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THE CONSTRUCTIONS AND DEFORMATIONS OF NIJENHUIS OPERATORS ON 3-HOM-LIE ALGEBRAS

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Abstract. In this paper, we study Nijenhuis operators on 3-Hom-Lie algebras and provide some examples. Next, we give various constructions of Nijenhuis operators according to constructions of 3-Hom-Lie algebras. Furthermore, we define a cohomology of Nijenhuis operators on 3-Hom-Lie algebras with coefficients in a suitable representation. Finally, as an application, we study formal deformations of Nijenhuis operators that are generated by the above-defined cohomology.

1. Introduction

The notion of Hom-Lie algebras was introduced by Hartwig, Larsson, and Silvestrov in [9] as part of a study of deformations of Witt and Virasoro algebras. Some q-deformations of the Witt and the Virasoro algebras have the structure of a Hom-Lie algebra [9]. Because of the close relation to discrete and deformed vector fields and differential calculus [14], many people paid special attention to this algebraic structure. In particular, the author studied representations and cohomologies of Hom-Lie algebras in [20], the authors introduced the notion of a Hom-Lie bialgebra and gave an equivalent description of the Hom-Lie bialgebras, the matched pairs and the Manin triples of Hom-Lie algebras in [21]. These provide a good starting point for discussion and further research.

The notion of n-Lie algebras was introduced by Filippov in [6]. The authors studied Nijenhuis operators on n-Lie algebras in [18]. Recently, the authors studied Reynolds operators on n-Lie algebras in [11]. 3-Lie algebras were special types of n-Lie algebras and played an important role in string theory [4]. Many more properties and structures of 3-Lie algebras have been developed, see [7, 10, 12, 13, 17, 22, 25, 26] and references cited therein. Hom-type generalizations of n-ary Nambu-Lie algebras, called n-ary Hom-Nambu-Lie algebras, were introduced by Arnlind, Makhlouf, and Silvestrov in [3]. Since then, the

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authors studied the cohomologies adapted to central extensions and deformations in [1], the authors studied the constructions of 3-Hom-Lie algebras from Hom-Lie algebras in [2], the authors provided the constructions of Rota-Baxter multiplicative 3-ary Hom-Nambu-Lie algebras in [24], the authors constructed the representations and module-extensions of 3-Hom-Lie algebras in [16], and the authors introduced the notion of a 3-Hom-Lie bialgebra and gave an equivalent description of the 3-Hom-Lie bialgebras, the matched pairs and the Manin triples of 3-Hom-Lie algebras in [8]. Recently, we studied relative Rota-Baxter operators on 3-Hom-Lie algebras in [15]. See [23] for more details about deformations of n-Hom-Lie algebras.

Recently, Mishra and Naolekar [19] studied $\mathcal{O}$-operators, also known as relative or generalized Rota-Baxter operators on Hom-Lie algebras. Later, Das and Sen [5] studied Nijenhuis operators on Hom-Lie algebras. Motivated by the work of [5,19], it is natural and meaningful to study Nijenhuis operators on 3-Hom-Lie algebras. This becomes our motivation for writing the present paper. This paper is organized as follows. In Section 2, we will recall some basic notions and facts about 3-Hom-Lie algebras. In Section 3, we study Nijenhuis operators on 3-Hom-Lie algebras and provide some examples. In Section 4, we give various constructions of Nijenhuis operators according to constructions of 3-Hom-Lie algebras. In Section 5, we define a cohomology of Nijenhuis operators on 3-Hom-Lie algebras with coefficients in a suitable representation. In Section 6, we study formal deformations of Nijenhuis operators that are generated by the above-defined cohomology.

Throughout this paper, all vector spaces, (multi)linear maps are over an algebraically closed field $\mathbb{K}$ of characteristic 0.

2. Preliminaires

In this section, we will recall some basic notions and facts about 3-Hom-Lie algebras from [1,16,23].

Definition. A 3-Hom-Lie algebra is a triple $(L, [\cdot,\cdot,\cdot], \alpha)$ consisting of a vector space L, a 3-ary skew-symmetric operation $[\cdot,\cdot,\cdot] : \wedge^3 L \to L$ and an algebra morphism $\alpha : L \to L$ satisfying the following 3-Hom-Jacobi identity

$$\begin{aligned}&[\alpha(x), \alpha(y), [u, v, w]]\\ =& [[x, y, u], \alpha(v), \alpha(w)] + [\alpha(u), [x, y, v], \alpha(w)] + [\alpha(u), \alpha(v), [x, y, w]],\end{aligned}$$

for any $x, y, u, v, w \in L$.

Example 2.1. Let $(L, [\cdot,\cdot,\cdot])$ be a 3-Lie algebra and $\alpha : L \to L$ a 3-Lie algebra morphism. Then $(L, [\cdot,\cdot,\cdot]_\alpha = \alpha \circ [\cdot,\cdot,\cdot], \alpha)$ is a 3-Hom-Lie algebra induced by the 3-Lie algebra morphism α.

Definition. A representation of a 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$ on the vector space V with respect to $A \in \mathfrak{gl}(V)$ is a bilinear map $\rho : \wedge^2 L \to \mathfrak{gl}(V)$, such that for any $x, y, z, u \in L$, the following equalities are satisfied:

$$\begin{aligned}
\rho(\alpha(x), \alpha(y)) \circ A &= A \circ \rho(x, y),\\
\rho([x, y, z], \alpha(u)) \circ A &= \rho(\alpha(y), \alpha(z))\rho(x, u)\\
&\quad + \rho(\alpha(z), \alpha(x))\rho(y, u) + \rho(\alpha(x), \alpha(y))\rho(z, u),\\
\rho(\alpha(x), \alpha(y))\rho(z, u) &= \rho(\alpha(z), \alpha(u))\rho(x, y)\\
&\quad + \rho([x, y, z], \alpha(u)) \circ A + \rho(\alpha(z), [x, y, u]) \circ A.
\end{aligned}$$

Then (V, A, ρ) is called a representation of L, or V is an L-module.

Define $\mathrm{ad} : \wedge^2 L \to \mathfrak{gl}(L)$ by

$$\mathrm{ad}_{x,y} z = [x, y, z], \qquad \forall x, y, z \in L.$$

Then (L, ad, α) is a representation of the 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$ on L with respect to α, which is called the adjoint representation.

