu_B^R(x)$, $\nu_A^L(x) \geq \nu_B^L(x)$ for all $x \in X$.

(ii) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

(iii) The complement $A^c$ of $A$ is defined by $\mu_{A^c}(x) = \nu_A(x)$ and $\nu_{A^c}(x) = \mu_A(x)$ for all $x \in X$.

(iv) For a family of IVIFSs $\{A_i : i \in \Gamma\}$, the union $\bigcup_{i \in \Gamma} A_i$ and the intersection $\bigcap_{i \in \Gamma} A_i$ are respectively defined by

\[
\mu_{\bigcup_{i \in \Gamma} A_i}(x) = \bigcup_{i \in \Gamma} \mu_{A_i}(x), \quad \nu_{\bigcup_{i \in \Gamma} A_i}(x) = \bigcap_{i \in \Gamma} \nu_{A_i}(x),
\]

\[
\mu_{\bigcap_{i \in \Gamma} A_i}(x) = \bigcap_{i \in \Gamma} \mu_{A_i}(x), \quad \nu_{\bigcap_{i \in \Gamma} A_i}(x) = \bigcup_{i \in \Gamma} \nu_{A_i}(x)
\]

for all $x \in X$.

3. $([r, s], [t, u])$-interval-valued intuitionistic fuzzy alpha closed and open sets

**Definition 3.1.** [12]. An interval-valued intuitionistic gradation of openness (for short, IVIGO) on $X$, which is also called an interval-valued intuitionistic smooth topology on $X$, is an ordered pair $(\tau, \tau^*)$ of mappings $\tau = [\tau^L, \tau^U] : I^X \to D(I)$ and $\tau^* = [\tau^{*L}, \tau^{*U}] : I^X \to D(I)$ satisfying the following conditions:

(IVIGO1) $\tau^L(A) \leq \tau^U(A)$, $\tau^{*L}(A) \leq \tau^{*U}(A)$ and $\tau^U(A) + \tau^{*U}(A) \leq 1$ for each $A \in I^X$.

(IVIGO2) $\tau(0_X) = \tau(1_X) = 1$ and $\tau^*(0_X) = \tau^*(1_X) = 0$.

(IVIGO3) $\tau^L(A \cap B) \geq \tau^L(A) \wedge \tau^L(B)$, $\tau^U(A \cap B) \geq \tau^U(A) \wedge \tau^U(B)$ and $\tau^{*L}(A \cap B) \leq \tau^{*L}(A) \vee \tau^{*L}(B)$, $\tau^{*U}(A \cap B) \leq \tau^{*U}(A) \vee \tau^{*U}(B)$ for each $A, B \in I^X$.

(IVIGO4) $\tau^L(\bigcup_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \tau^L(A_i)$, $\tau^U(\bigcup_{i \in \Gamma} A_i) \geq \bigwedge_{i \in \Gamma} \tau^U(A_i)$ and $\tau^{*L}(\bigcup_{i \in \Gamma} A_i) \leq \bigvee_{i \in \Gamma} \tau^{*L}(A_i)$, $\tau^{*U}(\bigcup_{i \in \Gamma} A_i) \leq \bigvee_{i \in \Gamma} \tau^{*U}(A_i)$ for each subfamily $\{A_i : i \in \Gamma\} \subseteq I^X$.

The triple $(X, \tau, \tau^*)$ is called an interval-valued intuitionistic smooth topological space (for short, IVISTS). $\tau$ and $\tau^*$ may be interpreted as interval-valued gradation of openness and interval-valued gradation of nonopenness, respectively.

**Definition 3.2.** [12]. Let $(X, \tau, \tau^*)$ be an IVISTS, $A \in I^X$ and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$.

(i) $A$ is called an $([r, s], [t, u])$-interval-valued intuitionistic fuzzy open set (for short, $([r, s], [t, u])$-IVIFOS) if $\tau(A) \geq [r, s]$ and $\tau^*(A) \leq [t, u]$. 

(ii) $A$ is called an $([r, s], [t, u])$-interval-valued intuitionistic fuzzy closed set (for short, $([r, s], [t, u])$-IVIFCS) if $\tau(A^c) \geq [r, s]$ and $\tau^*(A^c) \leq [t, u]$.

**Definition 3.3.** [12]. Let $(X, \tau, \tau^*)$ be an IVISTS, $A \in I^X$ and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$. The $([r, s], [t, u])$-interval-valued intuitionistic fuzzy closure and $([r, s], [t, u])$-interval-valued intuitionistic fuzzy interior of $A$ are defined by

$$\text{cl}_{[r,s],[t,u]}(A) = \cap \{K \in I^X : A \subseteq K, \ K \text{ is an } ([r, s], [t, u])\text{-IVIFCS}\},$$

$$\text{int}_{[r,s],[t,u]}(A) = \cup \{G \in I^X : G \subseteq A, \ G \text{ is an } ([r, s], [t, u])\text{-IVIFOS}\}.$$

**Definition 3.4.** Let $(X, \tau, \tau^*)$ be an IVISTS, $A \in I^X$ and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$.

(i) $A$ is called an $([r, s], [t, u])$-interval-valued intuitionistic fuzzy $\alpha$-closed set (for short, $([r, s], [t, u])$-IVIF$\alpha$CS) if $\text{cl}_{[r,s],[t,u]}(\text{int}_{[r,s],[t,u]}(\text{cl}_{[r,s],[t,u]}(A))) \subseteq A$.

(ii) $A$ is called an $([r, s], [t, u])$-interval-valued intuitionistic fuzzy $\alpha$-open set (for short, $([r, s], [t, u])$-IVIF$\alpha$OS) if $A^c$ is an $([r, s], [t, u])$-IVIF$\alpha$CS, or equivalently, $A \subseteq \text{int}_{[r,s],[t,u]}(\text{cl}_{[r,s],[t,u]}(\text{int}_{[r,s],[t,u]}(A)))$.

Note that if $A$ is an $([r, s], [t, u])$-IVIFCS then $A$ is an $([r, s], [t, u])$-IVIF$\alpha$CS and that if $A$ is an $([r, s], [t, u])$-IVIFOS then $A$ is an $([r, s], [t, u])$-IVIF$\alpha$OS.

**Example 3.5.** Every $([r, s], [t, u])$-IVIF$\alpha$CS need not be an $([r, s], [t, u])$-IVIFCS and every $([r, s], [t, u])$-IVIF$\alpha$OS need not be an $([r, s], [t, u])$-IVIFOS.

