J. Korean Math. Soc. **62** (2025), No. 1, pp. 1–31
https://doi.org/10.4134/JKMS.j230346
pISSN: 0304-9914 / eISSN: 2234-3008

HARMONIC FUNCTIONS AND END NUMBERS ON SMOOTH METRIC MEASURE SPACES

Xuenan Fu and Jia-Yong Wu

Abstract. In this paper, we study properties of functions on smooth metric measure space $(M, g, e^{-f}dv)$. We prove that any simply connected, negatively curved smooth metric measure space with a small bound of $|\nabla f|$ admits a unique f-harmonic function for a given boundary value at infinity. We also prove a sharp L^2_f-decay estimate for a Schrödinger equation under certain positive spectrum.

As applications, we discuss the number of ends on smooth metric measure spaces. We show that the space with finite f-volume has a finite number of ends when the Bakry-Émery Ricci tensor and the bottom of Neumann spectrum satisfy some lower bounds. We also show that the number of ends with infinite f-volume is finite when the Bakry-Émery Ricci tensor is bounded below by certain positive spectrum. Finally we study the dimension of the first L^2_f-cohomology of the smooth metric measure space.

1. Introduction

A smooth metric measure space (for short, SMMS), denoted by $(M, g, e^{-f}dv)$, is a complete Riemann manifold (M, g) coupled with a weighted measure $e^{-f}dv$ for some weight function f and the Riemann volume element dv on (M, g). SMMSs are natural extensions of Riemann manifolds and are characterized by collapsed measured Gromov-Hausdorff limits [22]. On $(M, g, e^{-f}dv)$, the f-Laplacian, self-adjoint with respect to $e^{-f}dv$, is defined by

$$\Delta_f := \Delta - \langle \nabla f, \nabla \rangle,$$

and the weighted volume (or f-volume) of M (if it exists) is defined by $V_f(M) := \int_M e^{-f}dv$. A function u is called f-harmonic if $\Delta_f u = 0$, and f-subharmonic if $\Delta_f u \geq 0$. For $0 < p < \infty$, u is called L^p_f-integrable if the weighted L^p-norm (or L^p_f-norm) $(\int_M |u|^p e^{-f}dv)^{1/p}$ is finite. For $0 < m < \infty$, the m-dimensional

Received July 2, 2023; Accepted September 9, 2024.
2020 *Mathematics Subject Classification.* Primary 53C21, 58J32; Secondary 58J05, 58J90.
Key words and phrases. Smooth metric measure space, Bakry-Émery Ricci tensor, harmonic function, Dirichlet problem, end, spectrum, Sobolev inequality, cohomology.

Bakry-Émery Ricci tensor [3] is defined by

$$\mathrm{Ric}_f^m := \mathrm{Ric} + \mathrm{Hess}\, f - \tfrac{1}{m} df \otimes df,$$

where Ric is the Ricci tensor of (M, g) and Hess is the Hessian with respect to metric g. For $m = \infty$, we have the (∞-dimensional) Bakry-Émery Ricci tensor

$$\mathrm{Ric}_f := \mathrm{Ric} + \mathrm{Hess}\, f.$$

Clearly, $\mathrm{Ric}_f^m \geq c$ implies $\mathrm{Ric}_f \geq c$, but not vice versa. If a SMMS satisfies $\mathrm{Ric}_f^m = \lambda\, g$ for some $\lambda \in \mathbb{R}$, it is called an m-quasi-Einstein manifold. When $m = 1$, it is the so-called static manifold in general relativity. When $0 < m < \infty$, $(M^n \times F^m, g_M + e^{-2f/m} g_F)$ is a warped product Einstein manifold [5], where (F^m, g_F) is an Einstein manifold. In particular, if $\mathrm{Ric}_f = \lambda\, g$, it is a gradient Ricci soliton, which arises in the Ricci flow [11].

In this paper, we shall study function theoretic properties and the number of ends on SMMSs. First, we give a sufficient condition such that the SMMS admits f-harmonic functions. Then we give a sharp L^2_f-decay estimate for the Schrödinger operator. As applications, we study finitely many ends for various types of SMMSs. We also study the dimension estimate of the space of L^2_f-harmonic one-forms.

Recently, the study of relations among the curvature and function on SMMSs has been active. Brighton [4] proved that there is no non-constant bounded f-harmonic function when $\mathrm{Ric}_f \geq 0$. Munteanu-Wang [23] proved that any positive f-harmonic function of sublinear growth is constant. Pigola-Rigoli-Setti [26] showed that any nonnegative L^p_f-integrable ($p > 1$) f-subharmonic function is constant. In [35,36], many L^p_f-Liouville theorems are obtained under various types of SMMSs. Among these results, we see that many SMMSs do not admit non-constant f-harmonic function. So we raise a natural question.

Question. What types of SMMSs do they admit non-constant f-harmonic functions?

Anderson [1] and Sullivan [30] independently proved that any complete, simply connected, negatively curved manifold admits many bounded harmonic functions. Indeed, they showed that there is a geometric compactification of manifold M by adding an $(n-1)$-sphere $S_\infty(M)$ at infinity, such that the Dirichlet problem of the Laplacian equation on $M \cup S_\infty(M)$ can be solved for any given data on $S_\infty(M)$. Here we prove a weighted version of their result.

Theorem 1.1. *Let $(M, g, e^{-f}dv)$ be an n-dimensional simply connected SMMS with sectional curvature K_M satisfying*

$$-b^2 \leq K_M \leq -a^2 < 0,$$

where a and b are two positive constants. If $|\nabla f| \leq \alpha$ for some constant $\alpha < a/4$, then, for any $\varphi \in C(S_\infty(M))$, there exists a unique f-harmonic function $u \in C^\infty(M) \cap C(\overline{M})$ with $u = \varphi$ on $S_\infty(M)$, where $\overline{M} := M \cup S_\infty(M)$.

The proof of theorem uses the Rauch-Toponogov comparison, the cone topology of $M \cup S_\infty(M)$ and the classical Perron method; see Section 2. We remark that the assumption $\alpha < a/4$ is used in the construction of barrier functions, and we do not know if it can be removed or not.

In Section 3, we review some relevant facts needed for our further arguments. We first recall Qian's volume comparison [27]. Then we give an improved Bochner formula for f-harmonic functions. Finally we recall certain f-mean value inequalities on SMMSs.

In Section 4, we study L^2_f-decay estimates for a Schrödinger operator. The proof uses Hua-Lu's argument [12], which avoids tedious iteration in [18–20]. Recall that an end E of M is an unbounded component of $M\backslash\Omega$ for some compact set $\Omega \subset M$. Without loss of generality, we assume Ω is a geodesic ball $B(p,R)$ with center p and radius $R > 0$. In particular, the end E is a manifold with boundary. Let $\lambda_{1,f}(E)$ be the infimum of the L^2_f-spectrum of f-Laplacian on E with Dirichlet boundary conditions. We now state a special result of this Section 4, which is a weighted version of Hua-Lu's estimate [12].

Theorem 1.2. *Let $(M, g, e^{-f}dv)$ be a SMMS. Suppose E is an end of M with $\lambda_{1,f}(E) > \sigma$ for some constant σ. Let u be a nonnegative function in E satisfying $\Delta_f u \geq -\sigma u$. If there exists a sequence $R_i \to \infty$ such that*

$$\int_{E(R_i+1)\backslash E(R_i)} u^2 e^{-2ar} e^{-f} dv \to 0, \tag{1}$$

where $a = \sqrt{\lambda_{1,f}(E) - \sigma}$, then

$$\int_{E(\rho+1)\backslash E(\rho)} u^2 e^{-f} dv \leq \frac{a+1}{a^2} e^{-2a\rho} \int_{E(R_0+1)\backslash E(R_0)} e^{2ar} u^2 e^{-f} dv$$

for all $\rho > R_0 + 1$, where $E(R) = B(p,R) \cap E$.

Remark 1.3. The condition (1) of theorem is a little weaker than Dung-Khanh-Son's condition $\int_{E(R)} u^2 e^{-2ar} e^{-f} dv = o(R)$ in [8]. Indeed, as in [12], for $0 < s < \infty$ and sufficiently large R,

$$\int_{E(R)} u^2 e^{-2ar} e^{-f} dv \geq \sum_{i=2}^{[\frac{R}{s}]-1} \int_{E(si+1)\backslash E(si)} u^2 e^{-2ar} e^{-f} dv.$$

Hence if $\int_{E(R)} u^2 e^{-2ar} e^{-f} dv = o(R)$, then there exists a sequence $R_i \to \infty$, $i \to \infty$, such that $\int_{E(R_i+1)\backslash E(R_i)} u^2 e^{-2ar} e^{-f} dv \to 0$, $i \to \infty$. On the other hand, we will give a specific example (see Example 4.3) to demonstrate that our condition (1) is best possible.

In Section 5, we provide some sufficient conditions for an end of SMMS being f-parabolic or f-non-parabolic. Meanwhile we give a precise decay estimate for f-volumes of annuluses on SMMSs with positive spectrum.

In Section 6, we apply Theorem 1.2 to get an estimate for the number of ends of SMMS with finite f-volume when Ric_f^m is bounded below; see Theorem 6.1.

This result generalizes Li-Wang's manifold case [20] and Dung-Khanh-Son's weighted version [8].

In Section 7, we discuss the number of ends with infinite f-volume. We say that f satisfies first-order derivative property with constant $\alpha \geq 0$ for a point $p \in M$ if $|\langle \nabla f, \nabla u \rangle| \leq \alpha |\nabla u|$ for any f-harmonic function u on M, and $\partial_r f \geq -\alpha$ along all minimal geodesic segments r from p. In particular, if $|\nabla f| \leq \alpha$, then f obviously satisfies first-order derivative property with constant α for any point $x \in M$.

Theorem 1.4. *Let $(M, g, e^{-f}dv)$ be an n-dimensional SMMS. Suppose that there exists a geodesic ball $B(p, R_0) \subset M$ such that $\lambda_{1,f}(M \backslash B(p, R_0)) > 0$. On $M \backslash B(p, R_0)$, assume that f satisfies first-order derivative property with constant $\alpha \geq 0$ for the point p. If*

$$\mathrm{Ric}_f \geq -\frac{k-1}{k-2}\lambda_{1,f}(M \backslash B(p, R_0)) + \frac{\alpha^2}{k-n} + \epsilon$$

for some $\epsilon > 0$ and some $k \geq n + \frac{\alpha}{2}\lambda_{1,f}(M \backslash B(p, R_0))^{-1/2}$, then M has finitely many ends with infinite f-volume.

Our proof indicates that there exists a number C depending on n, R_0, α, ϵ, $\lambda_{1,f}(M \backslash B(p, R_0))$, $V_f(p, 2R_0)$, $\sup_{B(p,3R_0)} |\mathrm{Ric}_f|$, $\inf_{x \in B(p,2R_0)} V_f(x, R_0)$ and $\sup_{B(p,3R_0)} |\nabla f|$, such that the number of ends is at most C; see Section 7.

When $\alpha = 0$ (and $k = n$), Theorem 1.4 recovers Li-Wang's result [18]. The assumption on k ensures an improved Bochner formula in Lemma 3.2 and an exponentially decay property (see (20)) in our argument. The proof involves Li-Tam's theory [17], L^2_f-decay estimates (Lemma 4.4), an improved Bochner formula (Lemma 3.2), a f-mean value inequality (Proposition 3.3), etc. These discussion is provided in Section 7.

In Section 8, we study the dimension estimate of the space of L^2_f-harmonic one-forms. We get that

Theorem 1.5. *Let $(M, g, e^{-f}dv)$ be an n-dimensional SMMS with a Sobolev inequality*

$$\left(\int_M |\phi|^\nu e^{-f} dv\right)^{2/\nu} \leq C_s \int_M |\nabla \phi|^2 e^{-f} dv$$

for all $\phi \in C_0^\infty(M)$, where $\nu > 2$ and $C_s > 0$ are constants. Suppose that there exists a constant $\alpha \geq 0$ such that $|\langle \nabla f, \omega \rangle| \leq \alpha |\omega|$ for all f-harmonic one-form ω on M. If

$$\mathrm{Ric}_f \geq -\tau(x) + \frac{\alpha^2}{k-n}$$

for some $k \geq n$ and some non-negative smooth function $\tau \in C^\infty(M)$ satisfying $\int_M \tau^{\frac{\nu}{\nu-2}} e^{-f} dv < \infty$, then M has finitely many ends.