Let (V, A, ρ) be a representation of a 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$. The cohomology of L with coefficients in V is the cohomology of the cochain complex $\{C^*(L, V), \delta\}$, where

$$\begin{aligned}
C^n_{\alpha,A}(L, V) = \{f \in \mathrm{Hom}(\underbrace{\wedge^2 L \otimes \cdots \otimes \wedge^2 L}_{n-1} \wedge L, V), \quad (n \geq 1),\\
A \circ f = f \circ \alpha^{\otimes^{2n-1}}\}
\end{aligned}$$

and the coboundary operator $\delta : C^n_{\alpha,A}(L, V) \to C^{n+1}_{\alpha,A}(L, V)$ given by

$$\begin{aligned}
&(\delta f)(\mathfrak{X}_1, \mathfrak{X}_2, \ldots, \mathfrak{X}_n, x_{n+1})\\
&= \sum_{1\leq j<k\leq n} (-1)^j f(\alpha(\mathfrak{X}_1), \ldots, \hat{\mathfrak{X}}_j, \ldots, \alpha(\mathfrak{X}_{k-1}), [x_j, y_j, x_k] \wedge \alpha(y_k)\\
&\quad + \alpha(x_k) \wedge [x_j, y_j, y_k], \alpha(\mathfrak{X}_{k+1}), \ldots, \alpha(\mathfrak{X}_n), \alpha(x_{n+1}))\\
&\quad + \sum_{j=1}^{n} (-1)^j f(\alpha(\mathfrak{X}_1), \ldots, \hat{\mathfrak{X}}_j, \ldots, \alpha(\mathfrak{X}_n), [x_j, y_j, x_{n+1}])\\
&\quad + \sum_{j=1}^{n} (-1)^{j+1} \rho(\alpha^{n-1}(\mathfrak{X}_j)) f(\mathfrak{X}_1, \ldots, \hat{\mathfrak{X}}_j, \ldots, \mathfrak{X}_n, x_{n+1})\\
&\quad + (-1)^{n+1}\big(\rho(\alpha^{n-1}(y_n), \alpha^{n-1}(x_{n+1})) f(\mathfrak{X}_1, \ldots, \hat{\mathfrak{X}}_j, \ldots, \mathfrak{X}_{n-1}, x_n)\\
&\quad + \rho(\alpha^{n-1}(x_{n+1}), \alpha^{n-1}(x_n)) f(\mathfrak{X}_1, \ldots, \hat{\mathfrak{X}}_j, \ldots, \mathfrak{X}_{n-1}, y_n)\big)
\end{aligned}$$

for $\mathfrak{X}_i = x_i \wedge y_i \in L \wedge L, i = 1, \ldots, n$ and $x_{n+1} \in L$. The corresponding cohomology groups are denoted by $H^*_{3\mathrm{HLie}}(L, V)$.

3. Nijenhuis operators

In this section, we first recall Nijenhuis operators on a 3-Hom-Lie algebra in [15] and provide some new examples.

Definition. Let $(L,[\cdot,\cdot,\cdot],\alpha)$ be a 3-Hom-Lie algebra. A linear operator $N: L\to L$ is called a Nijenhuis operator if N satisfies: for any $x,y,z\in L$,

$$N\circ\alpha=\alpha\circ N, \tag{3.1}$$

$$\begin{aligned}[][Nx,Ny,Nz] &= N([Nx,Ny,z]+[x,Ny,Nz]+[Nx,y,Nz]\\ &\quad -N[Nx,y,z]-N[x,Ny,z]-N[x,y,Nz]+N^2[x,y,z]).\end{aligned}\tag{3.2}$$

The vector space L carries a new 3-Hom-Lie algebra structure with bracket

$$\begin{aligned}[][x,y,z]_N &:= [Nx,Ny,z]+[x,Ny,Nz]+[Nx,y,Nz]-N[Nx,y,z]\\ &\quad -N[x,Ny,z]-N[x,y,Nz]+N^2[x,y,z],\ \text{for } x,y,z\in L.\end{aligned}$$

We denote this deformed 3-Hom-Lie algebra by $(L,[\cdot,\cdot,\cdot]_N,\alpha)$.

Example 3.1. The identity map $\mathrm{id}: L\to L$ is a Nijenhuis operator.

Example 3.2. Consider 3-dimensional 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ with basis $\{e_1,e_2,e_3\}$ and the nonzero multiplication is given by

$$[e_1,e_2,e_3]=e_1,\ \alpha(e_1)=-e_1,\ \alpha(e_2)=e_2,\ \alpha(e_3)=e_3.$$

Let $N: L\to L$ be a linear map given by $\begin{pmatrix} d & 0 & 0\\ 0 & c & f\\ 0 & 0 & b\end{pmatrix}$ where $dcb\neq 0$. Then we have

$$[Ne_1,Ne_2,Ne_3]=dcbe_1.$$

Further, we have

$$\begin{aligned}[][Ne_1,Ne_2,e_3]&=dce_1,\ [Ne_1,e_2,e_3]=de_1,\\ [Ne_1,e_2,Ne_3]&=dbe_1,\ [e_1,Ne_2,e_3]=ce_1,\\ [e_1,Ne_2,Ne_3]&=cbe_1,\ [e_1,e_2,Ne_3]=be_1.\end{aligned}$$

It is easy to obtain that

$$\begin{aligned}&[Ne_1,Ne_2,Ne_3]\\ &=N([Ne_1,Ne_2,e_3]+[Ne_1,e_2,Ne_3]+[e_1,Ne_2,Ne_3]\\ &\quad -N([Ne_1,e_2,e_3]+[e_1,Ne_2,e_3]+[e_1,e_2,Ne_3])+N^2[e_1,e_2,e_3]).\end{aligned}$$

Thus, N is a Nijenhuis operator on $(L,[\cdot,\cdot,\cdot],\alpha)$.

Example 3.3. Let $(L,[\cdot,\cdot,\cdot])$ be a 3-Lie algebra and $\alpha: L\to L$ a 3-Lie algebra morphism. If N is a Nijenhuis operator on the 3-Lie algebra L commuting with α, then N is also a Nijenhuis operator on the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot]_\alpha,\alpha)$ in Example 2.2.

Example 3.4. A complex 3-Hom-Lie algebra is a 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$ together with a linear map $J: L\to L$ commuting with α that satisfies $J^2=-\mathrm{id}$ and

$$J[x,y,z]=-[Jx,Jy,Jz]+[Jx,y,z]+[x,Jy,z]+[x,y,Jz]$$

$$+ J[Jx, Jy, z] + J[x, Jy, Jz] + J[Jx, y, Jz].$$

It follows from $J^2 = -\mathrm{id}$ and the last identity that J is a Nijenhuis operator.