Let $X = \{a, b\}$. Define $F_1, F_2, F_3, F_4 \in I^X$ as follows:

$F_1 = \{(a, 0.4), (b, 0.4)\}$, $F_2 = \{(a, 0.5), (b, 0.6)\}$, $F_3 = \{(a, 0.5), (b, 0.4)\}$, $F_4 = \{(a, 0.6), (b, 0.6)\}$.

Define $\tau, \tau^* : I^X \to D(I)$ as follows:

$$\tau(A) = \begin{cases} 1 & \text{if } A \in \{0_X, 1_X\}, \\ [0.7, 0.8] & \text{if } A = F_2, \\ [0.4, 0.5] & \text{if } A = F_1, \\ 0 & \text{otherwise.} \end{cases}$$

$$\tau^*(A) = \begin{cases} 0 & \text{if } A \in \{0_X, 1_X\}, \\ [0.1, 0.2] & \text{if } A = F_2, \\ [0.3, 0.4] & \text{if } A = F_1, \\ 1 & \text{otherwise.} \end{cases}$$
Let \([r, s] = [0.5, 0.6]\) and \([t, u] = [0.2, 0.3]\). Then \(F_1\) is an \(([r, s], [t, u])\)-IVIF\(\alpha\)CS, but \(F_1\) is not an \(([r, s], [t, u])\)-IVIF\(\alpha\)CS. Also \(F_4\) is an \(([r, s], [t, u])\)-IVIF\(\alpha\)OS, but \(F_4\) is not an \(([r, s], [t, u])\)-IVIF\(\alpha\)OS.

**Remark 3.6.** Let \((X, \tau, \tau^*)\) be an IVISTS, \(A \in I^X\) and \([r, s] \in D(I_0)\), \([t, u] \in D(I_1)\) with \(s + u \leq 1\). Then

(i) Any intersection of \(([r, s], [t, u])\)-IVIF\(\alpha\)CSs is an \(([r, s], [t, u])\)-IVIF\(\alpha\)CS.

(ii) Any union of \(([r, s], [t, u])\)-IVIF\(\alpha\)OSs is an \(([r, s], [t, u])\)-IVIF\(\alpha\)OS.

**Definition 3.7.** Let \((X, \tau, \tau^*)\) be an IVISTS, \(A \in I^X\) and \([r, s] \in D(I_0)\), \([t, u] \in D(I_1)\) with \(s + u \leq 1\). The \(([r, s], [t, u])\)-interval-valued intuitionistic fuzzy \(\alpha\)-closure and \(([r, s], [t, u])\)-interval-valued intuitionistic fuzzy \(\alpha\)-interior of \(A\) are defined by

\[
\alpha cl_{[r, s], [t, u]}(A) = \cap \{K \in I^X : A \subseteq K, \ K \text{ is an } ([r, s], [t, u])\text{-IVIF}\alpha\text{CS}\},
\]

\[
\alpha int_{[r, s], [t, u]}(A) = \cup \{G \in I^X : G \subseteq A, \ G \text{ is an } ([r, s], [t, u])\text{-IVIF}\alpha\text{OS}\}.
\]

Note that \(int_{[r, s], [t, u]}(A) \subseteq \alpha int_{[r, s], [t, u]}(A) \subseteq A \subseteq \alpha cl_{[r, s], [t, u]}(A) \subseteq cl_{[r, s], [t, u]}(A)\).

**Theorem 3.8.** Let \((X, \tau, \tau^*)\) be an IVISTS, \(A, B \in I^X\) and \([r, s] \in D(I_0)\), \([t, u] \in D(I_1)\) with \(s + u \leq 1\). Then

(i) \(\alpha cl_{[r, s], [t, u]}(0_X) = 0_X\).

(ii) \(A \subseteq \alpha cl_{[r, s], [t, u]}(A)\).

(iii) \(\alpha cl_{[r, s], [t, u]}(A) \subseteq \alpha cl_{[r, s], [t, u]}(B)\) if \(A \subseteq B\).

(iv) \(\alpha cl_{[r, s], [t, u]}(A \cup B) \supseteq \alpha cl_{[r, s], [t, u]}(A) \cup \alpha cl_{[r, s], [t, u]}(B)\),

\(\alpha cl_{[r, s], [t, u]}(A \cap B) \subseteq \alpha cl_{[r, s], [t, u]}(A) \cap \alpha cl_{[r, s], [t, u]}(B)\).

(v) \(A = \alpha cl_{[r, s], [t, u]}(A)\) if and only if \(A\) is an \(([r, s], [t, u])\)-IVIF\(\alpha\)CS.

(vi) \(\alpha cl_{[r, s], [t, u]}(\alpha cl_{[r, s], [t, u]}(A)) = \alpha cl_{[r, s], [t, u]}(A)\).

(vii) \(\alpha cl_{[r, s], [t, u]}(A^c) = (\alpha int_{[r, s], [t, u]}(A))^c\).

**Proof.** (i), (ii) and (iii) follow directly from Definition 3.7.

(iv) It follows directly from (iii).

(v) It follows directly from Definition 3.7 and Remark 3.6.

(vi) By Definition 3.7 and Remark 3.6, \(\alpha cl_{[r, s], [t, u]}(A)\) is an \(([r, s], [t, u])\)-IVIF\(\alpha\)CS. By (v), \(\alpha cl_{[r, s], [t, u]}(\alpha cl_{[r, s], [t, u]}(A)) = \alpha cl_{[r, s], [t, u]}(A)\).
(vii) By Definition 3.7, we have
\[
\alpha cl_{[r,s],[t,u]}(A^c) = \cap \{K \in I^X : A^c \subseteq K, \ K \text{ is an } ([r,s],[t,u])\text{-IVIF}\alpha OS\},
\]
\[
= \cap \{G^c \in I^X : A^c \subseteq G^c, \ G^c \text{ is an } ([r,s],[t,u])\text{-IVIF}\alpha OS\}
\]
\[
= (\cup \{G \in I^X : G \subseteq A, \ G \text{ is an } ([r,s],[t,u])\text{-IVIF}\alpha OS\})^c
\]
\[
= (\alpha int_{[r,s],[t,u]}(A))^c.
\]
Thus

\[ i \text{ is an } ([r, s], [t, u])\text{-IVIF}\alpha\text{-continuous mapping}. \]

(ii) \( cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(B)))) \subseteq f^{-1}(cl_{[r,s],[t,u]}(B)) \) for each \( B \in I^Y \).