The above Sobolev inequality implies that each end is f-non-parabolic; see [15]. If $\lambda_{1,f}(M) > 0$ and Ric_f has a more restriction, we are able to get a stronger result; see Theorem 8.2 in Section 8.

We mention that there are many related works concerning the number of ends on SMMSs. On one hand, many people generalize the Cheeger-Gromoll splitting theorem [6] to SMMSs, such as [7], [21], [33], [10], [31], [25] and [37]. From these results, we immediately get that each case has at most two ends. On the other hand, Wei-Wylie [33] proved that any SMMS with $\mathrm{Ric}_f > 0$ for bounded f has only one end. The second author [34] proved the finite number of ends when $\mathrm{Ric}_f \geq 0$ outside a compact set. Recently, Hua and the second author [13] established gap theorems for the number of ends of SMMSs.

2. Existence of f-harmonic functions

In this section, we will prove Theorem 1.1 in introduction. Following [1,2,28], we first recall a process how to give a compactification of a negatively curved manifold.

Let (M, g) be an n-dimensional complete simply connected manifold whose sectional curvature K_M satisfying $-b^2 \leq K_M \leq -a^2 < 0$, where a and b are positive constants. Let $H(c)$ denote the n-dimensional space form with constant curvature c. By the Cartan-Hadamard theorem, for any point $p \in M$, the exponential map $\exp_p : T_pM \to M$ is a diffeomorphism. This allows us to make large-scale comparisons of the geometry of M with $H(-a^2)$ and $H(-b^2)$ by detecting the behavior of Jacobi fields. For example, we can obtain the Rauch-Toponogov comparison, which roughly says that angles are smaller in M than in $H(-a^2)$, while distances are greater; the reverse holds in comparison to $H(-b^2)$. In particular, for the distance $r(x) = d(p, x)$ starting from p, we have (cf. [28])

$$a\coth(ar)(g - dr \otimes dr) \leq \mathrm{Hess}\, r \leq b\coth(br)(g - dr \otimes dr). \tag{2}$$

This implies all geodesic balls $B(p, r)$ are strictly convex. Moreover, for any three points $p, x_1, x_2 \in M$, let $r = d(p, x_1) = d(p, x_2)$ and the corresponding geodesic rays γ_1, γ_2 from x_1 and x_2, respectively, meet with angle θ at p. By the Rauch-Toponogov comparison,

$$2r + \tfrac{2}{a}(\ln\theta - 1) \leq d(x_1, x_2) \leq 2r + \tfrac{2}{b}(\ln\theta + 1) \tag{3}$$

for sufficiently large r and sufficiently small θ.

We recall the concept of the boundary of (M, g) with negative curvature at infinity. Given any point $p \in M$ and any unit vector $v \in T_pM$, there exists a geodesic ray starting from p with direction v. Two geodesic rays γ_1 and γ_2 in M are called equivalent, denoted by $\gamma_1 \sim \gamma_2$, if $d(\gamma_1(t), \gamma_2(t)) \leq c$ for all $t \geq 0$, where c is an absolute constant. The geometric boundary of M at infinity, denoted by $S_\infty(M)$, is defined to be the set of all equivalence classes of geodesic rays, i.e., $S_\infty(M) :=$ The set of all geodesic rays$/\sim$. By (3) we see that two geodesic rays starting from the same point are equivalent if and only if they are the same. We identify geometric boundary at infinity $S_\infty(M)$ with the set all geodesics starting from point p, hence with the $(n-1)$-dimensional unit sphere $S(p, 1)$ of T_pM.

A natural topology, the cone topology, is defined on $\overline{M} = M \cup S_\infty(M)$ as follows: let $C_p(v, \delta)$ be the cone about vector v of angle $\delta > 0$, i.e., $C_p(v, \delta) := \{x \in M | \angle(v, \overline{px}) < \delta\}$, where $\angle(v, \overline{px})$ denotes the angle at point p between vector v and the geodesic ray from p passing through x. We then define the truncated cone by $T_p(v, \delta, R) := C_p(v, \delta) \backslash B(p, R)$. All truncated cones $\{T_p(v, \delta, R) | p \in M, v \in T_x M, \delta > 0, R > 0\}$ together with geodesic balls $\{B(x, R) | x \in M, R > 0\}$ form a local basis for the cone topology of the space $\overline{M} = M \cup S_\infty(M)$. Such a topology gives a compactification of M. In general, the boundary at infinity $S_\infty(M)$ does not attach smoothly onto M. However, when sectional curvatures are bounded between two finite negative constants, using the Rauch-Toponogov comparison, we have that

Proposition 2.1 (Anderson-Schoen [2])**.** *Let (M, g) be an n-dimensional simply connected Riemann manifold with sectional curvature K_M satisfying $-b^2 \leq K_M \leq -a^2 < 0$, where a and b are two positive constants. The cone topology of $\overline{M} = M \cup S_\infty(M)$ defined as above is a C^α-structure, where $\alpha = a/b$.*

We now prove Theorem 1.1 by following the argument of [2].

Proof of Theorem 1.1. For a fixed point $p \in M$, $S_\infty(M)$ can be viewed as the $(n-1)$-dimensional unit sphere $S(p, 1)$ of $T_p M$. Note that any continuous function can be uniformly approximated by smooth functions on $S(p, 1) \cong S_\infty(M)$, where $S(p, 1)$ has its standard smooth structure. Moreover, the maximum principle implies that if there exists a series of f-harmonic functions $\{u_k \in C^\infty(M) \cap C(\overline{M})\}$ converges uniformly on $S_\infty(M)$, then it also converges uniformly on $\overline{M}$ to a f-harmonic function in the class $C^\infty(M) \cap C(\overline{M})$. Therefore, without loss of generality, we assume $\varphi \in C^\infty(S(p, 1))$.

Consider normal polar coordinates $\{(r, \theta) | r > 0, \theta \in S(p, 1)\}$ at p. Write $\varphi = \varphi(\theta)$, where $\theta \in S(p, 1)$. Then we extend φ to $M \backslash \{p\}$ by defining $\varphi(r, \theta) = \varphi(0, \theta)$, $\forall\, r > 0$. The extended function is also denoted by φ. We see that φ is smooth and bounded on $M \backslash \{p\}$. Set $\mathrm{osc}_{B(x,1)} := \sup_{y \in B(x,1)} |\varphi(y) - \varphi(x)|$. We shall prove the theorem by four steps.

Step 1: We will prove that $\mathrm{osc}_{B(x,1)} \varphi = O(e^{-ar(x)})$, where $r(x) = d(p, x)$ is a distance function starting from $p \in M$. Indeed, by the definition of φ, for any $y \in B(p, 1)$,

$$|\varphi(y) - \varphi(x)| = |\varphi(\theta') - \varphi(\theta)| \leq C|\theta' - \theta|,$$

where θ' and θ denotes the spherical coordinates of y and x, respectively. By (3), we have $2r(x) + \frac{2}{a}(\ln|\theta' - \theta| - 1) \leq d(x, y) < 1$, namely, $|\theta' - \theta| \leq C_1(a) e^{-ar(x)}$. Therefore, $\mathrm{osc}_{B(x,1)} \varphi \leq C(p, a, \varphi) e^{-ar(x)}$, where $C(p, a, \varphi)$ is a constant depending on p, a and φ.

Step 2: We shall take the average $\overline{\varphi}$ of φ satisfying $\Delta_f \overline{\varphi} = O(e^{(4\alpha - a) r(x)})$, where $\alpha < a/4$. To achieve it, we choose a cut-off function $\chi \in C_0^\infty(\mathbb{R})$ such that

$$\chi(t)=\begin{cases}1 & t\in[-1/2,1/2],\\ 0 & t\in(-\infty,-1]\cup[1,\infty),\\ 0\le\chi(t)\le 1 & t\in(-1,-1/2)\cup(1/2,1).\end{cases}$$

Set $\overline{\varphi}(x):=\frac{\int_M \chi(\rho_x^2(y))\varphi(y)e^{-f(y)}dv(y)}{\int_M \chi(\rho_x^2(y))e^{-f(y)}dv(y)}$, where $\rho_x(y):=d(x,y)$. Then,

$$\begin{aligned}|\overline{\varphi}(x)-\varphi(x)| &= \frac{\left|\int_{B(x,1)}\chi(\rho_x^2(y))\cdot\big(\varphi(y)-\varphi(x)\big)e^{-f(y)}dv(y)\right|}{\int_{B(x,1)}\chi(\rho_x^2(y))e^{-f(y)}dv(y)}\\ &\le \sup_{y\in B(x,1)}|\varphi(y)-\varphi(x)|\\ &= \operatorname{osc}_{B(x,1)}\varphi\\ &= O(e^{-ar(x)}).\end{aligned}\tag{4}$$

On the other hand, we will give an upper bound of $\Delta_f\,\overline{\varphi}$. Direct computation shows that

$$\begin{aligned}&\Delta_f\,\overline{\varphi}(x_0)\\ &=\Delta_f\big(\overline{\varphi}(x)-\varphi(x_0)\big)\Big|_{x=x_0}\\ &=\int_M \Delta_f\left[\frac{\chi(\rho_x^2(y))}{\int_M\chi(\rho_x^2(y))e^{-f(y)}dv(y)}\right]\Big(\varphi(y)-\varphi(x_0)\Big)e^{-f(y)}dv(y)\Bigg|_{x=x_0}.\end{aligned}\tag{5}$$

To estimate the right hand side of (5), set $\Psi:=\int_M\chi(\rho_y^2(x))e^{-f(y)}dv(y)$ and $\Phi:=\chi(\rho_y^2(x))$. Compute that

$$\Delta_f\left(\frac{\Phi}{\Psi}\right)=\frac{1}{\Psi^2}\left(\Psi\Delta_f\Phi-2\nabla\Phi\cdot\nabla\Psi-\Phi\Delta_f\Psi\right)+2\frac{\Phi}{\Psi^3}|\nabla\Phi|^2.\tag{6}$$

Notice that $\nabla\Phi=\chi'(\rho^2)\cdot 2\rho\nabla\rho$ and $\Delta_f\Phi=4\rho^2\chi''(\rho^2)+2\chi'(\rho^2)+2\rho\chi'(\rho^2)\Delta_f\rho$, where $\rho=\rho_y(x)=d(y,x)$. Since $K_M\ge -b^2$, by the Laplacian comparison, when $\rho=\rho_y(x)\le 1$,

$$\rho\Delta_f\rho=\rho\Delta\rho-\rho\nabla f\nabla\rho\le(n-1)(1+b)+\alpha.$$

Hence, when $\rho\le 1$, by the above equalities and the definition of χ, we have

$$|\nabla\Phi|\le 5\quad\text{and}\quad \Delta_f\Phi\le C(n,b,\alpha)\tag{7}$$

for some $C(n,b,\alpha)$ depending on n, b and α.