Definition. A linear map $T : L \to L$ is called a modified Rota-Baxter operator of weight $\lambda \in \mathbb{K}$ on a 3-Hom-Lie algebra $(L, [\cdot, \cdot, \cdot], \alpha)$ if T satisfies the following conditions

$$\begin{aligned} T \circ \alpha &= \alpha \circ T, \\ [Tx, Ty, Tz] &= T([Tx, Ty, z] + [x, Ty, Tz] + [Tx, y, Tz] - \lambda[x, y, z]) \\ &\quad + \lambda[Tx, y, z] + \lambda[x, Ty, z] + \lambda[x, y, Tz]. \end{aligned}$$

Definition. A linear map $T : L \to L$ is called a Rota-Baxter operator of weight λ on a 3-Hom-Lie algebra $(L, [\cdot, \cdot, \cdot], \alpha)$ if T satisfies the following conditions

$$\begin{aligned} T \circ \alpha &= \alpha \circ T, \\ [Tx, Ty, Tz] &= T([Tx, Ty, z] + [x, Ty, Tz] + [Tx, y, Tz] \\ &\quad + \lambda[Tx, y, z] + \lambda[x, Ty, z] + \lambda[x, y, Tz] + \lambda^2[x, y, z]). \end{aligned}$$

Proposition 3.5. *Let $(L, [\cdot, \cdot, \cdot], \alpha)$ be a 3-Hom-Lie algebra and $N : L \to L$ be a linear operator. Then*

(a) *If $N^2 = 0$, then N is a Nijenhuis operator if and only if N is a Rota-Baxter operator.*

(b) *If $N^2 = N$, then N is a Nijenhuis operator if and only if N is a Rota-Baxter operator of weight -1.*

(c) *If $N^2 = \pm\mathrm{id}$, then N is a Nijenhuis operator if and only if N is a modified Rota-Baxter operator of weight ∓ 1.*

Proof. (a) Let $N^2 = 0$. Suppose N is a Nijenhuis operator. Then for any $x, y, z \in L$, we have

$$\begin{aligned} [Nx, Ny, Nz] &= N\big([Nx, Ny, z] + [x, Ny, Nz] + [Nx, y, Nz] \\ &\qquad - N([Nx, y, z] + [x, Ny, z] + [x, y, Nz] - N[x, y, z])\big) \\ &= N([Nx, Ny, z] + [x, Ny, Nz] + [Nx, y, Nz]). \end{aligned}$$

Hence, N is a Rota-Baxter operator. Proof of the converse part is similar.

(b) Let $N^2 = N$. Suppose N is a Nijenhuis operator. Then for any $x, y, z \in L$, we have

$$\begin{aligned} &[Nx, Ny, Nz] \\ &= N\big([Nx, Ny, z] + [x, Ny, Nz] + [Nx, y, Nz] \\ &\quad - N([Nx, y, z] + [x, Ny, z] + [x, y, Nz] - N[x, y, z])\big) \\ &= N([Nx, Ny, z] + [x, Ny, Nz] + [Nx, y, Nz] - [Nx, y, z] - [x, Ny, z] \\ &\quad - [x, y, Nz] + [x, y, z]). \end{aligned}$$

Hence, N is a Rota-Baxter operator of weight -1. Similarly, the converse can be shown.

(c) Let $N^2 = \mathrm{id}$. Suppose N is a Nijenhuis operator. Then, for any $x, y, z \in L$, we have

$$\begin{aligned}
&[Nx, Ny, Nz] \\
&= N\big([Nx, Ny, z] + [x, Ny, Nz] + [Nx, y, Nz] \\
&\quad - N([Nx, y, z] + [x, Ny, z] + [x, y, Nz] - N[x, y, z])\big) \\
&= N([Nx, Ny, z] + [x, Ny, Nz] + [Nx, y, Nz] + [x, y, z]) - [Nx, y, z] \\
&\quad - [x, Ny, z] - [x, y, Nz].
\end{aligned}$$

Hence, N is a modified Rota-Baxter operator of weight -1. In a similar way the other cases can be shown. □

We recall the notion of relative Rota-Baxter operators on a 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$ from [8].

Definition. Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom-Lie algebra and (V, A, ρ) be a representation. A linear operator $T : V \to L$ is called a relative Rota-Baxter operator associated to (V, A, ρ) if T satisfies: for any $u, v, w \in V$,

$$\begin{aligned}
&\alpha \circ T = T \circ A, \\
&[Tu, Tv, Tw] = T(\rho(Tu, Tv)w + \rho(Tv, Tw)u + \rho(Tw, Tu)v).
\end{aligned}$$

Definition. Two relative Rota-Baxter operators $T_1, T_2 : V \to L$ are said to be compatible if for all $\mu, \nu \in \mathbb{K}$, the sum $\mu T_1 + \nu T_2 : V \to L$ is a relative Rota-Baxter operator.

The following result is straightforward.

Lemma 3.6. *Two relative Rota-Baxter operators T_1 and T_2 are compatible if and only if the following condition holds*:

$$\begin{aligned}
&[T_i(u), T_i(v), T_j(w)] + [T_i(u), T_j(v), T_i(w)] + [T_j(u), T_i(v), T_i(w)] \\
&= T_i(\rho(T_iu, T_jv)w + \rho(T_ju, T_iv)w + \rho(T_iv, T_jw)u \\
&\quad + \rho(T_jv, T_iw)u + \rho(T_iw, T_ju)v + \rho(T_jw, T_iu)v) \\
&\quad + T_j(\rho(T_iu, T_iv)w + \rho(T_iv, T_iw)u + \rho(T_iw, T_iu)v), \ (i, j = 1, 2).
\end{aligned}$$

The following proposition shows that a pair of compatible relative Rota-Baxter operators with an invertible condition are connected by a Nijenhuis operator.

Proposition 3.7. *Let T_1 and T_2 be two compatible relative Rota-Baxter operators in which T_2 (resp. T_1) is invertible. Then $N = T_1 \circ T_2^{-1}$ (resp. $N = T_2 \circ T_1^{-1}$) is a Nijenhuis operator on L.*

Proof. We only prove the case in which T_2 is invertible. The other case is similar. First observe that, for $N = T_1 \circ T_2^{-1}$, we have

$$\alpha \circ N = \alpha \circ T_1 \circ T_2^{-1} = T_1 \circ \alpha \circ T_2^{-1} = T_1 \circ T_2^{-1} \circ \alpha = N \circ \alpha.$$

Further, for any $x, y, z \in L$, there exists $u, v, w \in V$ such that $T_2(u) = x, T_2(v) = y, T_2(w) = z$. Therefore, we have

$$\begin{aligned}[Nx, Ny, Nz] &= [NT_2(u), NT_2(v), NT_2(w)] \\ &= [T_1(u), T_1(v), T_1(w)] \\ &= T_1(\rho(T_1u, T_1v)w + \rho(T_1v, T_1w)u + \rho(T_1w, T_1u)v). \end{aligned} \tag{3.3}$$