(iii) \( f(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(A)))) \subseteq cl_{[r,s],[t,u]}(f(A)) \) for each \( A \in I^X \).

**Proof.** (i)\(\Rightarrow\)(ii). Let \( B \in I^Y \). Then \( cl_{[r,s],[t,u]}(B) \) is an \((r, s), [t, u]\)-IVIFCS of \( Y \). Since \( f \) is an \((r, s), [t, u]\)-IVIF\(\alpha\)-continuous mapping, 

\[ f^{-1}(cl_{[r,s],[t,u]}(B)) \supseteq cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(cl_{[r,s],[t,u]}(B)))))) \]

\[ \supseteq cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(B))))). \]

(ii)\(\Rightarrow\)(iii). Let \( A \in I^X \). Then \( f(A) \in I^Y \). By (ii),

\[ f^{-1}(cl_{[r,s],[t,u]}(f(A))) \supseteq cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(f(A)))))) \]

\[ \supseteq cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(A))). \]

Hence

\[ cl_{[r,s],[t,u]}(f(A)) \supseteq f(f^{-1}(cl_{[r,s],[t,u]}(f(A)))) \]

\[ \supseteq f(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(A))))). \]

(iii)\(\Rightarrow\)(i). Let \( B \) be an \((r, s), [t, u]\)-IVIFCS of \( Y \). Then \( cl_{[r,s],[t,u]}(B) = B \) and \( f^{-1}(B) \in I^X \). By (iii),

\[ f(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(B)))))) \subseteq cl_{[r,s],[t,u]}(f(f^{-1}(B))) \]

\[ \subseteq cl_{[r,s],[t,u]}(B) = B. \]

Hence

\[ cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(B)))) \]

\[ \subseteq f^{-1}(f(cl_{[r,s],[t,u]}(int_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(B))))))). \]

\[ \subseteq f^{-1}(B). \]

Thus \( f^{-1}(B) \) is an \((r, s), [t, u]\)-IVIF\(\alpha\)-CS of \( X \). Hence \( f \) is an \((r, s), [t, u]\)-IVIF\(\alpha\)-continuous mapping. \( \square \)

**Theorem 4.3.** Let \((X, \tau, \tau^*)\) and \((Y, \eta, \eta^*)\) be two IVISTs and \([r, s] \in D(I_0), [t, u] \in D(I_1)\) with \( s + u \leq 1 \) and let \( f : X \to Y \) be a mapping. Then the following statements are equivalent.

(i) \( f \) is an \((r, s), [t, u]\)-IVIF\(\alpha\)-continuous mapping.

(ii) \( f(cl_{[r,s],[t,u]}(A)) \subseteq cl_{[r,s],[t,u]}(f(A)) \) for each \( A \in I^X \).

(iii) \( cl_{[r,s],[t,u]}(f^{-1}(B)) \subseteq f^{-1}(cl_{[r,s],[t,u]}(B)) \) for each \( B \in I^Y \).
(iv) \( f^{-1}(\text{int}_{[r,s],[t,u]}(B)) \subseteq \alpha\text{int}_{[r,s],[t,u]}(f^{-1}(B)) \) for each \( B \in I^Y \).

Proof. (i)\( \Rightarrow \) (ii). Let \( A \in I^X \). Then \( cl_{[r,s],[t,u]}(f(A)) \) is an \((r, s), [t, u])\)-IVIFCS of \( Y \). Since \( f \) is an \((r, s), [t, u])\)-IVIF\(\alpha\)-continuous mapping, \( f^{-1}(cl_{[r,s],[t,u]}(f(A))) \) is an \((r, s), [t, u])\)-IVIF\(\alpha\)CS of \( X \). By Theorem 3.8,
\[
\alpha cl_{[r,s],[t,u]}(A) \subseteq \alpha cl_{[r,s],[t,u]}(f^{-1}(f(A)))
\]
\[
\subseteq \alpha cl_{[r,s],[t,u]}(f^{-1}(cl_{[r,s],[t,u]}(f(A))))
\]
\[
= f^{-1}(cl_{[r,s],[t,u]}(f(A))).
\]
Hence
\[
f(\alpha cl_{[r,s],[t,u]}(A)) \subseteq f(f^{-1}(cl_{[r,s],[t,u]}(f(A))))
\]
\[
\subseteq cl_{[r,s],[t,u]}(f(A)).
\]
(ii)\( \Rightarrow \) (iii). Let \( B \in I^Y \). Then \( f^{-1}(B) \in I^X \). By (ii),
\[
f(\alpha cl_{[r,s],[t,u]}(f^{-1}(B))) \subseteq cl_{[r,s],[t,u]}(f(f^{-1}(B))) \subseteq cl_{[r,s],[t,u]}(B).
\]
Hence
\[
\alpha cl_{[r,s],[t,u]}(f^{-1}(B)) \subseteq f^{-1}(f(\alpha cl_{[r,s],[t,u]}(f^{-1}(B)))) \subseteq f^{-1}(cl_{[r,s],[t,u]}(B)).
\]
(iii)\( \Rightarrow \) (iv). Let \( B \in I^Y \). By (iii) and Theorem 3.8,
\[
(\alpha\text{int}_{[r,s],[t,u]}(f^{-1}(B)))^c = \alpha cl_{[r,s],[t,u]}(f^{-1}(B^c))
\]
\[
\subseteq f^{-1}(cl_{[r,s],[t,u]}(B^c))
\]
\[
= (f^{-1}(\text{int}_{[r,s],[t,u]}(B)))^c.
\]
Hence \( f^{-1}(\text{int}_{[r,s],[t,u]}(B)) \subseteq \alpha\text{int}_{[r,s],[t,u]}(f^{-1}(B)) \).