Since $\mathrm{Ric}\ge-(n-1)b^2$, by the Bishop volume comparison, $V(B(x,1))\le C(n,b)$ for some constant $C(n,b)$ depending on n and b. Also, the assumption $|\nabla f|\le\alpha$ implies that, for any $y\in B(x,1)$,

$$|f(y)|-|f(p)|\le|f(y)-f(p)|\le\alpha\,d(p,y)\le\alpha\,(r(x)+1),$$

where $r(x)=d(p,x)$. Hence, $|f(y)|\leq |f(p)|+\alpha\,(r(x)+1)$ for any $y\in B(x,1)$. Combining these estimates, we get

$$\begin{aligned}|\nabla\Psi(x)| &\leq C\int_{B(x,1)} e^{-f(y)}dv(y)\\ &\leq C\, e^{|f(p)|+\alpha(r(x)+1)}\, V(B(x,1))\\ &\leq C(n,b,\alpha,f(p))e^{\alpha r(x)}\end{aligned}\tag{8}$$

and

$$|\Delta_f\Psi(x)|\leq C\int_{B(x,1)} e^{-f(y)}dv(y)\leq C(n,b,\alpha,f(p))\,e^{\alpha r(x)},\tag{9}$$

where $r(x)=d(p,x)$. We also need a lower bound of $\Psi(x)$. Since $K_M\leq 0$, by the volume comparison and simply connectedness, we have $V(x,1)\geq c(n)$. Hence,

$$\Psi(x)=\int_M \chi(\rho_x^2(y))e^{-f(y)}dv(y)\geq V_f(x,\tfrac{1}{2})\geq C(n,\alpha,f(p))\,e^{-\alpha r(x)},\tag{10}$$

where we used $-f(y)\geq -|f(p)|-\alpha(r(x)+1)$ for $y\in B(x,1)$ in the last inequality.

We now substitute (7), (8), (9) and (10) into (6), and finally obtain that

$$\Delta_f(\tfrac{\Phi}{\Psi})\leq C(n,b,\alpha,f(p))\,e^{3\alpha r(x)}.$$

Plugging this into (5) and using the estimate $|f(y)|\leq|f(p)|+\alpha(r(x)+1)$ for $y\in B(x,1)$, we obtain

$$\begin{aligned}|\Delta_f\overline{\varphi}(x)| &\leq C(n,b,\alpha,f(p))e^{3\alpha r(x)}\cdot \mathrm{osc}_{B(x,1)}\varphi\cdot\int_{B(x,1)} e^{-f(y)}dv(y)\\ &\leq C(n,a,b,\alpha,f(p),p,\varphi)\,e^{(4\alpha-a)r(x)},\end{aligned}$$

where we used $\mathrm{osc}_{B(x,1)}\varphi\leq C(p,a,\varphi)e^{-ar(x)}$ and $V(x,1)\leq C(n,b)$ in the last inequality.

Step 3: We will construct Barrier functions for f-harmonic functions.

Consider $g(x):=e^{-\delta r(x)}$, where δ is a positive constant, determined later. We have $\Delta_f\, g=-\delta e^{-\delta r}\Delta_f\, r+\delta^2e^{-\delta r}|\nabla r|^2$. Since $K_M\leq -a^2$, by the Laplacian comparison, we have

$$\begin{aligned}\Delta_f\, r &= \Delta\, r-\nabla f\cdot\nabla r\\ &\geq (n-1)a\coth(ar)-|\nabla f|\\ &\geq (n-1)a-\alpha.\end{aligned}$$

Since $\alpha<a/4$, then $\alpha-(n-1)a<0$. Therefore, for sufficiently small $\delta>0$, we get that

$$\begin{aligned}\Delta_f\, g &\leq \delta e^{-\delta r}\,[\alpha-(n-1)a]+\delta^2e^{-\delta r}\\ &= \delta e^{-\delta r}\,[\delta+\alpha-(n-1)a]\\ &< 0.\end{aligned}$$

Since $\Delta_f \overline{\varphi} = O(e^{(4\alpha-a)r(x)})$, where $\alpha < a/4$, as long as δ is sufficiently small, we are able to find a positive constant C such that $\Delta_f(Cg) \le -|\Delta_f \overline{\varphi}|$, that is, we have two barrier functions such that $\Delta_f(\overline{\varphi}+Cg) \le 0$ and $\Delta_f(\overline{\varphi}-Cg) \ge 0$.

Step 4: We will prove the theorem.

By the classical Perron method, there exists a f-harmonic function u satisfying $\overline{\varphi} - Cg \le u \le \overline{\varphi} + Cg$. Moreover u satisfies the boundary condition because that

$$\begin{aligned} |u-\varphi|(x) &\le |u-\overline{\varphi}+\overline{\varphi}-\varphi| \\ &\le Cg(x) + \overline{C}e^{-ar(x)} \\ &\le Ce^{-\delta r(x)} + \overline{C}e^{-ar(x)} \to 0 \end{aligned}$$

as $r(x) \to \infty$, where we used (4). This completes the theorem. □

3. Bochner formula and f-mean value inequality

In this section, we first recall Qian's f-volume comparison [27]. Then we give an improved Bochner formula for f-harmonic functions. Finally we provide f-mean value inequalities for a differential inequality. These results will be used in the study of the number of ends for SMMSs in the following sections.

Lemma 3.1 (Qian [27])**.** *If* $\mathrm{Ric}_f^m \ge -(n+m-1)K$ *for some constant* $K > 0$*, where* $0 < m < \infty$*, then for any point* $p \in M$*, we have*

$$\frac{V_f(p,R)}{V_f(p,r)} \le \frac{V_K^{n+m}(R)}{V_K^{n+m}(r)}$$

for any $R \ge r > 0$*, where* $V_K^{n+m}(R)$ *denotes the volume of geodesic ball* $B(o,R)$ *in the model space* M_K^{n+m}*, the simply connected space of dimension* $n+m$ *with constant sectional curvature* $-K$*.*

Inspired by Yau [38], we next give an improved Bochner formula for f-harmonic functions. We do not know whether the formula holds or not when f grows linearly.

Lemma 3.2. *Let* $(M, g, e^{-f}dv)$ *be an* n*-dimensional SMMS. If* $\mathrm{Ric}_f \ge -(n-1)\tau(x)$ *for some function* $\tau(x)$ *and* $|\langle\nabla u, \nabla f\rangle| \le \alpha|\nabla u|$ *for any* f*-harmonic function* u*, where* $\alpha \ge 0$ *is a constant, then for any* $k > n$*,*

$$\Delta_f|\nabla u|^p \ge \frac{1}{p}\left(\frac{k}{k-1}+p-2\right)\frac{|\nabla|\nabla u|^p|^2}{|\nabla u|^p} - p\left[\frac{\alpha^2}{k-n}+(n-1)\tau(x)\right]|\nabla u|^p,$$

where $p > 0$*. In particular, if* $p = \frac{k-2}{k-1}$*, then*

$$\Delta_f|\nabla u|^p \ge -\frac{k-2}{k-1}\left[\frac{\alpha^2}{k-n}+(n-1)\tau(x)\right]|\nabla u|^p. \tag{11}$$

Proof. Choose a local orthogonal frame $\{e_1, e_2, \ldots, e_n\}$ near a point $x \in M$ so that $\nabla u = |\nabla u| e_1$. Using the classical Bochner formula, $\mathrm{Ric}_f \geq -(n-1)\tau(x)$ and the f-harmonicity of u, we compute that

$$\Delta_f |\nabla u|^2 = 2u_{ij}^2 + 2\mathrm{Ric}_f(\nabla u, \nabla u) \geq 2u_{ij}^2 - 2(n-1)\tau(x)|\nabla u|^2.$$

Observe $|\nabla|\nabla u|^2|^2 = 4\sum_{j=1}^n (\sum_{i=1}^n u_i u_{ij})^2 = 4u_1^2 \sum_{i=1}^n u_{1i}^2 = 4|\nabla u|^2 \sum_{i=1}^n u_{1i}^2$ and

$$\begin{aligned} u_{ij}^2 &\geq u_{11}^2 + 2\sum_{s=2}^n u_{1s}^2 + \sum_{s=2}^n u_{ss}^2 \\ &\geq u_{11}^2 + 2\sum_{s=2}^n u_{1s}^2 + \frac{1}{n-1}\left(\Delta u - u_{11}\right)^2 \\ &= u_{11}^2 + 2\sum_{s=2}^n u_{1s}^2 + \frac{1}{n-1}\left(f_i u_i - u_{11}\right)^2. \end{aligned}$$

By the Cauchy-Schwarz inequality, for any $k > n$,

$$\begin{aligned} u_{ij}^2 &\geq u_{11}^2 + 2\sum_{s=2}^n u_{1s}^2 + \frac{1}{n-1}\left(\frac{u_{11}^2}{1+\frac{k-n}{n-1}} - \frac{(f_i u_i)^2}{\frac{k-n}{n-1}}\right) \\ &\geq \frac{k}{k-1}\sum_{i=1}^n u_{1i}^2 - \frac{\alpha^2}{k-n}|\nabla u|^2, \end{aligned}$$

where we used $|\langle \nabla u, \nabla f\rangle| \leq \alpha |\nabla u|$. Combining these inequalities,

$$\begin{aligned} \Delta_f |\nabla u|^2 &\geq \frac{k}{2(k-1)}|\nabla u|^{-2}|\nabla|\nabla u|^2|^2 - 2\left(\frac{\alpha^2}{k-n} + (n-1)\tau(x)\right)|\nabla u|^2 \\ &= \frac{2k}{k-1}|\nabla|\nabla u||^2 - 2\left(\frac{\alpha^2}{k-n} + (n-1)\tau(x)\right)|\nabla u|^2. \end{aligned}$$

Using the above inequality and $\nabla|\nabla u|^p = \frac{p}{2}|\nabla u|^{p-2}\nabla|\nabla u|^2 = p|\nabla u|^{p-1}\cdot\nabla|\nabla u|$, we get

$$\begin{aligned} \Delta_f |\nabla u|^p &= \frac{p-2}{p}\frac{(\nabla|\nabla u|^p)^2}{|\nabla u|^p} + \frac{p}{2}|\nabla u|^{p-2}\Delta_f|\nabla u|^2 \\ &\geq \frac{p-2}{p}\frac{(\nabla|\nabla u|^p)^2}{|\nabla u|^p} \\ &\quad + p|\nabla u|^{p-2}\left[\frac{k|\nabla u|^{2-2p}(\nabla|\nabla u|^p)^2}{(k-1)p^2} - \left(\frac{\alpha^2}{k-n} + (n-1)\tau(x)\right)|\nabla u|^2\right] \\ &= \frac{1}{p}\left(\frac{k}{k-1} + p - 2\right)\frac{(\nabla|\nabla u|^p)^2}{|\nabla u|^p} - p\left(\frac{\alpha^2}{k-n} + (n-1)\tau(x)\right)|\nabla u|^p. \end{aligned}$$

Then (11) follows by letting $p = \frac{k-2}{k-1}$. □

Finally we give a f-mean value inequality for a differential inequality; it will be used in the proof of Theorem 1.4.

Proposition 3.3. *Let $(M, g, e^{-f}dv)$ be n-dimensional SMMS. Assume* $\mathrm{Ric}_f \geq -(n-1)K$ *for some constant $K \geq 0$. If v is a non-negative function defined on $B(p, \rho)$ satisfying*

$$\Delta_f v \geq -\lambda v,$$

for some constant $\lambda \geq 0$, then for any $0 < \delta < 1$ and $k > 0$, there exists a constant $C > 0$ depending only on n, k, δ, $k\lambda\rho^2$, $\sqrt{K}\rho$, $A'\rho$ such that

$$\sup_{B(p,\delta\rho)} (v^k) \leq \frac{C}{V_f(p,\rho)} \int_{B(p,\rho)} v^k e^{-f} dv,$$

where $A' := A'(\rho) = \sup_{x \in B(p,\rho)} |\nabla f(x)|$.