On the other hand, we have

$$\begin{aligned} & N([Nx, Ny, z] + [x, Ny, Nz] + [Nx, y, Nz]) \\ & - N^2([Nx, y, z] + [x, Ny, z] + [x, y, Nz]) + N^3[x, y, z] \\ = {} & T_1 \circ T_2^{-1}([T_1(u), T_1(v), T_2(w)] + [T_2(u), T_1(v), T_1(w)] \\ & + [T_1(u), T_2(v), T_1(w)]) - (T_1 \circ T_2^{-1})^2([T_1(u), T_2(v), T_2(w)] \\ & + [T_2(u), T_1(v), T_2(w)] + [T_2(u), T_2(v), T_1(w)]) \\ & + (T_1 \circ T_2^{-1})^3([T_2(u), T_2(v), T_2(w)]) \\ = {} & T_1 \circ T_2^{-1}(T_1(\rho(T_1u, T_2v)w + \rho(T_2u, T_1v)w + \rho(T_1v, T_2w)u \\ & + \rho(T_2v, T_1w)u + \rho(T_1w, T_2u)v + \rho(T_2w, T_1u)v) + T_2(\rho(T_1u, T_1v)w \\ & + \rho(T_1v, T_1w)u + \rho(T_1w, T_1u)v)) - (T_1 \circ T_2^{-1})^2(T_2(\rho(T_1u, T_2v)w \\ & + \rho(T_2u, T_1v)w + \rho(T_1v, T_2w)u + \rho(T_2v, T_1w)u + \rho(T_1w, T_2u)v \\ & + \rho(T_2w, T_1u)v) + T_1(\rho(T_2u, T_2v)w + \rho(T_2v, T_2w)u \\ & + \rho(T_2w, T_2u)v)) + (T_1 \circ T_2^{-1})^3(T_2(\rho(T_2u, T_2v)w + \rho(T_2v, T_2w)u \\ & + \rho(T_2w, T_2u)v)) \\ = {} & T_1(\rho(T_1u, T_1v)w + \rho(T_1v, T_1w)u + \rho(T_1w, T_1u)v). \end{aligned} \tag{3.4}$$

It follows from (3.3) and (3.4) that N is a Nijenhuis operator. □

4. Constructions of Nijenhuis operators

In this section, according to constructions of 3-Hom-Lie algebras, we give various constructions of Nijenhuis operators. We also give some examples of Nijenhuis operators on 3-Hom-Lie algebras.

4.1. Constructions of Nijenhuis operators on 3-Hom-Lie algebras from those on Hom-Lie algebras

Lemma 4.1 ([2]). *Let $(L, [\cdot,\cdot], \alpha)$ be a Hom-Lie algebra and L^* be the dual space of L. Suppose that $f \in L^*$ satisfies $f([x, y]) = 0$ and $f(\alpha(x))f(y) = f(\alpha(y))f(x)$ for all $x, y \in L$. Then there is a 3-Hom-Lie algebra structure on L given by*

$$[x, y, z]_f = f(x)[y, z] + f(y)[z, x] + f(z)[x, y], \qquad \forall x, y, z \in L. \tag{4.1}$$

Theorem 4.2. *Let N be a Nijenhuis operator on a Hom-Lie algebra $(L, [\cdot,\cdot], \alpha)$. Suppose that $f \in L^*$ satisfies $f([x, y]) = 0$ and $f(\alpha(x))f(y) = f(\alpha(y))f(x)$ for* *all $x, y \in L$. Then N is also a Nijenhuis operator on the 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot]_f, \alpha)$.*

Proof. By Lemma 4.1, $(L, \{\cdot,\cdot,\cdot\}_{f,D}, \alpha)$ is a 3-Hom-Lie algebra with the multiplication $[\cdot,\cdot,\cdot]_f$ defined in (4.1). Now for any $x, y, z \in L$, we have $\alpha \circ N = N \circ \alpha$ and

$$\begin{aligned}
&[Nx, Ny, Nz]_f \\
&= f(Nx)[Ny, Nz] + f(Ny)[Nz, Nx] + f(Nz)[Nx, Ny] \\
&= f(Nx)([Ny, z] + [y, Nz] - N[y, z]) + f(Ny)([Nz, x] + [z, Nx] - N[z, x]) \\
&\quad + f(Nz)([Nx, y] + [x, Ny] - N[x, y]).
\end{aligned}$$

Applying (3.2) and (4.1), we obtain

$$\begin{aligned}
&N\Big([Nx, Ny, z]_f + [x, Ny, Nz]_f + [Nx, y, Nz]_f \\
&\quad - N[Nx, y, z]_f - N[x, Ny, z]_f - N[x, y, Nz]_f + N^2[x, y, z]_f\Big) \\
&= N\Big(f(Nx)[Ny, z] + f(Ny)[z, Nx] + f(z)[Nx, Ny] + f(x)[Ny, Nz] \\
&\quad + f(Ny)[Nz, x] + f(Nz)[x, Ny] + f(Nx)[y, Nz] + f(y)[Nz, Nx] \\
&\quad + f(Nz)[Nx, y] - N(f(Nx)[y, z] + f(y)[z, Nx] + f(z)[Nx, y] \\
&\quad + f(x)[Ny, z] + f(Ny)[z, x] + f(z)[x, Ny] + f(x)[y, Nz] \\
&\quad + f(y)[Nz, x] + f(Nz)[x, y]) + N^2(f(x)[y, z] + f(y)[z, x] + f(z)[x, y])\Big) \\
&= [Nx, Ny, Nz]_f + N(f(x)[Ny, Nz] + f(y)[Nz, Nx] + f(z)[Nx, Ny]) \\
&\quad - Nf(x)[Ny, Nz] - Nf(y)[Nz, Nx] - Nf(z)[Nx, Ny]) \\
&= [Nx, Ny, Nz]_f.
\end{aligned}$$

And the proof is finished. □

By Lemma 4.1 and Theorem 4.2, we have

Example 4.3. Let $(L, [\cdot,\cdot])$ be a Lie algebra and $\alpha : L \to L$ a Lie algebra morphism, N be a Nijenhuis operator on a Lie algebra $(L, [\cdot,\cdot])$ with $\alpha \circ N = N \circ \alpha$. Suppose that $f \in L^*$ satisfies $f([x, y]_\alpha) = 0$ and $f(\alpha(x))f(y) = f(\alpha(y))f(x)$ for all $x, y \in L$. Then N is also a Nijenhuis operator on the 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot]_f, \alpha)$, where $[\cdot,\cdot]_\alpha = \alpha \circ [\cdot,\cdot]$ and

$$[x, y, z]_f = f(x)[y, z]_\alpha + f(y)[z, x]_\alpha + f(z)[x, y]_\alpha, \quad \forall x, y, z \in L.$$