(iv)\( \Rightarrow \) (i). Let \( B \) be an \((r, s), [t, u])\)-IVIFOS of \( Y \). Then \( \text{int}_{[r,s],[t,u]}(B) = B \) and \( f^{-1}(B) \in I^X \). By (iv),
\[
f^{-1}(B) = f^{-1}(\text{int}_{[r,s],[t,u]}(B)) \subseteq \alpha\text{int}_{[r,s],[t,u]}(f^{-1}(B)) \subseteq f^{-1}(B).
\]
Thus \( f^{-1}(B) = \alpha\text{int}_{[r,s],[t,u]}(f^{-1}(B)) \). By Theorem 3.8, \( f^{-1}(B) \) is an \((r, s), [t, u])\)-IVIF\(\alpha\)OS of \( X \). Hence \( f \) is an \((r, s), [t, u])\)-IVIF\(\alpha\)-continuous mapping.

\[\square\]

**Theorem 4.4.** Let \( (X, \tau, \tau^*) \) and \( (Y, \eta, \eta^*) \) be two IVISTs and \( [r, s] \in D(I_0), [t, u] \in D(I_1) \) with \( s + u \leq 1 \) and let \( f : X \to Y \) be a bijective mapping. Then \( f \) is an \((r, s), [t, u])\)-IVIF\(\alpha\)-continuous mapping if and only if \( \text{int}_{[r,s],[t,u]}(f(A)) \subseteq f(\alpha\text{int}_{[r,s],[t,u]}(A)) \) for each \( A \in I^X \).
Proof. Let $f$ be an $([r, s], [t, u])$-IVIFα-continuous mapping and let $A \in I^X$. Then $\text{int}_{[r, s], [t, u]}(f(A))$ is an $([r, s], [t, u])$-IVIFOS of $Y$. Since $f$ is an $([r, s], [t, u])$-IVIFα-continuous mapping, $f^{-1}(\text{int}_{[r, s], [t, u]}(f(A)))$ is an $([r, s], [t, u])$-IVIFαOS of $X$. By Theorem 3.8 and injectivity of $f$,

$$f^{-1}(\text{int}_{[r, s], [t, u]}(f(A))) = \alpha\text{int}_{[r, s], [t, u]}(f^{-1}(\text{int}_{[r, s], [t, u]}(f(A)))) \subseteq \alpha\text{int}_{[r, s], [t, u]}(f^{-1}(f(A))) = \alpha\text{int}_{[r, s], [t, u]}(A).$$

By surjectivity of $f$,

$$\text{int}_{[r, s], [t, u]}(f(A)) = f(f^{-1}(\text{int}_{[r, s], [t, u]}(f(A)))) \subseteq f(\alpha\text{int}_{[r, s], [t, u]}(A)).$$

Conversely, let $B$ be an $([r, s], [t, u])$-IVIFOS of $Y$. Then $\text{int}_{[r, s], [t, u]}(B) = B$ and $f^{-1}(B) \in I^X$. By hypothesis and surjectivity of $f$,

$$B = \text{int}_{[r, s], [t, u]}(B) = \text{int}_{[r, s], [t, u]}(f(f^{-1}(B))) \subseteq f(\alpha\text{int}_{[r, s], [t, u]}(f^{-1}(B))) \subseteq f(f^{-1}(B)) = B.$$

Thus $B = f(\alpha\text{int}_{[r, s], [t, u]}(f^{-1}(B)))$. By injectivity of $f$,

$$f^{-1}(B) = f^{-1}(f(\alpha\text{int}_{[r, s], [t, u]}(f^{-1}(B)))) = \alpha\text{int}_{[r, s], [t, u]}(f^{-1}(B)).$$

By Theorem 3.8, $f^{-1}(B)$ is an $([r, s], [t, u])$-IVIFαOS of $X$. Hence $f$ is an $([r, s], [t, u])$-IVIFα-continuous mapping. □

From Theorem 4.3 and Theorem 4.4, we can obtain the following corollary.

**Corollary 4.5.** Let $(X, \tau, \tau^*)$ and $(Y, \eta, \eta^*)$ be two IVISTSSs and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$ and let $f : X \to Y$ be a bijective mapping. Then the following statements are equivalent.

(i) $f$ is an $([r, s], [t, u])$-IVIFα-continuous mapping.
(ii) $f(\text{ocl}_{[r, s], [t, u]}(A)) \subseteq \text{cl}_{[r, s], [t, u]}(f(A))$ for each $A \in I^X$.
(iii) $\text{ocl}_{[r, s], [t, u]}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_{[r, s], [t, u]}(B))$ for each $B \in I^Y$.
(iv) $f^{-1}(\text{int}_{[r, s], [t, u]}(B)) \subseteq \alpha\text{int}_{[r, s], [t, u]}(f^{-1}(B))$ for each $B \in I^Y$.
(v) $\text{int}_{[r, s], [t, u]}(f(A)) \subseteq f(\alpha\text{int}_{[r, s], [t, u]}(A))$ for each $A \in I^X$. 

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5. \([r, s], [t, u]\)-interval-valued intuitionistic fuzzy alpha generalized closed and open sets

**Definition 5.1.** Let \(X, \tau, \tau^*\) be an IVISTS, \(A \in I^X\) and \([r, s] \in D(I_0), [t, u] \in D(I_1)\) with \(s + u \leq 1\).

(i) \(A\) is called an \([r, s], [t, u]\)-interval-valued intuitionistic fuzzy \(\alpha\)-generalized closed set (for short, \([r, s], [t, u]\)-IVIFGCS) if \(\alpha cl_{[r, s], [t, u]}(A) \subseteq U\) whenever \(A \subseteq U\) and \(U\) is an \([r, s], [t, u]\)-IVIFOS.

(ii) \(A\) is called an \([r, s], [t, u]\)-interval-valued intuitionistic fuzzy \(\alpha\)-generalized open set (for short, \([r, s], [t, u]\)-IVIFGOS) if \(A^c\) is an \([r, s], [t, u]\)-IVIFGCS, or equivalently, \(U \subseteq \alpha int_{[r, s], [t, u]}(A)\) whenever \(U \subseteq A\) and \(U\) is an \([r, s], [t, u]\)-IVIFCS.

Note that if \(A\) is an \([r, s], [t, u]\)-IVIFGCS then \(A\) is an \([r, s], [t, u]\)-IVIFCS and that if \(A\) is an \([r, s], [t, u]\)-IVIFGOS then \(A\) is an \([r, s], [t, u]\)-IVIFOS.

**Example 5.2.** Every \([r, s], [t, u]\)-IVIFGCS need not be an \([r, s], [t, u]\)-IVIFCS and every \([r, s], [t, u]\)-IVIFGOS need not be an \([r, s], [t, u]\)-IVIFOS.