Proof. Let $u(x,t) = e^{-\lambda t} v(x)$ and then $\Delta_f u - u_t \geq 0$ on $B(p,\rho) \times [0, \infty)$. Fix $k > 0$. Let u be a smooth non-negative subsolution of the f-heat equation in cylinder $Q = B(p,\rho) \times (s - \rho^2, s)$, $s \in \mathbb{R}$. Recall that by Proposition 2.7 and Remark 2.5 in [36], we showed that there exist constants $c_i(n,k)$, $i = 1, 2, 3$, such that

$$\sup_{Q_\delta}(u^k) \leq \frac{c_1 e^{c_2(A' + \sqrt{K})\rho + c_3 A' \sqrt{K} \rho^2}}{(1-\delta)^{2+n} \rho^2 V_f(p,\rho)} \int_Q u^k e^{-f} dv dt,$$

where $0 < \delta < 1$ and $Q_\delta = B(p, \delta\rho) \times (s - \delta\rho^2, s)$, where $A' := A'(\rho) = \sup_{x \in B(p,\rho)} |\nabla f(x)|$. Letting $s = 2\rho^2$, then for any $(x,t) \in B(p,\delta\rho) \times (\rho^2, 2\rho^2)$,

$$e^{-k\lambda t} v^k \leq \frac{c_1 e^{c_2(A' + \sqrt{K})\rho + c_3 A' \sqrt{K} \rho^2}}{(1-\delta)^{2+n} \rho^2 V_f(p,\rho)} \int_{\rho^2}^{2\rho^2} e^{-k\lambda t} dt \int_{B(p,\rho)} u^k e^{-f} dv,$$

that is,

$$\sup_{B(p,\delta\rho)} (v^k) \leq \frac{c_1 e^{c_2(A' + \sqrt{K})\rho + c_3 A' \sqrt{K} \rho^2}}{(1-\delta)^{2+n} V_f(p,\rho)} \cdot \frac{1 - e^{-k\lambda\rho^2}}{k\lambda\rho^2} \int_{B(p,\rho)} v^k e^{-f} dv.$$

This implies the result. □

4. L^2_f-decay estimate

In this section, we give a weighted version of L^2-decay estimates in [18–20]. Then we apply this result to get f-volume estimates for ends.

On $(M, g, e^{-f}dv)$, let $\lambda_{1,f}(M) = \inf \mathrm{Spec}(\Delta_f)$ denote the infimum of the spectrum of the f-Laplacian. It can be characterized by

$$\lambda_{1,f}(M) = \inf \left\{ \frac{\int_M |\nabla\phi|^2 e^{-f} dv}{\int_M \phi^2 e^{-f} dv} \middle| \phi \in H^{1,2}_{f,0}(M) \right\},$$

where the infimum is taken over all compactly supported functions in $H^{1,2}_f(M)$. In particular, $\lambda_{1,f}(M) = \inf_{i \to \infty} \lambda_{1,f}(\Omega_i)$ for any compact exhaustion $\{\Omega_i\}$ of M, where $\lambda_{1,f}(\Omega_i)$ is the Dirichlet first eigenvalue of Δ_f on Ω_i. Assume that E

is an end of M with respect to the compact set $B(p, R_0)$. Let $\lambda_{1,f}(E)$ denote the infimum of the Dirichlet spectrum of Δ_f on E. If $\lambda_{1,f}(E) > 0$, then

$$\lambda_{1,f}(E) \int_E \phi^2 e^{-f} dv \leq \int_E |\nabla \phi|^2 e^{-f} dv \tag{12}$$

for any compactly supported function $\phi \in H^{1,2}_{f,0}(E)$.

Another important quantity of the spectrum is the infimum of the essential spectrum $\lambda_{e,f}(M)$ of Δ_f. It has property that $\mu_{1,f}(M) \leq \lambda_{e,f}(M)$ and $\lambda_{1,f}(M) \leq \lambda_{e,f}(M)$. Inversely, for any $\epsilon > 0$, there exists a compact set $\Omega \subset M$ such that $\lambda_{e,f}(M) \leq \lambda_{1,f}(\overline{\Omega}) + \epsilon$ for any compact set $\overline{\Omega} \subset M \backslash \Omega$. From this, we easily see that the results proved in this paper with the assumption of $\lambda_{1,f}(E) > 0$ can be stated as $\lambda_{e,f}(M) > 0$.

We consider a general situation for L^2_f-decay estimates on SMMSs. Let V be a potential function on M, and $\Delta_f - V(x)$ be a weighted Schrödinger operator. Assume that there exists a positive function ϱ on M such that the weighted f-Poincaré type inequality

$$\int_M \varrho \, \phi^2 e^{-f} dv \leq \int_M |\nabla \phi|^2 e^{-f} dv + \int_M V(x) \phi^2 e^{-f} dv \tag{13}$$

holds for $\phi \in C_0^\infty(M)$. If $\varrho = \lambda_{1,f}(M)$ and $V = 0$, then (13) recovers to (12). So (13) is a generalization of the condition $\lambda_{1,f}(M) > 0$. If there exists a positive function h on M such that $(\Delta_f + \varrho - V)h \leq 0$, then (13) is valid. For the existence issue of ϱ in (13), the reader are referred to [19] for nice discussions.

We introduce the ϱ-metric by $ds_\varrho{}^2 = \varrho \, ds_M^2$. Using this metric, we consider the ϱ-distance function to be $r_\varrho(x, y) = \inf_\gamma \ell_\varrho(\gamma)$, the infimum of length of all smooth curves γ joining x and y with respect to ds_ϱ^2. For a fixed point $p \in M$, let $r_\varrho(x) = r_\varrho(p, x)$ be the ϱ-distance to p, and then $|\nabla r_\varrho|^2(x) = \varrho(x)$.

Definition. We say that $(M, g, e^{-f} dv)$ has property $(\mathcal{P}^f_{\varrho,V})$ if there exists a positive function ϱ such that (13) holds and the ϱ-metric is complete.

Let $B_\varrho(p, R) = \{x \in M | r_\varrho(p, x) < R\}$ be a geodesic ball centered at $p \in M$ with radius R with respect to $ds_\varrho{}^2$. Clearly, if $\varrho = 1$, this geodesic ball returns to the usual geodesic ball $B(p, R)$ with respect to the background metric ds_M^2. When p is a fixed point, we often suppress the dependency of p and write $B_\varrho(R) = B_\varrho(p, R)$ and $B(R) = B(p, R)$. If E is an end of M, we let $E_\varrho(R) = B_\varrho(p, R) \cap E$ and $E(R) = B(R) \cap E$. Now We give an improved weighted version of Li-Wang's decay estimate [19].

Theorem 4.1. *Let $(M, ds_\varrho^2, e^{-f} dv)$ be an n-dimensional SMMS with property $(\mathcal{P}^f_{\varrho,V})$. Let u be a nonnegative function in an end E with $(\Delta_f - V(x))\, u \geq 0$. If there exists a sequence $R_i \to \infty$ such that $\int_{E_\varrho(R_i+1) \backslash E_\varrho(R_i)} \varrho \, e^{-2r_\varrho} u^2 e^{-f} dv \to 0$, then*

$$\int_{E_\varrho(\rho+1) \backslash E_\varrho(\rho)} \varrho \, u^2 e^{-f} dv \leq 3e^{-2\rho} \int_{E_\varrho(R_0+1) \backslash E_\varrho(R_0)} \varrho \, e^{2r_\varrho} u^2 e^{-f} dv$$

for all $\rho > R_0 + 1$.

Remark 4.2. In [19], Li-Wang required u to satisfy $\int_{E_\varrho(R)} \varrho\, e^{-2r_\varrho} u^2 dv = o(R)$. In [12], Hua-Lu extended Li-Wang's result [18] when $\lambda_{1,f}(E) > 0$. Combining their arguments, our condition of u generalizes previous cases. We remark that Dung-Sung [9] also proved a weighted version under a stronger condition of u.

Proof of Theorem 4.1. Recall that Li-Wang [19] used a delicate iterated technique to prove the decay estimate. Here we will apply the argument of [12] to give a simplified proof without iterated argument. In the following, all integrals are discussed with respect to the weighted measure $e^{-f}dv$, and we omit the notation $e^{-f}dv$ for simplicity.

For a smooth function ϕ with compact support in E, we have

$$\int_E |\nabla(\phi u)|^2 = \int_E |\nabla\phi|^2 u^2 - \int_E \phi^2 u \Delta_f u.$$

Using $\Delta_f u \geq V\, u$ and the fact $\int_E (\varrho - V)(\phi u)^2 \leq \int_E |\nabla(\phi u)|^2$, we get $\int_E \varrho\, \phi^2 u^2 \leq \int_E |\nabla\phi|^2 u^2$. Replacing ϕ by ϕe^h for some Lispchitz function h and expanding $\nabla(\phi\, e^h)$,

$$\int_E \varrho\, \phi^2 e^{2h} u^2 \leq \int_E |\nabla\phi|^2 e^{2h} u^2 + 2\int_E \phi\, e^{2h} \langle\nabla\phi, \nabla h\rangle u^2 + \int_E \phi^2 |\nabla h|^2 e^{2h} u^2. \tag{14}$$

As in [12], we will choose suitable cut-off functions ϕ and h in (14) to prove our result. Recall that Hua-Lu's construction of ϕ and h in [12] is as follows; see Figure 1 for a concrete description. Let $R_0 > 0$ and $R > R_0 + 1$. Define

$$\phi(t) = \begin{cases} 0 & t \in [0, R_0) \cup [R+1, \infty), \\ t - R_0 & t \in [R_0, R_0+1), \\ 1 & t \in [R_0+1, R), \\ -t + R + 1 & t \in [R, R+1). \end{cases}$$

Let $R_0 + 1 < \rho < R - 1$. For some constant $a > 0$, define

$$h(t) = \begin{cases} at & t \in [0, \rho), \\ a\rho & t \in [\rho, \rho+1), \\ a\rho - a(t - \rho - 1) & t \in [\rho+1, \infty). \end{cases}$$

In our situation, $\phi(r_\varrho(x))$ is given by

$$\phi(r_\varrho(x)) = \begin{cases} 0 & E_\varrho(R_0) \cup (E \backslash E_\varrho(R+1)), \\ r_\varrho(x) - R_0 & E_\varrho(R_0+1) \backslash E_\varrho(R_0), \\ 1 & E_\varrho(R_0+1) \backslash E_\varrho(R), \\ -r_\varrho(x) + R + 1 & E_\varrho(R+1) \backslash E_\varrho(R). \end{cases}$$

For $R_0 + 1 < \rho < R - 1$ and for some $a > 0$, $h(r_\varrho(x))$ is defined by

$$h(r_\varrho(x)) = \begin{cases} a\, r_\varrho(x) & E_\varrho(\rho), \\ a\, \rho & E_\varrho(\rho+1) \backslash E_\varrho(\rho), \\ a\, \rho - a\,(r_\varrho(x) - \rho - 1) & E \backslash E_\varrho(\rho+1). \end{cases}$$

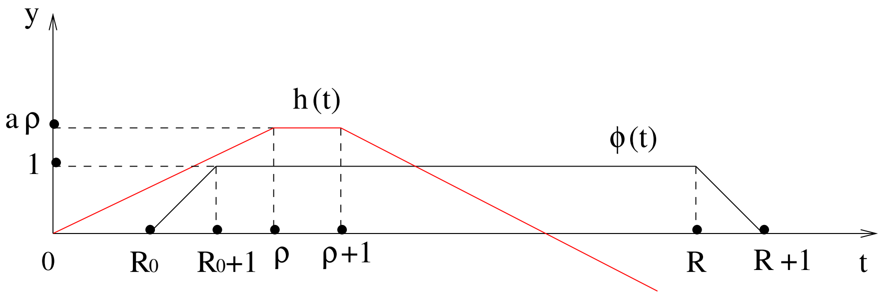

FIGURE 1. Definition of cut-off functions $\phi(t)$ and $h(t)$

Using the above two cut-off functions, we have that

$$\left|\int_E \phi\, e^{2h}\langle\nabla\phi,\nabla h\rangle u^2\right| \le \int_{E_\varrho(R_0+1)\setminus E_\varrho(R_0)} a\varrho\, e^{2h}u^2 + \int_{E_\varrho(R+1)\setminus E_\varrho(R)} a\varrho\, e^{2h}u^2.$$

Substituting this into (14) yields

$$\begin{aligned}\int_E \varrho\,\phi^2 e^{2h}u^2 \le& \int_{E_\varrho(R_0+1)\setminus E_\varrho(R_0)} (2a+1)\varrho\, e^{2h}u^2 + \int_{E_\varrho(R+1)\setminus E_\varrho(R)} (2a+1)\varrho\, e^{2h}u^2\\ &+\int_{E\setminus(E_\varrho(\rho+1)\setminus E_\varrho(\rho))} a^2\varrho\,\phi^2 e^{2h}u^2.\end{aligned}$$

Let $a=1$. By definitions of ϕ and h, the above inequality is simplified as

$$\begin{aligned}e^{2\rho}\int_{E_\varrho(\rho+1)\setminus E_\varrho(\rho)} \varrho\, u^2 \le& \,3\int_{E_\varrho(R_0+1)\setminus E_\varrho(R_0)} \varrho\, e^{2r_\varrho}\, u^2\\ &+3e^{4\rho+2}\int_{E_\varrho(R+1)\setminus E_\varrho(R)} \varrho\, e^{-2r_\varrho}\, u^2.\end{aligned}$$

For any a fixed ρ, by the decay condition of u, there exists a sequence $R_i\to\infty$ such that $\int_{E_\varrho(R_i+1)\setminus E_\varrho(R_i)} \varrho\, e^{-2r_\varrho}u^2 \to 0$. Taking $R=R_i$ and letting $i\to\infty$ gives the conclusion. □

In the rest of this section, we consider L^2_f-decay estimates for generalized subharmonic functions when $\lambda_{1,f}(E)>0$.