Example 4.4. Let $(L, [\cdot,\cdot], \alpha)$ be the 3-dimensional Hom-Lie algebra given by

$$[e_1, e_2] = e_2, \quad \alpha(e_1) = e_1, \quad \alpha(e_2) = -e_2, \quad \alpha(e_3) = e_3,$$

where $\{e_1, e_2, e_3\}$ is a basis of L. By Lemma 4.1, the trace function $f \in L^*$, where $\begin{cases} f(e_1) = 1, \\ f(e_2) = 0, \\ f(e_3) = 1 \end{cases}$ induces a 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot]_f, \alpha)$ defined with the same basis by

$$[e_1, e_2, e_3] = e_2.$$

Consider a linear map $N : L \to L$ defined by $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ with respect to the basis $\{e_1, e_2, e_3\}$. Define

$$\begin{aligned} Ne_1 &= a_{11}e_1 + a_{21}e_2 + a_{31}e_3, \\ Ne_2 &= a_{12}e_1 + a_{22}e_2 + a_{32}e_3, \\ Ne_3 &= a_{13}e_1 + a_{23}e_2 + a_{33}e_3. \end{aligned}$$

In order to satisfy $\alpha \circ N = N \circ \alpha$, we need

$$a_{21} = a_{12} = a_{32} = a_{23} = 0.$$

In order to obtain that N is a Nijenhuis operator on the Hom-Lie algebra L, we need

$$[Ne_i, Ne_j] = N([Ne_i, e_j] + [e_i, Ne_j] - N[e_i, e_j]), \quad i, j = 1, 2, 3.$$

By a straightforward computation, we conclude that N is a Nijenhuis operator on the Hom-Lie algebra L if and only if

$$a_{21} = a_{12} = a_{32} = a_{23} = 0.$$

By Theorem 4.2, $N = \begin{pmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & 0 \\ a_{31} & 0 & a_{33} \end{pmatrix}$ is also a Nijenhuis operator on the 3-Hom-Lie algebra $(L, [\cdot, \cdot, \cdot]_f, \alpha)$.

4.2. Constructions of Nijenhuis operators on 3-Hom-Lie algebras from Nijenhuis operators on commutative Hom-associative algebras

Lemma 4.5 ([24]). *Let $(L, \cdot, \alpha)$ be a commutative Hom-associative algebra and D be a derivation and $f \in L^*$ satisfies $f(D(x) \cdot y) = f(x \cdot D(y))$ and $f(\alpha(x))f(y) = f(\alpha(y))f(x)$ for all $x, y \in L$. Then $(L, \{\cdot, \cdot, \cdot\}_{f,D}, \alpha)$ is a 3-Hom-Lie algebra, where the bracket is given*

$$\{x, y, z\}_{f,D} = \begin{vmatrix} f(x) & f(y) & f(z) \\ D(x) & D(y) & D(z) \\ x & y & z \end{vmatrix} \tag{4.2}$$

for any $x, y, z \in L$.

Proposition 4.6. *With the same assumptions as Lemma 4.5. Let N be a Nijenhuis operator on $(L, \cdot, \alpha)$ satisfying $D \circ N = N \circ D$. Then N is a Nijenhuis operator on the 3-Lie algebra $(L, \{\cdot, \cdot, \cdot\}_{f,D}, \alpha)$, where the bracket is given by* (4.2).

Proof. For all $x, y \in L$, define $[x,y]_D = D(x)\cdot y - D(y)\cdot x$. By direct calculations, we can verify that $(L, [\cdot,\cdot]_D, \alpha)$ is a Hom-Lie algebra. Furthermore, assume that N is a Nijenhuis operator on $(L, \cdot, \alpha)$ satisfying $D \circ N = N \circ D$. Then we have $\alpha \circ N = N \circ \alpha$ and

$$\begin{aligned}
&[Nx, Ny]_D \\
&= D(Nx)\cdot Ny - D(Ny)\cdot Nx \\
&= ND(x)\cdot Ny - Nx\cdot ND(y) \\
&= N(Dx\cdot Ny + NDx\cdot y - N(Dx\cdot y)) - N(Nx\cdot Dy + x\cdot NDy - N(x\cdot Dy)) \\
&= N([Nx, y]_D + [x, Ny]_D - N[x,y]_D),
\end{aligned}$$

which implies that N is a Nijenhuis operator on the Hom-Lie algebra $(L, [\cdot,\cdot]_D, \alpha)$. By Theorem 4.2, N is a Nijenhuis operator on the 3-Hom-Lie algebra $(L, \{\cdot,\cdot,\cdot\}_{f,D}, \alpha)$. □

By Lemma 4.5 and Proposition 4.6, we have

Example 4.7. Let $(L, \cdot)$ be a commutative algebra and $\alpha : L \to L$ an algebra morphism, D be a derivation on a commutative algebra $(L, \cdot)$ with $\alpha \circ D = D \circ \alpha$ and N be a Nijenhuis operator on a commutative algebra $(L, \cdot)$ with $\alpha \circ N = N \circ \alpha$. Suppose that $f \in L^*$ satisfies $f(D(x)\cdot_\alpha y) = f(x \cdot_\alpha D(y))$ and $f(\alpha(x))f(y) = f(\alpha(y))f(x)$ for all $x, y \in L$. Then N is also a Nijenhuis operator on the 3-Hom-Lie algebra $(L, \{\cdot,\cdot,\cdot\}_{f,D}, \alpha)$, where $\cdot_\alpha = \alpha \circ \cdot$ and

$$\begin{aligned}
[x,y]_{\alpha,D} &= D(x)\cdot_\alpha y - D(y)\cdot_\alpha x, \\
\{x,y,z\}_{f,D} &= f(x)[y,z]_{\alpha,D} + f(y)[z,x]_{\alpha,D} + f(z)[x,y]_{\alpha,D}, \qquad \forall x,y,z \in L.
\end{aligned}$$

Let $(L, \cdot, \alpha)$ be a commutative Hom-associative algebra. For $x_i, y_i, z_i \in L, i = 1,2,3$, denote by

$$|\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}| = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix},$$

where $\overrightarrow{x}, \overrightarrow{y}$ and $\overrightarrow{z}$ denote the column vectors.

The following lemma is easy to check and we omit the proof.