Let \(X = \{a, b\}\). Define \(F_1, F_2, F_3, F_4 \in I^X\) as follows:

\[
F_1 = \{(a, 0.4), (b, 0.4)\}, \quad F_2 = \{(a, 0.5), (b, 0.6)\}, \quad F_3 = \{(a, 0.6), (b, 0.6)\}, \\
F_4 = \{(a, 0.5), (b, 0.4)\}.
\]

Define \(\tau, \tau^* : I^X \to D(I)\) as follows:

\[
\tau(A) = \begin{cases} 
1 & \text{if } A \in \{0_X, 1_X\}, \\
[0.7, 0.8] & \text{if } A = F_1, \\
[0.4, 0.5] & \text{if } A = F_2, \\
0 & \text{otherwise}.
\end{cases}
\]

\[
\tau^*(A) = \begin{cases} 
0 & \text{if } A \in \{0_X, 1_X\}, \\
[0.1, 0.2] & \text{if } A = F_1, \\
[0.3, 0.4] & \text{if } A = F_2, \\
1 & \text{otherwise}.
\end{cases}
\]

Let \([r, s] = [0.5, 0.6]\) and \([t, u] = [0.2, 0.3]\). Then \(F_2\) is an \([r, s], [t, u]\)-IVIFGCS, but \(F_2\) is not an \([r, s], [t, u]\)-IVIFCS. Also \(F_4\) is an \([r, s], [t, u]\)-IVIFGOS, but \(F_4\) is not an \([r, s], [t, u]\)-IVIFOS.
Example 5.3. The intersection of two \((r, s], [t, u])\)-IVIF\(\alpha\)GCSs need not be an \((r, s], [t, u])\)-IVIF\(\alpha\)GCS and the union of two \((r, s], [t, u])\)-IVIF\(\alpha\)GOSs need not be an \((r, s], [t, u])\)-IVIF\(\alpha\)GOS.

Let \(X = \{a, b, c\}\). Define \(G_1, G_2, G_3 \in I^X\) as follows:

\[
G_1 = \{(a, 1), (b, 0), (c, 0)\}, \quad G_2 = \{(a, 1), (b, 1), (c, 0)\}, \quad G_3 = \{(a, 1), (b, 0), (c, 1)\}.
\]

Define \(\tau, \tau^* : I^X \to D(I)\) as follows:

\[
\tau(A) = \begin{cases} 
1 & \text{if } A \in \{0_X, 1_X\}, \\
[0.7, 0.8] & \text{if } A = G_1, \\
[0.5, 0.6] & \text{if } A = G_2, \\
[0.3, 0.4] & \text{if } A = G_3, \\
0 & \text{otherwise}.
\end{cases}
\]

\[
\tau^*(A) = \begin{cases} 
0 & \text{if } A \in \{0_X, 1_X\}, \\
[0.1, 0.2] & \text{if } A = G_1, \\
[0.3, 0.4] & \text{if } A = G_2, \\
[0.5, 0.6] & \text{if } A = G_3, \\
1 & \text{otherwise}.
\end{cases}
\]

Let \([r, s] = [0.6, 0.7]\) and \([t, u] = [0.2, 0.3]\). Then the only \((r, s], [t, u])\)-IVIFOSs are \(0_X, 1_X\) and \(G_1\) and the only \((r, s], [t, u])\)-IVIFCSs are \(0_X, 1_X\) and \(G_1^\dagger\). Also \(G_1 \subseteq G_2\) and \(G_1 \subseteq G_3\). Let \(G_2 \subseteq U\) and let \(U\) be an \((r, s], [t, u])\)-IVIFOS. Then \(U = 1_X\) and so \(\alpha cl_{[r, s], [t, u]}(G_2) \subseteq 1_X = U\). Hence \(G_2\) is an \((r, s], [t, u])\)-IVIF\(\alpha\)GCS. Similarly, \(G_3\) is also an \((r, s], [t, u])\)-IVIF\(\alpha\)GCS. Now \(G_2 \cap G_3 = G_1\). \(cl_{[r, s], [t, u]}(int_{[r, s], [t, u]}(cl_{[r, s], [t, u]}(G_1))) = cl_{[r, s], [t, u]}(int_{[r, s], [t, u]}(1_X)) = cl_{[r, s], [t, u]}(1_X) = 1_X \not\subseteq G_1\). Hence \(G_1\) is not an \((r, s], [t, u])\)-IVIF\(\alpha\)CS. Let \(G_1 \subseteq U\) and let \(U\) be an \((r, s], [t, u])\)-IVIFOS. Then \(U = G_1\) or \(U = 1_X\). In the case \(U = G_1\), by Theorem 3.8(v) \(G_1 \not= \alpha cl_{[r, s], [t, u]}(G_1)\) since \(G_1\) is not an \((r, s], [t, u])\)-IVIF\(\alpha\)CS. Thus \(\alpha cl_{[r, s], [t, u]}(G_1) \supseteq G_1\). Hence \(\alpha cl_{[r, s], [t, u]}(G_1) \not\subseteq G_1 = U\). Thus \(G_1\) is not an \((r, s], [t, u])\)-IVIF\(\alpha\)GCS.

By taking the complementation in the above example, the union of two \((r, s], [t, u])\)-IVIF\(\alpha\)GOSs need not be an \((r, s], [t, u])\)-IVIF\(\alpha\)GOS.

Definition 5.4. Let \((X, \tau, \tau^*)\) be an IVISTS, \(A \in I^X\) and \([r, s] \in D(I_0), [t, u] \in D(I_1)\) with \(s + u \leq 1\). The \((r, s], [t, u])\)-interval-valued
intuitionistic fuzzy $\alpha$-generalized closure and $([r, s], [t, u])$-interval-valued intuitionistic fuzzy $\alpha$-generalized interior of $A$ are defined by

$$\alpha\text{gcl}_{[r, s], [t, u]}(A) = \cap\{K \in I^X : A \subseteq K, \ K \text{ is an } ([r, s], [t, u])\text{-IVIFoGCS}\},$$

$$\alpha\text{gint}_{[r, s], [t, u]}(A) = \cup\{G \in I^X : G \subseteq A, \ G \text{ is an } ([r, s], [t, u])\text{-IVIFoGSC}\}.$$

Note that $\alpha\text{in}_{[r, s], [t, u]}(A) \subseteq \alpha\text{gint}_{[r, s], [t, u]}(A) \subseteq A \subseteq \alpha\text{gcl}_{[r, s], [t, u]}(A) \subseteq \alpha\text{cl}_{[r, s], [t, u]}(A)$.

**Theorem 5.5.** Let $(X, \tau, \tau^*)$ be an IVISTS, $A, B \in I^X$ and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$. Then

(i) $\alpha\text{gcl}_{[r, s], [t, u]}(0_X) = 0_X$.