Similar to the manifold case, if SMMS has polynomial f-volume growth, then $\lambda_{1,f}(M)=0$, which is a weighted version of Cheng-Yau's spectrum estimate. Moreover, we have a weighted version of Brooks' spectrum estimate $\lambda_{e,f}(M)\le \frac14\tau_f^2(M)$, where the weighted volume entropy $\tau_f(M)$ is defined by

$$\tau_f(M):=\begin{cases}\limsup\limits_{R\to\infty}\frac{\ln V_f(p,R)}{R}, & V_f(M)=\infty,\\ \limsup\limits_{R\to\infty}\frac{-\ln V_f(M\setminus B(p,R))}{R}, & V_f(M)<\infty.\end{cases}$$

This result extends the weighted Cheng-Yau's eigenvalue estimate. Now we apply Theorem 4.1 to prove Theorem 1.2, which further improves Brooks' spectrum estimate.

Proof of Theorem 1.2. Since $\lambda_{1,f}(E) > 0$, then

$$\lambda_{1,f}(E)\int_E \phi^2 e^{-f}dv \le \int_E |\nabla\phi|^2 e^{-f}dv$$

for any compactly supported function ϕ in $H_f^{1,2}(E)$. This implies the Poincaré inequality

$$\int_E (\lambda_{1,f}(E)-\sigma)\,\phi^2 e^{-f}dv \le \int_E |\nabla\phi|^2 e^{-f}dv - \sigma\int_E \phi^2 e^{-f}dv.$$

Let $V=-\sigma$ and $\varrho = \lambda_{1,f}(E)-\sigma$ in Theorem 4.1. Meanwhile the distance function with respect to the metric $ds_\varrho{}^2$ is given by $r_\varrho(x) = a\,r(x)$, where $r(x)$ is the background distance function. The theorem then follows by the same argument of Theorem 4.1. □

Using Theorem 1.1, we give an example to indicate that the condition (1) is best possible.

Example 4.3. Consider an n-dimensional hyperbolic space form $\mathbb{H}^n$ with sectional curvature $K=-1$ and the weight function $f(x) = \theta\sin r(x)$, where $0<\theta<1/4$ is a constant and $r(x) = d(p,x)$ is a distance function starting from point $p\in\mathbb{H}^n$. Then $V_f(B(p,R)) \sim C_1(n,\theta)e^{(n-1)R}$. By the definition of $\lambda_{1,f}(\mathbb{H}^n)$, we have $\lambda_{1,f}(\mathbb{H}^n) \ge \lambda_1(\mathbb{H}^n)e^{-2\theta} = \frac{(n-1)^2}{4}e^{-2\theta}$. Since f is sublinear, by Theorem 1.2 of [24], we also have $\lambda_{1,f}(\mathbb{H}^n) \le \frac{(n-1)^2}{4}$. Let u be a non-constant bounded f-harmonic function on $\mathbb{H}^n$ with boundary value $u=2$ on $S_\infty(\mathbb{H}^n)$, for sufficiently large R, we then have

$$\begin{aligned}\int_{E(R+1)\setminus E(R)} u^2 e^{-2\sqrt{\lambda_{1,f}(\mathbb{H}^n)}\,r(x)}e^{-f}dv &\ge \int_{E(R+1)\setminus E(R)} e^{-(n-1)(R+1)}e^{-f}dv\\ &\ge C_2(n,\theta),\end{aligned}$$

where we used $V_f(B(p,R))\sim C_1(n,\theta)e^{(n-1)R}$ in the second inequality. On the other hand, if the conclusion of Theorem 1.2 is true, then for all $R>2$,

$$\int_{B(p,R+1)\setminus B(p,R)} u^2 e^{-f}dv \le Ce^{-\frac{2(n-1)}{e^\theta}R},$$

where C is a constant depending on n, θ and $\|u\|_{L_f^2(B(p,2))}$, which implies that

$$\begin{aligned}\int_{B(p,R+1)} u^2 e^{-f}dv &\le \int_{B(p,R)} u^2 e^{-f}dv + Ce^{-\frac{2(n-1)}{e^\theta}R}\\ &\le \int_{B(p,R-1)} u^2 e^{-f}dv + C\left[e^{-\frac{2(n-1)}{e^\theta}R} + e^{-\frac{2(n-1)}{e^\theta}(R-1)}\right]\end{aligned}$$

$$\vdots$$
$$\leq \int_{B(p,3)} u^2 e^{-f} dv + C \sum_{k=1}^{[R]+1} e^{-\frac{2(n-1)}{e^\theta}k}.$$

Letting $R \to \infty$ gives $u \in L^2_f(\mathbb{H}^n)$. Combining this with [26] implies that u must be constant. However, by Theorem 1.1, the SMMS $(\mathbb{H}^n, g_{\mathbb{H}^n}, e^{-f}dv)$, where $f = \theta \sin r(x)$ and $0 < \theta < 1/4$, has a non-constant bounded f-harmonic function with $u = 2$ on $S_\infty(\mathbb{H}^n)$.

Below we apply Theorem 1.2 to study a decay estimate for a class of f-harmonic functions on SMMSs with positive spectrum; this will be used in f-volume estimates for ends in Section 5. In particular, we get a decay estimate for f-Green's function.

Let $G_f(x,y)$ be a f-Green's function on $(M \times M)\backslash D$, where $D = \{(x,x)|x \in M\}$. Then $G_f(x,y) = G_f(y,x)$ and $\Delta_{f,y} G_f(x,y) = -\delta_{f,x}(y)$ for all $x \neq y$, where $\delta_{f,x}(y)$ is a weighted delta function, defined by $\int_M \phi(y)\delta_{f,x}(y)e^{-f}dv = \phi(x)$ for $\phi \in C_0^\infty(M)$. Similar to Li-Tam's construction for the Green's function on manifolds [16], it is not hard to see that every SMMS admits a f-Green's function. In [36], Wu and the second author gave a necessary and sufficient condition of the existence of positive f-Green's function. An end E is called to be f-non-parabolic if it admits a positive f-Green's function with the Neumann boundary condition on ∂E. Otherwise, it is called to be f-parabolic.

In the following lemma, we let K denote the space of bounded f-harmonic functions with finite weighted Dirichlet integral in $(M, g, e^{-f}dv)$.

Lemma 4.4. *Let E be an end of n-dimensional SMMS $(M, g, e^{-f}dv)$ with $\lambda_{1,f}(E) > 0$. Then for any f-harmonic function $u \in \mathrm{K}$, there exists a constant c such that $u - c$ must be in $L^2_f(E)$. Moreover, the function $u - c$ must satisfy the decay estimate*

$$\int_{E(\rho+1)\backslash E(\rho)} (u-c)^2 e^{-f} dv \leq C e^{-2\sqrt{\lambda_{1,f}(E)}\rho}$$

for some positive constant C depending on u, $\lambda_{1,f}(E)$ and n.

Proof. The proof is the weighted version of the manifold case [15]. We include the details for the reader's convenience. Following [17], for a f-non-parabolic end E_1, let u_ρ be a sequence of f-harmonic functions satisfying $\Delta_f u_\rho = 0$ on $B(p,\rho)$ with boundary condition $u_\rho = 1$ on $\partial B(p,\rho) \cap E_1$ and $u_\rho = 0$ on $\partial B(p,\rho)\backslash E_1$. The sequence of functions $\{u_\rho\}$ must have a subsequence that converges to a f-harmonic function $u \in \mathrm{K}$ on compact subsets of M. For any fixed end E, since u_ρ has the boundary value either 0 or 1 on $\partial E(\rho) = \partial B(p,\rho) \cap E$, by considering either u_ρ or $1 - u_\rho$, we may assume that u_ρ has the boundary value 0 on $\partial B(p,\rho)\cap E$. Define the function v_ρ by

$$v_\rho(x)=\begin{cases} u_\rho(x) & \text{on } E(\rho),\\ 0 & \text{on } E\backslash E(\rho).\end{cases}$$

Clearly, v_ρ is a nonnegative f-subharmonic function, which is compact supported on E. So v_ρ satisfies the assumption of Theorem 1.2 and the decay estimate holds for v_ρ. Then the decay estimate also holds for the above constructed function u by letting $\rho\to\infty$. The conclusion follows by the linear combinations of those functions u. □

Lemma 4.4 also holds for any function u with $c=0$ provided that u is the limit of a sequence of f-harmonic functions u_ρ on $E(\rho)$ satisfying $u_\rho=0$ on $\partial E(\rho)$ regardless of their boundary values on ∂E.

5. f-parabolicity and f-non-parabolicity

In this section, we discuss some conditions on function ϱ for an end being f-parabolic or f-non-parabolic. When $\varrho=\lambda_{1,f}(M)$, we have some special conclusions; these results will be used in the proof of our main results in introduction.

We first provide f-volume estimates for ends, generalizing the manifold cases [18, 19]. Since the proofs are the same as the manifold cases, we omit the detailed proof here.

Theorem 5.1. *Let E be an end of n-dimensional SMMS $(M,g,e^{-f}dv)$ with the property $(\mathcal{P}^f_{\varrho,0})$ for some positive function ϱ on M.*

(1) *If E is f-parabolic, then $\int_E \varrho\, e^{-f}dv<\infty$ and $\int_{E\backslash E_\varrho(\rho)}\varrho\, e^{-f}dv\le C_1\exp(-2\rho)$ for some constant $C_1>0$ and for ρ sufficiently large.*

(2) *If E is f-non-parabolic, then $\int_{E_\varrho(\rho+1)\backslash E_\varrho(\rho)}\varrho\, e^{-f}dv\ge C_2\exp(2\rho)$ for some constant $C_2>0$ and for ρ sufficiently large.*

When $\varrho=\lambda_{1,f}(E)>0$, by a similar argument of [18], we have f-volume estimates for ends by using Theorem 1.2 and Lemma 4.4. Let $V_{E,f}(\rho)$ be the f-volume of $E(\rho)$, i.e., $V_{E,f}(\rho)=\int_{E(\rho)}e^{-f}dv$, and the f-volume of the end E is denoted by $V_{E,f}(\infty)$.

Corollary 5.2. *Let E be an end of n-dimensional SMMS $(M,g,e^{-f}dv)$ with $\lambda_{1,f}(E)>0$.*

(1) *If E is f-parabolic, then*

$$V_{E,f}(\infty)-V_{E,f}(\rho)\le C_1e^{-2\sqrt{\lambda_{1,f}(E)}\rho} \tag{15}$$

for some constant $C_1>0$ depending on E.

(2) *If E is f-non-parabolic, then*

$$V_{E,f}(\rho)\ge C_2e^{2\sqrt{\lambda_{1,f}(E)}\rho}$$

for all $\rho\ge R_0+1$ and some constant $C_2>0$ depending on E.