Lemma 4.8. *Let N be a Nijenhuis operator on a commutative Hom-associative algebra $(L, \cdot, \alpha)$ and $N(\overrightarrow{x}), N(\overrightarrow{y}), N(\overrightarrow{z})$ denote the images of the column vectors. Then we have*

$$\begin{aligned}
&|N(\overrightarrow{x}), N(\overrightarrow{y}), N(\overrightarrow{z})| \\
&= N(|N(\overrightarrow{x}), N(\overrightarrow{y}), \overrightarrow{z}| + c.p.) - N^2(|N(\overrightarrow{x}), N(\overrightarrow{y}), \overrightarrow{z}| + c.p.) \\
&\quad + N^3(|\overrightarrow{x}, \overrightarrow{y}, \overrightarrow{z}|).
\end{aligned}$$

Definition ([16])**.** Let $(L, \cdot, \alpha)$ be a commutative Hom-associative algebra. If a linear map $\omega : L \to L$ satisfying for every $a, b \in L$,

$$\omega(a\cdot b) = \omega(a)\cdot\omega(b), \text{ and } \omega^2(a) = a,$$

then ω is called an involution of L.

Lemma 4.9 ([16]). *Let $(L, \cdot, \alpha)$ be a commutative Hom-associative algebra, D be a derivation of L and $\omega : L \to L$ be an involution of L satisfying*

$$\omega \circ D + D \circ \omega = 0, \text{ and } \alpha \circ \omega = \omega \circ \alpha.$$

Then $(L, [\cdot, \cdot, \cdot]_{\omega,D}, \alpha)$ is a 3-Hom-Lie algebra, where the bracket is given

$$[x, y, z]_{\omega,D} = \begin{vmatrix} \omega(x) & \omega(y) & \omega(z) \\ x & y & z \\ D(x) & D(y) & D(z) \end{vmatrix}, \tag{4.3}$$

for any $x, y, z \in L$.

Theorem 4.10. *With the same assumptions as Lemma 4.10. Let N be a Nijenhuis operator on a commutative Hom-associative algebra $(L, \cdot, \alpha)$ satisfying $N \circ D = D \circ N$ and $N \circ \omega = \omega \circ N$. Then N is a Nijenhuis operator on the 3-Hom-Lie algebra $(L, [\cdot, \cdot, \cdot]_{\omega,D}, \alpha)$, where the bracket is given by (4.3).*

Proof. For any $x, y, z \in L$, we have $\alpha \circ N = N \circ \alpha$ and

$$\begin{aligned}
&[Nx, Ny, Nz]_{\omega,D} \\
&= \begin{vmatrix} \omega N(x) & \omega N(y) & \omega N(z) \\ Nx & Ny & Nz \\ DN(x) & DN(y) & DN(z) \end{vmatrix} \\
&= \begin{vmatrix} N\omega(x) & N\omega(y) & N\omega(z) \\ Nx & Ny & Nz \\ ND(x) & ND(y) & ND(z) \end{vmatrix} \\
&= N \left(\begin{vmatrix} N\omega(x) & N\omega(y) & \omega(z) \\ Nx & Ny & z \\ ND(x) & ND(y) & D(z) \end{vmatrix} + c.p. \right) \\
&\quad - N^2 \left(\begin{vmatrix} N\omega(x) & \omega(y) & \omega(z) \\ Nx & y & z \\ ND(x) & D(y) & D(z) \end{vmatrix} + c.p. \right) \\
&\quad + N^3 \left(\begin{vmatrix} \omega(x) & \omega(y) & \omega(z) \\ x & y & z \\ D(x) & ND(y) & D(z) \end{vmatrix} \right) \\
&= N \left(\begin{vmatrix} \omega N(x) & \omega N(y) & \omega(z) \\ Nx & Ny & z \\ DN(x) & DN(y) & D(z) \end{vmatrix} + c.p. \right) \\
&\quad - N^2 \left(\begin{vmatrix} \omega N(x) & \omega(y) & \omega(z) \\ Nx & y & z \\ DN(x) & D(y) & D(z) \end{vmatrix} + c.p. \right) + N^3 \left(\begin{vmatrix} \omega(x) & \omega(y) & \omega(z) \\ x & y & z \\ D(x) & D(y) & D(z) \end{vmatrix} \right) \\
&= N \left([Nx, Ny, z]_{\omega,D} + c.p. \right) - N^2 \left([Nx, y, z]_{\omega,D} + c.p. \right) + N^3 \left([x, y, z]_{\omega,D} \right).
\end{aligned}$$

Thus, N is a Nijenhuis operator on the 3-Hom-Lie algebra $(L, [\cdot, \cdot, \cdot]_{\omega,D}, \alpha)$. $\square$

5. Cohomology of Nijenhuis operators

In this section, we construct a representation of the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot]_N,\alpha)$ on the vector space L, and define the cohomology of Nijenhuis operators on 3-Hom-Lie algebras.

Lemma 5.1. *Let $N:L\to L$ be a Nijenhuis operator on a 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot],\alpha)$. Define $\rho_N:L\wedge L\to\mathfrak{gl}(L)$ by*

$$\begin{aligned}\rho_N(x_1,x_2)(x)=[N(x_1),N(x_2),x]-N\big([N(x_1),x_2,x]\\+[x_1,N(x_2),x]-N[x_1,x_2,x]\big),\end{aligned}$$

where $x_1,x_2,x\in L$. Then (L,α,ρ_N) is a representation of the 3-Hom-Lie algebra $(L,[\cdot,\cdot,\cdot]_N,\alpha)$.

Proof. For any $x_1,x_2,x_3,x_4,x\in L$, it is easy to check that $\rho_N(\alpha(x_1),\alpha(x_2))\circ\alpha=\alpha\circ\rho_N(x_1,x_2)$ holds. Further, we have