(ii) $A \subseteq \alpha\text{gcl}_{[r, s], [t, u]}(A)$.

(iii) $\alpha\text{gcl}_{[r, s], [t, u]}(A) \subseteq \alpha\text{gcl}_{[r, s], [t, u]}(B)$ if $A \subseteq B$.

(iv) $\alpha\text{gcl}_{[r, s], [t, u]}(A \cup B) \supseteq \alpha\text{gcl}_{[r, s], [t, u]}(A) \cup \alpha\text{gcl}_{[r, s], [t, u]}(B)$,

(v) $\alpha\text{gcl}_{[r, s], [t, u]}(A \cap B) \subseteq \alpha\text{gcl}_{[r, s], [t, u]}(A) \cap \alpha\text{gcl}_{[r, s], [t, u]}(B)$.

(vi) $A = \alpha\text{gcl}_{[r, s], [t, u]}(A)$ if $A$ is an $([r, s], [t, u])$-IVIFoGCS.

(vii) $\alpha\text{gcl}_{[r, s], [t, u]}(A^c) = (\alpha\text{in}_{[r, s], [t, u]}(A))^c$.

**Proof.** (i), (ii) and (iii) follow directly from Definition 5.4.

(iv) It follows directly from (iii).

(v) It follows directly from Definition 5.4.

(vi) By (ii) and (iii), $\alpha\text{gcl}_{[r, s], [t, u]}(A) \subseteq \alpha\text{gcl}_{[r, s], [t, u]}(\alpha\text{gcl}_{[r, s], [t, u]}(A))$. Suppose that $\alpha\text{gcl}_{[r, s], [t, u]}(\alpha\text{gcl}_{[r, s], [t, u]}(A)) \not\subseteq \alpha\text{gcl}_{[r, s], [t, u]}(A)$. Then there exists $x \in X$ such that $(\alpha\text{gcl}_{[r, s], [t, u]}(A))(x) < (\alpha\text{gcl}_{[r, s], [t, u]}(\alpha\text{gcl}_{[r, s], [t, u]}(A)))(x)$.

Choose $a \in (0, 1)$ with $(\alpha\text{gcl}_{[r, s], [t, u]}(A))(x) < a < (\alpha\text{gcl}_{[r, s], [t, u]}(\alpha\text{gcl}_{[r, s], [t, u]}(A)))(x)$. Since $(\alpha\text{gcl}_{[r, s], [t, u]}(A))(x) < a$, by Definition 5.4 there exists an $([r, s], [t, u])$-IVIFoGCS $K$ such that $A \subseteq K$ and $K(x) < a$. Since $K$ is an $([r, s], [t, u])$-IVIFoGCS with $A \subseteq K$, $\alpha\text{gcl}_{[r, s], [t, u]}(A) \subseteq K$ and also $\alpha\text{gcl}_{[r, s], [t, u]}(\alpha\text{gcl}_{[r, s], [t, u]}(A)) \subseteq K$. Hence $(\alpha\text{gcl}_{[r, s], [t, u]}(\alpha\text{gcl}_{[r, s], [t, u]}(A)))(x) \leq K(x) < a$. This is a contradiction. Hence $\alpha\text{gcl}_{[r, s], [t, u]}(\alpha\text{gcl}_{[r, s], [t, u]}(A)) \subseteq \alpha\text{gcl}_{[r, s], [t, u]}(A)$. Therefore $\alpha\text{gcl}_{[r, s], [t, u]}(\alpha\text{gcl}_{[r, s], [t, u]}(A)) = \alpha\text{gcl}_{[r, s], [t, u]}(A)$.

(vii) By Definition 5.4, we have

$$\alpha\text{gcl}_{[r, s], [t, u]}(A^c) = \cap\{K \in I^X : A^c \subseteq K, \ K \text{ is an } ([r, s], [t, u])\text{-IVIFoGCS}\}$$

$$= \cap\{G^c \in I^X : A^c \subseteq G^c, \ G^c \text{ is an } ([r, s], [t, u])\text{-IVIFoGCS}\}$$

$$= (\cup\{G \in I^X : G \subseteq A, \ G \text{ is an } ([r, s], [t, u])\text{-IVIFoGSC}\})^c$$

$$= (\alpha\text{gint}_{[r, s], [t, u]}(A))^c.$$
Theorem 5.6. Let \((X, \tau, \tau^*)\) be an IVISTS, \(A, B \in I^X\) and \([r, s] \in D(I_0), [t, u] \in D(I_1)\) with \(s + u \leq 1\). Then

(i) \(\alpha g_{int}[r, s][t, u](1_X) = 1_X\).

(ii) \(\alpha g_{int}[r, s][t, u](A) \subseteq A\).

(iii) \(\alpha g_{int}[r, s][t, u](A) \subseteq \alpha g_{int}[r, s][t, u](B)\) if \(A \subseteq B\).

(iv) \(\alpha g_{int}[r, s][t, u](A \cup B) \supseteq \alpha g_{int}[r, s][t, u](A) \cup \alpha g_{int}[r, s][t, u](B)\).

(v) \(A = \alpha g_{int}[r, s][t, u](A)\) if \(A\) is an \([r, s], [t, u]\)-IVIF\(\alpha\)GOS.

(vi) \(\alpha g_{int}[r, s][t, u](\alpha g_{int}[r, s][t, u](A)) = \alpha g_{int}[r, s][t, u](A)\).

(vii) \(\alpha g_{int}[r, s][t, u](A^e) = (\alpha g_{cl}[r, s][t, u](A))^e\).

Proof. The proof is similar to Theorem 5.5.

6. \([r, s], [t, u]\)-interval-valued intuitionistic fuzzy alpha generalized continuous mappings

Definition 6.1. Let \((X, \tau, \tau^*)\) and \((Y, \eta, \eta^*)\) be two IVISTSs and \([r, s] \in D(I_0), [t, u] \in D(I_1)\) with \(s + u \leq 1\) and let \(f : X \to Y\) be a mapping. \(f\) is called an \([r, s], [t, u]\)-interval-valued intuitionistic fuzzy \(\alpha\)-generalized continuous mapping (for short, \([r, s], [t, u]\)-IVIF\(\alpha\)G continuous mapping) if \(f^{-1}(B)\) is an \([r, s], [t, u]\)-IVIF\(\alpha\)CS of \(X\) for each \([r, s], [t, u]\)-IVIFCS \(B\) of \(Y\).