Remark 5.3. Corollary 5.2 implies that if $\lambda_{1,f}(E) > 0$, then an end E must either be f-non-parabolic or have finite f-volume.

Theorem 1.2 also implies that a precise decay estimate for annuluses, which will be used to estimate the number of ends on SMMSs in the following section.

Corollary 5.4. *Let $(M, g, e^{-f}dv)$ be n-dimensional SMMS. Assume E is an end of M with respect to a compact ball $\overline{B(p, R_0)}$ such that $\lambda_{1,f}(E) > 0$. If E is a f-parabolic end, then*

$$\begin{aligned} &V_f(p, \rho+1) - V_f(p, \rho) \\ &\le \frac{\sqrt{\lambda_{1,f}(E)}+1}{\lambda_{1,f}(E)} e^{2\sqrt{\lambda_{1,f}(E)}(R_0+1-\rho)} \Big[V_f(p, R_0+1) - V_f(p, R_0)\Big] \end{aligned}$$

for all $\rho > R_0 + 1$.

Proof. By (15), if $\lambda_{1,f}(E) > 0$, then E is f-parabolic if and only if its f-volume is finite. Using this fact and letting $u = 1$ in Theorem 1.2, we get that

$$\begin{aligned} &V_{E,f}(\rho+1) - V_{E,f}(\rho) \\ &\le \frac{\sqrt{\lambda_{1,f}(E)}+1}{\lambda_{1,f}(E)} e^{2\sqrt{\lambda_{1,f}(E)}(R_0+1-\rho)} \left[V_{E,f}(R_0+1) - V_{E,f}(R_0)\right] \end{aligned}$$

for all $\rho > R_0+1$. In particular, let $E = M\backslash B(p, R_0)$ and the result follows. □

6. Ends on SMMS with finite f-volume

In this section, we apply Theorem 1.2 to estimate the number of ends on SMMSs with finite f-volume. Let $N(M)$ be the number of ends of M, that is, $N(M)$ is the number of unbounded connected component of $M\backslash B(p, R)$ when $R \to \infty$. Clearly, $N(M)$ is independent of the base point p and the metric of M, and it is a topological invariant. The bottom of Neumann spectrum $\mu_{1,f}(M)$ of f-Laplacian is defined by

$$\mu_{1,f}(M) := \inf\left\{\frac{\int_M |\nabla\phi|^2 e^{-f}dv}{\int_M \phi^2 e^{-f}dv}\middle|\phi \in H_f^{1,2}(M) \text{ and } \int_M \phi e^{-f}dv = 0\right\},$$

where $H_f^{1,2}(M)$ is the weighted Hilbert space. Note that $\mu_{1,f}(M)$ might not be an eigenvalue, but it is linked with the first Dirichlet eigenvalue by $\mu_{1,f}(M) \le \max\{\lambda_{1,f}(\Omega_1)), \lambda_{1,f}(\Omega_2)\}$, where Ω_1 and Ω_2 are two disjoint domains of M, and $\lambda_{1,f}(\Omega_i))$ is the first Dirichlet eigenvalue of f-Laplacian on Ω_i, $i = 1, 2$. We have the following main result of this section.

Theorem 6.1. *Let $(M, g, e^{-f}dv)$ be an n-dimensional SMMS with $V_f(M) < \infty$ and a base point $p \in M$. For $0 < m < \infty$, assume that $\mathrm{Ric}_f^m \ge -(n+m-1)K$ for some constant $K > 0$. If $\mu_{1,f}(M) \ge \frac{1}{4}(n+m-1)^2K$, then there exists a* *constant $C(n+m,K)$ depending on $n+m$ and K such that the number of ends of M satisfies*

$$N(M) \leq C(n+m,K)\left(\frac{V_f(M)}{V_f(p,1)}\right)^2 \ln\left(\frac{V_f(M)}{V_f(p,1)}\right),$$

where $V_f(p,1)$ denotes the weighted volume of the geodesic ball $B(p,1)$.

When f is constant, Theorem 6.1 returns to Li-Wang's result [20]. Dung-Khanh-Son proved the finite number of ends when $\mathrm{Ric}_f \geq -(n-1)$ and $|\nabla f| \leq \alpha$ (see Theorem 1.2 in [8]). We observe that $\mathrm{Ric}_f \geq -(n-1)$ and $|\nabla f| \leq \alpha$ implies $\mathrm{Ric}_f^{\alpha} \geq -(n+\alpha-1)$. Therefore, Dung-Khanh-Son's result is regarded as a special case of Theorem 6.1.

The proof of Theorem 6.1 is similar to Li-Wang's argument [20]. It relies on Qian's volume comparison [27] (Lemma 3.1), L_f^2-decay estimates of f-subharmonic functions (Theorem 1.2) and upper bounds of the first Dirichlet eigenvalue (Lemma 6.3). We begin with a key proposition for preparing our proof.

Proposition 6.2. *Let $(M,g,e^{-f}dv)$ be an n-dimensional SMMS with $V_f(M) < \infty$. Assume $\mathrm{Ric}_f^m \geq -(n+m-1)K$ for some constants $K > 0$. If there exist a point $p \in M$ and $R_0 > 0$ such that $\lambda_{1,f}(M \backslash B(p,R_0)) \geq \frac{1}{4}(n+m-1)^2K$, then there exists a constant $C(n+m,K)$ depending only on $n+m$ and K such that*

$$N(M) \leq C(n+m,K)e^{(n+m-1)\sqrt{K}R_0} \cdot \frac{V_f(M)}{V_f(p,1)}.$$

Munteanu-Wang [24] and Su-Zhang [29] independently proved that if $\mathrm{Ric}_f \geq -(n-1)$, $|\nabla f| \leq \alpha$ and $\lambda_{1,f}(M) \geq \frac{1}{4}(n-1+\alpha)^2$, then such SMMS has at most two ends. Since $\lambda_{1,f}(M \backslash B(p,R_0)) \geq \lambda_{1,f}(M)$, Proposition 6.2 may be regarded as an extension of their results about number estimates of ends.

Proof. For any $y \in \partial B(p,R+1)$, where $R > 2(R_0+1)$, then $B(p,1) \subset B(y,R+2)$. Under the assumption of Proposition 6.2, by Lemma 3.1, for $R \geq \tau_0$ and $0 < \tau_0 \leq 1$, we have

$$V_f(y,R) \leq C(n+m,K)e^{(n+m-1)\sqrt{K}(R+2)}V_f(y,\tau_0).$$

So $V_f(p,1) \leq V_f(y,R+2) \leq C(n+m,K)e^{(n+m-1)\sqrt{K}(R+2)}V_f(y,\tau_0)$ and hence

$$V_f(y,\tau_0) \geq C(n+m,K)^{-1}e^{-(n+m-1)\sqrt{K}(R+2)}V_f(p,1) \tag{16}$$

for $y \in \partial B(p,R+1)$.

Adapting Li-Wang's argument [20], let $N(R)$ be the number of ends with respect to $B(p,R)$, that is, $M \backslash B(p,R)$ has $N(R)$ unbounded components. Then there exists $N(R)$ number of points $\{y_i \in \partial B(p,R+1)\}$ and $0 < r_0(R) \leq 1$ such that $B(y_i,r_0(R)) \cap B(y_j,r_0(R)) = \emptyset$ for $i \neq j$. Applying (16) to each of the y_i with $\tau_0 = r_0(R)$, we get

$$
\begin{aligned}
&\frac{N(R)V_f(p,1)}{Ce^{(n+m-1)\sqrt{K}(R+2)}}\\
&\le \sum_{i=1}^{N(R)} V_f(y_j, r_0(R))\\
&\le V_f(p,R+2)-V_f(p,R)\\
&=[V_f(p,R+2)-V_f(p,R+1)]+[V_f(p,R+1)-V_f(p,R)]\\
&\le c(n+m,K)e^{(n+m-1)\sqrt{K}(R_0-R)}\left[V_f(p,R_0+1)-V_f(p,R_0)\right]
\end{aligned}
$$

for $R > 2(R_0+1)$, where we used Corollary 5.4 and the eigenvalue assumption in the last inequality. The above estimate further reduces to

$$
N(R) \le C(n+m,K)e^{(n+m-1)\sqrt{K}R_0}\cdot\frac{V_f(p,R_0+1)-V_f(p,R_0)}{V_f(p,1)} \tag{17}
$$

for all $R > 2(R_0+1)$. Letting $R \to \infty$ in (17), the proposition follows. □

Proposition 6.2 shows that if we can control the size of R_0 in terms of the first Dirichlet eigenvalue below, then SMMS has finitely many ends (without the finite assumption on $V_f(M)$). At present we do not know how to control the R_0. However, the following lemma allows us to estimate $\lambda_{1,f}(B(p,R))$ in terms of the f-volume of its geodesic ball, which is the same as the manifold case [19].

Lemma 6.3. *Let $\lambda_{1,f}(B(p,R))$ be a first Dirichlet eigenvalue of Δ_f in $B(p,R)$, $R > 2$, in an n-dimensional SMMS $(M,g,e^{-f}dv)$. Then for any $0<\delta<1$,*

$$
\lambda_{1,f}(B(p,R)) \le \frac{1}{4\delta^2(R-1)^2}\left\{\ln\left[\left(\frac{81}{1-\delta}\right)\left(\frac{V_f(p,R)}{V_f(p,1)}\right)\right]\right\}^2.
$$

In particular, $\lambda_{1,f}(M) \le \frac{1}{4}(\liminf_{R\to\infty}\frac{\ln V_f(p,R)}{R})^2$.

In the end of this section we use Lemma 6.3 and Proposition 6.2 to prove Theorem 6.1. The argument is the same as [20] and we provide the detail for the completeness.

Proof of Theorem 6.1. Let $p \in M$ be a fixed point. For any $0<\delta<1$, let

$$
R_0 = \frac{1}{(n+m-1)\sqrt{K}\delta}\ln\left[\left(\frac{81}{1-\delta}\right)\left(\frac{V_f(M)}{V_f(p,1)}\right)\right]+1.
$$

Substituting R_0 into Lemma 6.3 gives $\lambda_{1,f}(B(p,R_0)) \le \frac{1}{4}(n+m-1)^2K$. Now we claim that $\lambda_{1,f}(M\backslash B(p,R_0)) \ge \frac{1}{4}(n+m-1)^2K$. If this claim is not true, by the variational principle, we get $\mu_1(M) < \frac{1}{4}(n+m-1)^2K$, which contradicts our assumption. Under the above spectrum condition, Proposition 6.2 implies $N(M) \le C(n+m,K)e^{(n+m-1)\sqrt{K}R_0}\cdot\frac{V_f(M)}{V_f(p,1)}$. Putting the value of R_0 into this estimate and letting $\delta = 1-(\ln\frac{V_f(M)}{V_f(p,1)})^{-1}$, we get the desired estimate. □

7. Ends with infinite f-volume

In this section, we prove Theorem 1.4. It states that if the gradient of weight function is bounded and the Bakry-Émery Ricci tensor has a lower bound at infinity, then the SMMS has finitely many ends with infinite f-volume.

As is well known, f-harmonic functions are characterized as critical points of Dirichlet energy $\int_M |\nabla u|^2 e^{-f} dv$. If u is f-harmonic and $\int_M |\nabla u|^2 e^{-f} dv < \infty$, then u is said to be finite f-energy.