$$\begin{aligned}
&\rho_N(\alpha(x_2),\alpha(x_3))\rho_N(x_1,x_4)(x)+\rho_N(\alpha(x_3),\alpha(x_1))\rho_N(x_2,x_4)(x)\\
&\quad+\rho_N(\alpha(x_1),\alpha(x_2))\rho_N(x_3,x_4)(x)\\
&=\rho_N(\alpha(x_2),\alpha(x_3))([N(x_1),N(x_4),x]\\
&\quad-N\big([N(x_1),x_4,x]+[x_1,N(x_4),x]-N[x_1,x_4,x]\big))\\
&\quad+\rho_N(\alpha(x_3),\alpha(x_1))([N(x_2),N(x_4),x]\\
&\quad-N\big([N(x_2),x_4,x]+[x_2,N(x_4),x]-N[x_2,x_4,x]\big))\\
&\quad+\rho_N(\alpha(x_1),\alpha(x_2))([N(x_3),N(x_4),x]\\
&\quad-N\big([N(x_3),x_4,x]+[x_3,N(x_4),x]-N[x_3,x_4,x]\big))\\
&=[N(x_2),N(x_3),[N(x_1),N(x_4),x]]\\
&\quad-N\big([N(x_2),x_3,[N(x_1),N(x_4),x]]+[x_2,N(x_3),[N(x_1),N(x_4),x]]\\
&\quad-N[x_2,x_3,[N(x_1),N(x_4),x]]\big)-[N(x_2),N(x_3),N\big([N(x_1),x_4,x]\\
&\quad+[x_1,N(x_4),x]-N[x_1,x_4,x]\big)]+N\big([N(x_2),x_3,N\big([N(x_1),x_4,x]\\
&\quad+[x_1,N(x_4),x]-N[x_1,x_4,x]\big)]+[x_2,N(x_3),N\big([N(x_1),x_4,x]+[x_1,N(x_4),x]\\
&\quad-N[x_1,x_4,x]\big)]-N[x_2,x_3,N\big([N(x_1),x_4,x]+[x_1,N(x_4),x]-N[x_1,x_4,x]\big)]\big)\\
&\quad+[N(x_3),N(x_1),[N(x_2),N(x_4),x]]-N\big([N(x_3),x_1,[N(x_2),N(x_4),x]]\\
&\quad+[x_3,N(x_1),[N(x_2),N(x_4),x]]-N[x_3,x_1,[N(x_2),N(x_4),x]]\big)\\
&\quad-[N(x_3),N(x_1),N\big([N(x_2),x_4,x]+[x_2,N(x_4),x]-N[x_2,x_4,x]\big)]\\
&\quad+N\big([N(x_3),x_1,N\big([N(x_2),x_4,x]+[x_2,N(x_4),x]-N[x_2,x_4,x]\big)]\\
&\quad+[x_3,N(x_1),N\big([N(x_2),x_4,x]+[x_2,N(x_4),x]-N[x_2,x_4,x]\big)]\\
&\quad-N[x_3,x_1,N\big([N(x_2),x_4,x]+[x_2,N(x_4),x]-N[x_2,x_4,x]\big)]\big)\\
&\quad+[N(x_1),N(x_2),[N(x_3),N(x_4),x]]-N\big([N(x_1),x_2,[N(x_3),N(x_4),x]]\\
&\quad+[x_1,N(x_2),[N(x_3),N(x_4),x]]-N[x_1,x_2,[N(x_3),N(x_4),x]]\big)\\
&\quad-[N(x_1),N(x_2),N\big([N(x_3),x_4,x]+[x_3,N(x_4),x]-N[x_3,x_4,x]\big)]
\end{aligned}$$

$$
\begin{aligned}
&+ N\big([N(x_1), x_2, N\big([N(x_3), x_4, x] + [x_3, N(x_4), x] - N[x_3, x_4, x]\big)]\\
&+ [x_1, N(x_2), N\big([N(x_3), x_4, x] + [x_3, N(x_4), x] - N[x_3, x_4, x]\big)]\\
&- N[x_1, x_2, N\big([N(x_3), x_4, x] + [x_3, N(x_4), x] - N[x_3, x_4, x]\big)]\big)\\
= {}& [[Nx_1, Nx_2, Nx_3], N\alpha(x_4), \alpha(x)] - N\big([[Nx_1, Nx_2, Nx_3], \alpha(x_4), \alpha(x)]\\
&+ [[x_1, x_2, x_3]_N, N\alpha(x_4), \alpha(x)] - N[[x_1, x_2, x_3]_N, \alpha(x_4), \alpha(x)]\big)\\
= {}& \rho_N([x_1, x_2, x_3]_N, \alpha(x_4))\alpha(x).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
&\rho_N(\alpha(x_1), \alpha(x_2))\rho_N(x_3, x_4)x\\
= {}& \rho_N(\alpha(x_3), \alpha(x_4))\rho_N(x_1, x_2)x + \rho_N([x_1, x_2, x_3]_N, \alpha(u))\alpha(x)\\
&+ \rho_N(\alpha(x_3), [x_1, x_2, x_4]_N)\alpha(x).
\end{aligned}
$$

And the proof is finished. □

Let $\delta_N : C^n_{\alpha,\alpha}(L, L) \to C^{n+1}_{\alpha,\alpha}(L, L)$ be the corresponding coboundary operator of the 3-Hom-Lie algebra $(L, [\cdot, \cdot, \cdot]_N, \alpha)$ with coefficients in the representation (L, α, ρ_N). More precisely, $\delta_N : C^n_{\alpha,\alpha}(L, L) \to C^{n+1}_{\alpha,\alpha}(L, L)$ is given by

$$
\begin{aligned}
&(\delta_N f)(\mathfrak{X}_1, \mathfrak{X}_2, \dots, \mathfrak{X}_n, x_{n+1})\\
= {}& \sum_{1\le j<k\le n} (-1)^j f(\alpha(\mathfrak{X}_1), \dots, \hat{\mathfrak{X}}_j, \dots, \alpha(\mathfrak{X}_{k-1}), [x_j, y_j, x_k]_N \wedge \alpha(y_k)\\
&+ \alpha(x_k) \wedge [x_j, y_j, y_k]_N, \alpha(\mathfrak{X}_{k+1}), \dots, \alpha(\mathfrak{X}_n), \alpha(x_{n+1}))\\
&+ \sum_{j=1}^{n} (-1)^j f(\alpha(\mathfrak{X}_1), \dots, \hat{\mathfrak{X}}_j, \dots, \alpha(\mathfrak{X}_n), [x_j, y_j, x_{n+1}]_N)\\
&+ \sum_{j=1}^{n} (-1)^{j+1} \rho_N(\alpha^{n-1}(\mathfrak{X}_j)) f(\mathfrak{X}_1, \dots, \hat{\mathfrak{X}}_j, \dots, \mathfrak{X}_n, x_{n+1})\\
&+ (-1)^{n+1}\big(\rho_N(\alpha^{n-1}(y_n), \alpha^{n-1}(x_{n+1})) f(\mathfrak{X}_1, \dots, \hat{\mathfrak{X}}_j, \dots, \mathfrak{X}_{n-1}, x_n)\\
&+ \rho_N(\alpha^{n-1}(x_{n+1}), \alpha^{n-1}(x_n)) f(\mathfrak{X}_1, \dots, \hat{\mathfrak{X}}_j, \dots, \mathfrak{X}_{n-1}, y_n)\big)
\end{aligned}
$$

for $\mathfrak{X}_i = x_i \wedge y_i \in \wedge^2 L$, $i = 1, \dots, n$ and $x_{n+1} \in L$.