Note that \(f : X \to Y\) is an \([r, s], [t, u]\)-IVIF\(\alpha\)G continuous mapping if and only if \(f^{-1}(B)\) is an \([r, s], [t, u]\)-IVIF\(\alpha\)GOS of \(X\) for each \([r, s], [t, u]\)-IVIFOS \(B\) of \(Y\) and that if \(f : X \to Y\) is an \([r, s], [t, u]\)-IVIF\(\alpha\)-continuous mapping then \(f : X \to Y\) is an \([r, s], [t, u]\)-IVIF\(\alpha\)G continuous mapping.

Example 6.2. Every \([r, s], [t, u]\)-IVIF\(\alpha\)G continuous mapping need not be an \([r, s], [t, u]\)-IVIF\(\alpha\)-continuous mapping.

Let \(X = \{a, b\}\) and \(Y = \{c, d\}\). Define \(F_1, F_2, F_3 \in I^X\) and \(G_1, G_2 \in I^Y\) as follows:

\(F_1 = \{(a, 0.4), (b, 0.4)\}\), \(F_2 = \{(a, 0.5), (b, 0.6)\}\), \(F_3 = \{(a, 0.6), (b, 0.6)\}\), \(G_1 = \{(c, 0.5), (d, 0.4)\}\), \(G_2 = \{(c, 0.5), (d, 0.6)\}\).
Define $\tau, \tau^*: I^X \to D(I)$, $\eta, \eta^*: I^Y \to D(I)$ as follows:

$$
\tau(A) = \begin{cases} 
1 & \text{if } A \in \{0_X, 1_X\}, \\
[0.7, 0.8] & \text{if } A = F_1, \\
[0.4, 0.5] & \text{if } A = F_2, \\
0 & \text{otherwise}.
\end{cases}
$$

$$
\tau^*(A) = \begin{cases} 
0 & \text{if } A \in \{0_X, 1_X\}, \\
[0.1, 0.2] & \text{if } A = F_1, \\
[0.3, 0.4] & \text{if } A = F_2, \\
1 & \text{otherwise}.
\end{cases}
$$

$$
\eta(B) = \begin{cases} 
[0.8, 0.9] & \text{if } B = G_1, \\
0 & \text{otherwise}.
\end{cases}
$$

$$
\eta^*(B) = \begin{cases} 
0 & \text{if } B \in \{0_Y, 1_Y\}, \\
[0.1, 0.2] & \text{if } B = G_1, \\
1 & \text{otherwise}.
\end{cases}
$$

Define the mapping $f : (X, \tau, \tau^*) \to (Y, \eta, \eta^*)$ by $f(a) = c, f(b) = d$ and let $[r, s] = [0.5, 0.6]$ and $[t, u] = [0.2, 0.3]$. Then $f$ is an $([r, s], [t, u])$-IVIFαG continuous mapping, but $f$ is not an $([r, s], [t, u])$-IVIFα-continuous mapping.

**Definition 6.3.** An IVISTS $(X, \tau, \tau^*)$ is called an *interval-valued intuitionistic fuzzy alpha T$_{1/2}^*$ space* (for short, IVIFαT$_{1/2}^*$ space) if for each $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$, every $([r, s], [t, u])$-IVIFαGCS in $X$ is an $([r, s], [t, u])$-IVIFCS in $X$.

**Theorem 6.4.** Let $(X, \tau, \tau^*)$ be an IVIFαT$_{1/2}^*$ space and $(Y, \eta, \eta^*)$ an IVISTS and $[r, s] \in D(I_0)$, $[t, u] \in D(I_1)$ with $s + u \leq 1$ and let $f : X \to Y$ be a mapping. Then the following statements are equivalent.

(i) $f$ is an $([r, s], [t, u])$-IVIFαG continuous mapping.

(ii) $f^{-1}(\text{int}_{[r,s],[t,u]}(B)) \subseteq \text{int}_{[r,s],[t,u]}(\mathcal{C}_{[r,s],[t,u]}(\text{int}_{[r,s],[t,u]}(f^{-1}(B))))$ for each $B \in I^Y$.

**Proof.** (i)$\Rightarrow$(ii). Let $B \in I^Y$. Then $\text{int}_{[r,s],[t,u]}(B)$ is an $([r, s], [t, u])$-IVIFOS of $Y$. Since $f$ is an $([r, s], [t, u])$-IVIFαG continuous mapping, $f^{-1}(\text{int}_{[r,s],[t,u]}(B))$ is an $([r, s], [t, u])$-IVIFαGOS of $X$. Since $X$ is an
IVIFαT_{1/2} space, $f^{-1}(\text{int}_{[r,s],[t,u]}(B))$ is an $([r,s],[t,u])$-IVIF of X. Hence

$$f^{-1}(\text{int}_{[r,s],[t,u]}(B)) = \text{int}_{[r,s],[t,u]}(f^{-1}(\text{int}_{[r,s],[t,u]}(B)))$$
$$\subseteq \text{int}_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(\text{int}_{[r,s],[t,u]}(B)))) = \text{int}_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(\text{int}_{[r,s],[t,u]}(f^{-1}(\text{int}_{[r,s],[t,u]}(B)))))$$
$$\subseteq \text{int}_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(\text{int}_{[r,s],[t,u]}(f^{-1}(B)))) .$$

(ii)$\Rightarrow$(i). Let $B$ be an $([r,s],[t,u])$-IVIFCS of Y. Then $B^c$ is an $([r,s],[t,u])$-IVIFOS of Y and so $\text{int}_{[r,s],[t,u]}(B^c) = B^c$. By hypothesis,

$$f^{-1}(B^c) = f^{-1}(\text{int}_{[r,s],[t,u]}(B^c)) \subseteq \text{int}_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(\text{int}_{[r,s],[t,u]}(f^{-1}(B^c)))) .$$

Thus $f^{-1}(B^c)$ is an $([r,s],[t,u])$-IVIFOS of X. Since every $([r,s],[t,u])$-IVIFαOS is an $([r,s],[t,u])$-IVIFαGOS, $f^{-1}(B^c)$ is an $([r,s],[t,u])$-IVIFαGOS of X. Hence $f^{-1}(B)$ is an $([r,s],[t,u])$-IVIFαGCS of X. Therefore f is an $([r,s],[t,u])$-IVIFαG continuous mapping.

By taking the complement of the set $B \in I^Y$ in Theorem 6.4, we obtain the following corollary.