According to Li-Tam's theory [17], to count the number of f-non-parabolic ends, we only need to estimate the dimension of the space of bounded f-harmonic functions with finite f-energy. For the convenience of the reader, we briefly recall Li-Tam's theory. For our purpose, we assume M has at least two f-non-parabolic ends. Then there exists $R_0 > 0$ such that $M \backslash B(p, R_0)$ has at least two disjoint f-non-parabolic ends E_1 and E_2. For E_1, we can construct a non-constant bounded f-harmonic function with finite f-energy as follows. For $R \geq R_0$, let u_R be the solution to the following Cauchy problem

$$\begin{cases} \Delta_f u_R = 0 & \text{on } B(p,R), \\ u_R = 1 & \text{on } \partial E_1(R) := E_1 \cap B(p,R), \\ u_R = 0 & \text{on } \partial B(p,R) \backslash E_1. \end{cases}$$

Clearly, $\partial E_2(R) \subset (\partial B(p,R) \backslash E_1)$. Since E_1 and E_2 are f-non-parabolic, the sequence of functions u_R has a subsequence that converges to an f-harmonic function u. Moreover, u has the property that $\sup_M u = \sup_{E_1} u = 1$ and $\inf_M u = \inf_{E_i} u = 0$ for the other f-non-parabolic ends E_i. In particular, u is bounded and has finite f-energy. We can use this construction on each f-non-parabolic end and get as many linear independent f-harmonic functions, adding the constant function, as the number of f-non-parabolic ends. As in Lemma 4.4, let K denote the space of bounded f-harmonic functions with finite f-energy. If $(M, g, e^{-f} dv)$ (or all its ends) has positive spectrum, then the number of infinite f-volume ends is less than dim K.

From the above discussion, we see that if we can estimate dim K, then Theorem 1.4 immediately follows. To estimate dim K, we need the following lemma.

Lemma 7.1. *Under the same assumptions of Theorem 1.4, for each $u \in \mathrm{K}$, the function $g := |\nabla u|^{\frac{k-2}{k-1}}$, where $k \geq n + \frac{\alpha}{2}\lambda_{1,f}(M \backslash B(p,R_0))^{-1/2}$, satisfies $\int_{B(p,2R)\backslash B(p,R)} g^2 e^{-f} dv \to 0$ for sufficient large R.*

Proof. For simplicity we omit the notation $e^{-f} dv$ in the following integrals. By the Hölder inequality, we have

$$\int_{B(p,2R)\backslash B(p,R)} g^2 \leq \left[\int_{B(p,2R)\backslash B(p,R)} \exp\left(-2(k-2)\sqrt{\lambda_{1,f}}\, r\right) \right]^{\frac{1}{k-1}} \tag{18}$$

$$\times\left[\int_{B(p,2R)\backslash B(p,R)}|\nabla u|^2\exp\left(2\sqrt{\lambda_{1,f}}\,r\right)\right]^{\frac{k-2}{k-1}},$$

where $\lambda_{1,f}=\lambda_{1,f}(M\backslash B(p,R_0))$. Below we shall estimate the first term of the right hand side of (18). Under our assumption $\partial_r f\geq-\alpha$, we have Wei-Wylie's comparison [33] of the weighted area of geodesic sphere

$$A_f(B(p,r))\leq Ce^{\alpha r}A_\Theta(B(r))\leq C\exp\left[\left(\alpha+(n-1)\sqrt{-\Theta}\right)r\right],$$

where $A_\Theta(B(r))$ denotes the area of geodesic sphere of radius r in the n-dimensional constant curvature space form with non-negative sectional curvature given by

$$\Theta:=-\frac{k-1}{(n-1)(k-2)}\lambda_{1,f}(M\backslash B(p,R_0))+\frac{\alpha^2}{(n-1)(k-n)}+\frac{\epsilon}{n-1}.$$

Using this estimate, we have

$$\begin{aligned}
&\int_{B(p,2R)\backslash B(p,R)}\exp\left(-2(k-2)\sqrt{\lambda_{1,f}}r\right)\\
(19)\quad&\leq C\int_R^{2R}\exp\left(-2(k-2)\sqrt{\lambda_{1,f}}r\right)\exp\left(\left(\alpha+(n-1)\sqrt{-\Theta}\right)r\right)dr\\
&< C\int_R^{2R}\exp\left[\left(-2(k-2)\sqrt{\lambda_{1,f}}+\alpha+\sqrt{\frac{(k-1)(n-1)}{k-2}}\sqrt{\lambda_{1,f}}\right)r\right]dr.
\end{aligned}$$

Since $k\geq n+\frac{\alpha}{2}\lambda_{1,f}(M\backslash B(p,R_0))^{-1/2}$, we directly check that

$$-2(k-2)\sqrt{\lambda_{1,f}}+\alpha+\sqrt{\frac{(k-1)(n-1)}{k-2}}\sqrt{\lambda_{1,f}}<0 \tag{20}$$

and the right hand side of (19) exponentially decays to 0. Moreover, by Lemma 4.4, the second term of the right hand side of (18) can be estimated by

$$\int_{B(p,2R)\backslash B(p,R)}|\nabla u|^2\exp\left(2\sqrt{\lambda_{1,f}}\,r\right)\leq CR$$

for sufficiently large R. Combining these, the desired conclusion follows. □

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. From the preceding discussion, it sufficient to estimate dimK. Under our theorem assumptions, by Lemma 3.2, $g=|\nabla u|^{\frac{k-2}{k-1}}$ satisfies

$$\Delta_f g\geq\left(\tfrac{k-2}{k-1}\epsilon-\lambda_{1,f}\right)g \tag{21}$$

on $M\backslash B(p,R_0)$, where $k\geq n+\frac{\alpha}{2}\lambda_{1,f}^{-1/2}$ and $\lambda_{1,f}:=\lambda_{1,f}(M\backslash B(p,R_0))$.

On the other hand, we consider any non-negative smooth cut-off function ψ supported in $M\backslash B(p,R_0)$. Then,

$$\int_M |\nabla(\psi g)|^2 = \int_M |\nabla\psi|^2 g^2 + \int_M \psi^2|\nabla g|^2 + 2\int_M \psi g \nabla\psi\nabla g. \tag{22}$$

We remark that the above and the following integrals are all discussed with respect to the weighted measure $e^{-f}dv$. Here we suppress the notation. Since

$$2\int_M \psi g\nabla\psi\nabla g = -\int_M \psi^2|\nabla g|^2 - \int_M \psi^2 g\,\Delta_f g,$$

by (22) and (21), we have

$$\begin{aligned}\int_M |\nabla(\psi g)|^2 &= \int_M |\nabla\psi|^2 g^2 - \int_M \psi^2 g\,\Delta_f g\\ &\le \int_M |\nabla\psi|^2 g^2 + \left(\lambda_{1,f} - \tfrac{k-2}{k-1}\epsilon\right)\int_M \psi^2 g^2.\end{aligned} \tag{23}$$

The variational property of $\lambda_{1,f}$ gives $\lambda_{1,f}\int_M \psi^2 g^2 \le \int_M |\nabla(\psi g)|^2$ for any non-negative smooth cut-off function ψ supported in $M\backslash B(p,R_0)$. Combining this with (23) yields

$$\tfrac{k-2}{k-1}\epsilon\int_M \psi^2 g^2 \le \int_M |\nabla\psi|^2 g^2$$

for any non-negative smooth cut-off function ψ supported in $M\backslash B(p,R_0)$. In particular we choose ψ such that

$$\psi(x) = \begin{cases} 0 & x\in B(p,R_0),\\ 1 & x\in B(p,R)\backslash B(p,2R_0),\\ 0 & x\in M\backslash B(p,2R),\end{cases}$$

$|\nabla\psi| \le C/R_0$ on $B(p,2R_0)\backslash B(p,R_0)$ and $|\nabla\psi|\le C/R$ on $B(p,2R)\backslash B(p,R)$ for some constant $C>0$. Then,

$$\tfrac{k-2}{k-1}\epsilon\int_{M\backslash B(p,2R_0)} g^2 \le \frac{C}{R_0^2}\int_{B(p,2R_0)\backslash B(p,R_0)} g^2,$$

where we used Lemma 7.1. Using this, we have

$$\begin{aligned}\int_{B(p,3R_0)} g^2 &= \int_{B(p,2R_0)} g^2 + \int_{B(p,2R_0)\backslash B(p,2R_0)} g^2\\ &\le \left(1+\frac{C(k-1)}{(k-2)\epsilon R_0^2}\right)\int_{B(p,2R_0)} g^2.\end{aligned} \tag{24}$$

On $B(p,3R_0)\subset M$, $\mathrm{Ric}_f \ge -\Lambda$, where $\Lambda := \sup_{B(p,3R_0)}|\mathrm{Ric}_f|$. Also f is smooth on M and hence $|\nabla f| \le A'(3R_0)$ for some constant $A'(3R_0)$ depending only on p and $3R_0$. Indeed, we can let $A'(3R_0) := \sup_{B(p,3R_0)}|\nabla f|$. Combining these restrictions and Lemma 3.2, g satisfies $\Delta_f g \ge -\lambda g$ for some constant

$\lambda(\Lambda, A'(3R_0), n, k) \geq 0$ on $B(p, 3R_0)$. By the f-mean value inequality, Proposition 3.3, for any $x \in B(p, 2R_0)$, we have

$$g^2(x) \leq \frac{C}{V_f(x, R_0)} \int_{B(x,R_0)} g^2 \leq \frac{C}{V_f(x, R_0)} \int_{B(p,3R_0)} g^2,$$

where C is a constant depending only on n, k, $A'(3R_0)$, Λ and R_0. Combining this with (24) gives that

$$\sup_{B(p,2R_0)} g^2 \leq C \int_{B(p,2R_0)} g^2,$$

where C is a constant depending only on n, k, R_0, ϵ, $\inf_{x \in B(p,2R_0)} V_f(x, R_0)$, $A'(3R_0)$ and Λ.

On the other hand, the Cauchy-Schwarz inequality gives that

$$\int_{B(p,2R_0)} g^2 \leq \left(\int_{B(p,2R_0)} |\nabla u|^2 \right)^{\frac{k-2}{k-1}} V_f(p, 2R_0)^{\frac{1}{k-1}}.$$

Therefore,

$$\sup_{B(p,2R_0)} |\nabla u|^2 \leq C \int_{B(p,2R_0)} |\nabla u|^2, \tag{25}$$

where C is a constant depending only on n, k, R_0, ϵ, $\inf_{x \in B(p,2R_0)} V_f(x, R_0)$, $A'(3R_0)$, Λ and $V_f(p, 2R_0)$. Here if function u is not constant, then it implies $\int_{B(p,2R_0)} |\nabla u|^2 \neq 0$ by unique continuation. Hence the bilinear form $\int_{B(p,2R_0)} \langle \nabla u_1 \nabla u_2 \rangle$ is non-degenerate on the linear space of one forms $\widetilde{\mathrm{K}} := \{du | u \in \mathrm{K}\}$. Using Lemma 11 of [14], there exists $du_0 \in \widetilde{\mathrm{K}} \backslash \{0\}$ such that

$$\dim \widetilde{\mathrm{K}} \int_{B(p,2R_0)} |du_0|^2 \leq V_f(p, 2R_0) \cdot \Big(\min\{n, \dim \widetilde{\mathrm{K}}\} \Big) \cdot \sup_{B(p,2R_0)} |du_0|^2,$$

which, combining with (25), gives $\dim \mathrm{K} = \dim \widetilde{\mathrm{K}} + 1 \leq C$, where constant C depends on n, k, R_0, ϵ, $A'(3R_0)$, Λ, $\inf_{x \in B(p,2R_0)} V_f(x, R_0)$ and $V_f(p, 2R_0)$. □

8. Dimension of $H^1(L^2_f(M))$

In this section, we study the dimension of space of L^2_f-harmonic one-forms. A differential form ω is said to be L^2_f if $\int_M |\omega|^2 e^{-f} dv < \infty$. The formal adjoint of exterior derivative d with respect to the L^2_f-inner product is given by $\delta_f := \delta + \iota_{\nabla f}$, where $\iota_{\nabla f}$ is the interior product with the vector field ∇f. The associated f-Hodge Laplacian is defined by $\Delta_f := -(d\delta_f + \delta_f d)$. The first L^2_f-cohomology of M, denoted by $H^1(L^2_f(M))$, is the space of L^2_f-harmonic one-forms, that is, it is the set of all L^2_f one-forms satisfying $\Delta_f \omega = 0$.