For any $\mathfrak{X} = x \wedge y \in \wedge^2 L$ with $\alpha(x) = x, \alpha(y) = y$, we define $d_N(\mathfrak{X}) : L \to L$ by

$$
d_N(\mathfrak{X})(z) = N[\mathfrak{X}, z] - [\mathfrak{X}, Nz], \quad \forall z \in L.
$$

Proposition 5.2. *Let N be a Nijenhuis operator on a 3-Hom-Lie algebra $(L, [\cdot, \cdot, \cdot], \alpha)$. Then $d_N(\mathfrak{X})$ is a 1-cocycle on the 3-Hom-Lie algebra $(L, [\cdot, \cdot, \cdot]_N, \alpha)$ with coefficients in (L, α, ρ_N).*

Proof. By direct calculation we can get the conclusion. □

Theorem 5.3. *Let N be a Nijenhuis operator on a 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$ with respect to a representation (L, ρ_N, α). Define the set of n-cochains by*

$$C^n_N(L,L) = \begin{cases} C^n_{\alpha,\alpha}(L,L), & n \geq 2, \\ L \wedge L, & n = 1. \end{cases}$$

Define $D_N : C^n_N(L,L) \to C^{n+1}_N(L,L)$ by

$$D_T = \begin{cases} \delta_N, & n \geq 2, \\ d_N, & n = 1. \end{cases}$$

We denote by

$$Z^n_N(L,L) = \{f \in C^n_N(L,L) \mid D_N f = 0\}$$

and

$$B^n_N(L,L) = \{D_N g \mid g \in C^{n-1}_N(L,L)\}$$

the space of n-cocycles and n-coboundaries, respectively. The corresponding quotients

$$H^n_N(L,L) := Z^n_N(L,L)/B^n_N(L,L), \text{ for } n \geq 1$$

are called the n-th cohomology group, and the cohomology of the cochain complex $(\oplus_{n=1}^{\infty} C^n_N(L,L), D_N)$ is taken to be the cohomology for the Nijenhuis operator N.

6. Formal deformations of Nijenhuis operators

In this section, we study formal deformations of Nijenhuis operators on 3-Hom-Lie algebras.

Let $(L, [\cdot,\cdot,\cdot], \alpha)$ be a 3-Hom-Lie algebra and $N : L \to L$ be a Nijenhuis operator on it. Consider the space $L[[t]]$ of formal power series in t with coefficients from L. Then $L[[t]]$ is a $\mathbb{K}[[t]]$-module. Moreover the trilinear bracket $[\cdot,\cdot,\cdot]$ and the linear map α can be extended to $L[[t]]$ by $\mathbb{K}[[t]]$-linearity. We denote extended structures by the same notation. With these extensions, $(L[[t]], [\cdot,\cdot,\cdot], \alpha)$ is a 3-Hom-Lie algebra over $\mathbb{K}[[t]]$.

Definition. A formal one-parameter deformation of N consists of a formal sum

$$N_t = N_0 + tN_1 + t^2 N_2 + \cdots \in \mathrm{Hom}(L,L)[[t]]$$

with $N_0 = N$ such that the $\mathbb{K}[[t]]$-linear map $N_t : L[[t]] \to L[[t]]$ is a Nijenhuis operator on the 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$.

Thus it follows from the above definition that N_t satisfies: $\alpha \circ N_t = N_t \circ \alpha$ and

$$\begin{aligned} & [N_t x, N_t y, N_t z] \\ = {} & N_t([N_t x, N_t y, z] + [x, N_t y, N_t z] + [N_t x, y, N_t z] - N_t[N_t x, y, z] \end{aligned}$$

$$- N_t[x, N_t y, z] - N_t[x, y, N_t z] + N_t \circ N_t[x, y, z]),$$

for all $x, y, z \in L$. These conditions are equivalent to $\alpha \circ N_n = N_n \circ \alpha$ and

$$\begin{aligned}
&\sum_{i+j+k=n} [N_i x, N_j y, N_k z] \\
= &\sum_{i+j+k=n} N_i([N_j x, N_k y, z] + [x, N_j y, N_k z] + [N_j x, y, N_k z] \\
&- N_j[N_k x, y, z] - N_j[x, N_k y, z] - N_j[x, y, N_k z] + N_j \circ N_k[x, y, z]),
\end{aligned}$$

for $n = 0, 1, 2, \ldots$ and $x, y, z \in L$, the conditions hold automatically as $N_0 = N$ is a Nijenhuis operator on the 3-Hom-Lie algebra $(L, [\cdot,\cdot,\cdot], \alpha)$. For $n = 1$, we get $\alpha \circ N_n = N_n \circ \alpha$ and

$$\begin{aligned}
&[N_1(z), N(y), N(z)] + [N(x), N_1(y), N(z)] + [N(x), N(y), N_1(z)] \\
= &\, N\left([x, N_1(y), N(z)] + [x, N(y), N_1(z)]\right) + N\left([N_1(z), y, N(z)]\right. \\
&\left. +[N(z), y, N_1(z)]\right) + N\left([N_1(x), N(y), z] + [N(x), N_1(y), z]\right) \\
&+ N_1\left([x, N(y), N(z)] + [N(x), y, N(z)] + [N(x), N(y), z]\right) \\
&- N^2\left([N_1(x), y, z] + [x, N(y), z] + [x, y, N_1(z)]\right) \\
&- N \circ N_1\left([x, N(y), z] + [N(x), y, z] + [x, y, N(z)]\right) \\
&+ N^2 \circ N_1([x, y, z]).
\end{aligned} \tag{6.1}$$

Hence it follows from (6.1) that N_1 is a 1-cocycle in the cohomology of N. This is called the infinitesimal of the deformation.

Definition. Two deformations $N_t = \sum_{i=0}^{\infty} t^i N_i$ and $N'_t = \sum_{i=0}^{\infty} t^i N'_i$ of a Nijenhuis operator N are said to be equivalent if there is an element $\mathfrak{X} = x \wedge y \in \wedge^2 L$ with $\alpha(x) = x, \alpha(y) = y$, and linear maps $\phi_i \in \mathrm{Hom}(L, L)$ for $i \geq 2$ such that

$$\phi_t = (\mathrm{id} + t[\mathfrak{X}, \cdot] + \sum_{i\geq 2} t^i \phi_i) : L[[t]] \to L[[t]]$$

is a morphism of Nijenhuis operators from N_t to N'_t.

Let ϕ_t be a morphism of Nijenhuis operators from N_t to N'_t. For any $z \in L$, we have

$$(N + tN'_1)(\mathrm{id} + t[\mathfrak{X}, \cdot])(z) = (\mathrm{id} + t[\mathfrak{X}, \cdot])(N + tN_1)(z) \pmod{t^2}.$$

By equating coefficients of t, we get

$$N_1(z) - N'_1(z) = N[\mathfrak{X}, z] - [\mathfrak{X}, Nz] = d_N(\mathfrak{X})(z).$$

As a consequence, we get the following

Theorem 6.1. *Let $N_t = \sum_{i=0}^{\infty} t^i N_i$ be a formal one-parameter deformation of N. Then the linear term N_1 is a 1-cocycle in the cohomology of N, whose cohomology class depends only on the equivalence class of the deformation.*

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