**Corollary 6.5.** Let $(X, \tau, \tau^*)$ be an IVIFαT_{1/2} space and $(Y, \eta, \eta^*)$ an IVISTS and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s + u \leq 1$ and let $f : X \rightarrow Y$ be a mapping. Then the following statements are equivalent.

(i) $f$ is an $([r,s],[t,u])$-IVIFαG continuous mapping.

(ii) $cl_{[r,s],[t,u]}(\text{int}_{[r,s],[t,u]}(cl_{[r,s],[t,u]}(f^{-1}(B)))) \subseteq f^{-1}(cl_{[r,s],[t,u]}(B))$ for each $B \in I^Y$.

**Definition 6.6.** An IVISTS $(X, \tau, \tau^*)$ is called an interval-valued intuitionistic fuzzy alpha $T_{1/2}$ space (for short, IVIFαT_{1/2} space) if for each $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s + u \leq 1$, every $([r,s],[t,u])$-IVIFαGCS in X is an $([r,s],[t,u])$-IVIFαCS in X.

**Theorem 6.7.** Let $(X, \tau, \tau^*)$ be an IVIFαT_{1/2} space and $(Y, \eta, \eta^*)$ an IVISTS and $[r,s] \in D(I_0)$, $[t,u] \in D(I_1)$ with $s + u \leq 1$ and let $f : X \rightarrow Y$ be a mapping. Then the following statements are equivalent.

(i) $f$ is an $([r,s],[t,u])$-IVIFαG continuous mapping.

(ii) $f(\text{agcl}_{[r,s],[t,u]}(A)) \subseteq \text{cl}_{[r,s],[t,u]}(f(A))$ for each $A \in I^X$.

(iii) $\text{agcl}_{[r,s],[t,u]}(f^{-1}(B)) \subseteq f^{-1}(\text{cl}_{[r,s],[t,u]}(B))$ for each $B \in I^Y$.

(iv) $f^{-1}(\text{int}_{[r,s],[t,u]}(B)) \subseteq \text{agint}_{[r,s],[t,u]}(f^{-1}(B))$ for each $B \in I^Y$. 


Proof. (i)⇒(ii). Let \( A \in I^X \). Then \( cl_{[r,s],[t,u]}(f(A)) \) is an \(([r,s],[t,u])\)-IVIFCS of \( Y \). Since \( f \) is an \(([r,s],[t,u])\)-IVIFG continuous mapping, \( f^{-1}(cl_{[r,s],[t,u]}(f(A))) \) is an \(([r,s],[t,u])\)-IVIFGCS of \( X \). Since \( A \subseteq f^{-1}(cl_{[r,s],[t,u]}(f(A))) \), by Definition 5.3 \( \alpha_{cl} \subseteq f^{-1}(cl_{[r,s],[t,u]}(f(A))) \).

Hence \( f(\alpha_{cl_{[r,s],[t,u]}(A)}) \subseteq f(f^{-1}(cl_{[r,s],[t,u]}(f(A)))) \subseteq cl_{[r,s],[t,u]}(f(A)) \).

(ii)⇒(iii). Let \( B \in I^Y \). Then \( f^{-1}(B) \in I^X \). By (ii),

\[
\alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} \subseteq cl_{[r,s],[t,u]}(f(f^{-1}(B)))
\subseteq cl_{[r,s],[t,u]}(B).
\]

Hence

\[
\alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} \subseteq f^{-1}(f(\alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))))).
\]

(iii)⇒(iv). Let \( B \in I^Y \). By (iii), \( \alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} \subseteq f^{-1}(cl_{[r,s],[t,u]}(B^c)) \).

Thus \( (\alpha_{int_{[r,s],[t,u]}(f^{-1}(B)))^c \subseteq (f^{-1}(int_{[r,s],[t,u]}(B)))^c \). Hence \( f^{-1}(int_{[r,s],[t,u]}(B)) \subseteq \alpha_{int_{[r,s],[t,u]}(f^{-1}(B))} \).

(iv)⇒(i). Let \( B \) be an \(([r,s],[t,u])\)-IVIFCS of \( Y \). Then \( f^{-1}(B) \in I^X \) and \( B^c \) is an \(([r,s],[t,u])\)-IVIFOS of \( Y \) and so \( int_{[r,s],[t,u]}(B^c) = B^c \). Let \( f^{-1}(B) \subseteq U \) and let \( U \) be an \(([r,s],[t,u])\)-IVIFOS of \( X \). By (iv),

\[
(f^{-1}(B))^c = f^{-1}(B^c) = f^{-1}(int_{[r,s],[t,u]}(B^c))
\subseteq \alpha_{int_{[r,s],[t,u]}(f^{-1}(B)))^c
= (\alpha_{cl_{[r,s],[t,u]}(f^{-1}(B)))^c.
\]

Hence \( \alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} \subseteq f^{-1}(B) \) and so \( \alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} = f^{-1}(B) \).

Since \( (X,\tau,\tau^*) \) is an IVIF\(\alpha\)T\(_{1/2}\) space, \( \alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} = \alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} \). Hence \( \alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} = f^{-1}(B) \subseteq \alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} \).

Hence \( \alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} \subseteq f^{-1}(B) \) and so \( \alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} = f^{-1}(B) \).

Since \( (X,\tau,\tau^*) \) is an IVIF\(\alpha\)T\(_{1/2}\) space, \( \alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} = \alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} \). Hence \( \alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} = f^{-1}(B) \subseteq \alpha_{cl_{[r,s],[t,u]}(f^{-1}(B))} \).

Thus \( f^{-1}(B) \) is an \(([r,s],[t,u])\)-IVIFGCS of \( X \). Therefore \( f \) is an \(([r,s],[t,u])\)-IVIFG continuous mapping.

\( \square \)

Corollary 6.8. Let \( (X,\tau,\tau^*) \) be an IVIF\(\alpha\)T\(_{1/2}\) space and \( (Y,\eta,\eta^*) \) an IVISTS and \( [r,s] \in D(I_0), [t,u] \in D(I_1) \) with \( s + u \leq 1 \) and let \( f : X \to Y \) be a mapping. Then \( f \) is an \(([r,s],[t,u])\)-IVIFG continuous mapping if and only if \( f(\alpha_{cl_{[r,s],[t,u]}(A)}) \subseteq cl_{[r,s],[t,u]}(f(A)) \) for each \( A \in I^X \).
References


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