When $\lambda_{1,f}(M) > 0$, we will study the dimension of $H^1(L^2_f(M))$ on SMMSs with a more restriction on Ric_f. Before stating the result, we recall a weighted version of Li-Wang's integral inequality [18].

Lemma 8.1. *Let $(M, g, e^{-f}dv)$ be an n-dimensional SMMS. Let u be a non-negative function in M satisfying*

$$u\Delta_f u \geq -au^2 + b|\nabla u|^2,$$

where a and $b \geq 0$ are constants. Then for any $\delta > 0$ and a smooth cut-off function ψ supported in M, we have

$$\int_M |\nabla(\psi u)|^2 e^{-f} dv \leq \frac{(1+\delta)a}{1+\delta(1+b)} \int_M (\psi u)^2 e^{-f} dv + \left(1 + \frac{\delta^2 b}{1+\delta(1+b)}\right) \int_M |\nabla \psi|^2 u^2 e^{-f} dv.$$

We now give an exact description for the first L^2_f-cohomology of M. This generalizes Li-Wang's result [18] to the weighted case. The result is stronger than Theorem 1.4.

Theorem 8.2. *Let $(M, g, e^{-f}dv)$ be an n-dimensional SMMS. Assume that $|\langle \nabla f, \omega\rangle| \leq \alpha|\omega|$ for any f-harmonic one-form ω, where $\alpha \geq 0$ is a constant. If $\lambda_{1,f}(M) > 0$ and*

$$\mathrm{Ric}_f \geq -\frac{k}{k-1}\lambda_{1,f}(M) + \frac{\alpha^2}{k-n} + \epsilon$$

for some $k \geq n$ and some $\epsilon > 0$, then $H^1(L^2_f(M)) = \{0\}$.

Remark 8.3. Since the exterior differential of a f-harmonic function with finite f-integral is an L^2_f harmonic one-form, the number of f-non-parabolic ends is less than $\dim H^1(L^2_f(M)) + 1$. If further $\lambda_{1,f}(M) > 0$, then number of infinite f-volume ends is less than $\dim H^1(L^2_f(M)) + 1$. Hence our estimate on $H^1(L^2_f(M))$ is stronger than the estimate on the number of ends with infinite f-volume.

Proof of Theorem 8.2. Let $\omega \in H^1(L^2_f(M))$. By Lemma 3.1 in [32], the length of ω, denoted by $u = |\omega|$, satisfies the Bochner type formula $u\Delta_f u + |\nabla u|^2 = \mathrm{Ric}_f(\omega, \omega) + |\nabla \omega|^2$. By Lemma 3.2 in [32], we see that, for any $\omega \in H^1(L^2_f(M))$,

$$|\nabla \omega|^2 \geq \frac{1}{n-1}\Big(|\nabla u| - \langle \nabla f, \omega\rangle\Big)^2 + |\nabla u|^2.$$

Using the assumption $|\langle \nabla f, \omega\rangle| \leq \alpha u$ and the Cauchy-Schwarz inequality, we further get

$$|\nabla \omega|^2 \geq \frac{1}{n-1}\left(\frac{|\nabla u|^2}{1+\frac{k-n}{n-1}} - \frac{\langle \nabla f, \omega\rangle^2}{\frac{k-n}{n-1}}\right) + |\nabla u|^2 \geq \frac{k}{k-1}|\nabla u|^2 - \frac{\alpha^2}{k-n}u^2.$$

for some $k \geq n$. Hence

$$u\Delta_f u \geq \mathrm{Ric}_f(\omega, \omega) + \frac{1}{k-1}|\nabla u|^2 - \frac{\alpha^2}{k-n}u^2. \tag{26}$$

Combining the curvature assumption of theorem, we have

$$u\Delta_f u \geq \left(-\frac{k}{k-1}\lambda_{1,f}(M)+\epsilon\right)u^2+\frac{1}{k-1}|\nabla u|^2.$$

Applying this inequality to Lemma 8.1 by letting $a=-\frac{k}{k-1}\lambda_{1,f}(M)+\epsilon$ and $b=\frac{1}{k-1}$, we get

$$\begin{aligned}\int_M |\nabla(\psi u)|^2 e^{-f}dv \leq &\left[\frac{k(1+\delta)\lambda_{1,f}(M)}{(k-1)+\delta k}-\frac{\epsilon(k-1)(1+\delta)}{(k-1)+\delta k}\right]\int_M(\psi u)^2e^{-f}dv\\ &+\frac{(k-1)+\delta(\delta+k)}{(k-1)+\delta k}\int_M|\nabla\psi|^2u^2e^{-f}dv.\end{aligned}$$

Below all integrals are considered with respect to $e^{-f}dv$. By the variational principle of $\lambda_{1,f}(M)$, we have $\lambda_{1,f}(M)\int_M(\psi u)^2 \leq \int_M|\nabla(\psi u)|^2$. Combining these yields

$$\big[\epsilon(k-1)(1+\delta)-\lambda_{1,f}(M)\big]\int_M(\psi u)^2 \leq \big[(k-1)+\delta(\delta+k)\big]\int_M|\nabla\psi|^2u^2.$$

For $R>0$, we let ψ be $\psi=1$ on $B(p,R)$, $\psi=0$ outside $B(p,2R)$ and $|\nabla\psi|\leq C/R$ on $B(p,2R)\backslash B(p,R)$. Hence, the above inequality becomes

$$[\epsilon(k-1)(1+\delta)-\lambda_{1,f}(M)]\int_{B(p,R)}u^2 \leq \frac{C\,[(k-1)+\delta(\delta+k)]}{R}\int_{B(p,2R)\backslash B(p,R)}u^2.$$

Since $u\in L^2_f(M)$, letting $R\to\infty$, the right hand side tends to 0 and hence $u\equiv 0$ by choosing δ sufficiently large such that $\epsilon(k-1)(1+\delta)>\lambda_{1,f}(M)$. The theorem follows. □

If a weak condition of Ric_f is given on SMMSs with certain Sobolev inequality, we also prove a finite dimensional property of $H^1(L^2_f(M))$, i.e., Theorem 1.5 in introduction.

Proof of Theorem 1.5. Since any end E of M satisfies the Sobolev inequality, from Chapter 20 of [15], we know that each end is f-non-parabolic. Moreover, by Remark 8.3, to show our theorem, it suffices to prove $\dim H^1(L^2_f(M))<\infty$.

From the proof of Theorem 8.2, we see that $u=|\omega|$, where $\omega\in H^1(L^2_f(M))$, satisfies the Bochner formula (26). Combining our curvature assumption, we get that

$$u\Delta_f u \geq -\tau(x)u^2+\frac{1}{k-1}|\nabla u|^2.$$

for some constant $k>n$. For a geodesic ball $B(p,R_0)$ and any non-negative supported function $\psi\in C_0^\infty(M\backslash B(p,R_0))$, multiplying by ψ^2 in the above inequality and integrating by parts, we have

$$\begin{aligned}&\frac{1}{k-1}\int_{M\setminus B(p,R_0)}\psi^2|\nabla u|^2\\ &\leq -\int_{M\setminus B(p,R_0)}2\psi u\nabla\psi\nabla u+\int_{M\setminus B(p,R_0)}\tau^2\psi^2u^2.\end{aligned} \tag{27}$$

Notice that,

$$\int_{M\setminus B(p,R_0)}2\psi u\nabla\psi\nabla u\leq\epsilon\int_{M\setminus B(p,R_0)}|\nabla u|^2\psi^2+\epsilon^{-1}\int_{M\setminus B(p,R_0)}|\nabla\psi|^2u^2, \tag{28}$$

where $\epsilon>0$ is a constant that will be determined later. Also notice that

$$\begin{aligned}\int_{M\setminus B(p,R_0)}\tau^2\psi^2u^2&\leq\left(\int_{M\setminus B(p,R_0)}\tau^{\frac{\nu}{\nu-2}}\right)^{\frac{\nu-2}{\nu}}\left(\int_{M\setminus B(p,R_0)}(\psi u)^p\right)^{\frac{2}{\nu}}\\ &\leq C_s\left(\int_{M\setminus B(p,R_0)}\tau^{\frac{\nu}{\nu-2}}\right)^{\frac{\nu-2}{\nu}}\int_{M\setminus B(p,R_0)}|\nabla(\psi u)|^2,\end{aligned}$$

where we used the Sobolev inequality in the last inequality. Since $\int_M\tau^{\frac{\nu}{\nu-2}}<\infty$, we can choose $R_0>1$ sufficiently large such that

$$\eta:=C_s\left(\int_{M\setminus B(p,R_0)}\tau^{\frac{\nu}{\nu-2}}\right)^{\frac{\nu-2}{\nu}}<\frac{1}{k-1}<1.$$

Therefore,

$$\int_{M\setminus B(p,R_0)}\tau^2\psi^2u^2\leq\eta\int_{M\setminus B(p,R_0)}|\nabla(\psi u)|^2. \tag{29}$$

Substituting (29) and (28) into (27) yields

$$\begin{aligned}&\frac{1}{k-1}\int_{M\setminus B(p,R_0)}|\nabla u|^2\psi^2\\ &\leq 2(\eta-1)\int_{M\setminus B(p,R_0)}\psi u\nabla\psi\nabla u+\eta\int_{M\setminus B(p,R_0)}|\nabla\psi|^2u^2+\eta\int_{M\setminus B(p,R_0)}|\nabla u|^2\psi^2\\ &\leq(\eta+(1-\eta)\epsilon)\int_{M\setminus B(p,R_0)}|\nabla u|^2\psi^2+\big((1-\eta)\epsilon^{-1}+\eta\big)\int_{M\setminus B(p,R_0)}|\nabla\psi|^2u^2.\end{aligned}$$

Since $\eta<\frac{1}{k-1}$, we may choose ϵ sufficiently small such that $\eta+(1-\eta)\epsilon<\frac{1}{k-1}$. Hence the preceding estimate implies that

$$\int_{M\setminus B(p,R_0)}|\nabla u|^2\psi^2\leq C_1\int_{M\setminus B(p,R_0)}|\nabla\psi|^2u^2$$

for some constant $C_1 > 0$. Moreover, the Sobolev inequality assumption implies that

$$\begin{aligned}\left(\int_{M\backslash B(p,R_0)}(\psi u)^\nu\right)^{2/\nu} &\le C_s\int_{M\backslash B(p,R_0)}|\nabla(\psi u)|^2\\ &\le 2C_s\int_{M\backslash B(p,R_0)}|\nabla\psi|^2u^2+2C_s\int_{M\backslash B(p,R_0)}|\nabla u|^2\psi^2.\end{aligned}$$

Combining the above two estimates, we obtain that

$$\left(\int_{M\backslash B(p,R_0)}(\psi u)^\nu\right)^{2/\nu} \le C_2\int_{M\backslash B(p,R_0)}|\nabla\psi|^2u^2 \tag{30}$$

for some constant $C_2 > 0$.

For any $R > 2R_0 > 2$, we choose ψ such that

$$\psi(x)=\begin{cases}0 & x\in B(p,R_0),\\ 1 & x\in B(p,R)\backslash B(p,2R_0),\\ 0 & x\in M\backslash B(p,2R).\end{cases}$$

The above cut-off function ψ simultaneously satisfies $|\nabla\psi| \le C_3/R_0 < C_3$ on $B(p,2R_0)\backslash B(p,R_0)$ and $|\nabla\psi|\le C_3/R$ on $B(p,2R)\backslash B(p,R)$ for some constant $C_3 > 0$. Applying this function ψ to (30) yields

$$\left(\int_{B(p,R)\backslash B(p,2R_0)}u^\nu\right)^{2/\nu} \le C_4\int_{B(p,2R_0)\backslash B(p,R_0)}u^2+\frac{C_4}{R^2}\int_{B(p,2R)\backslash B(p,R)}u^2$$

for some constant $C_4 > 0$. Since $u\in L^2_f(M)$, letting $R\to\infty$, the second term of the right hand side tends to 0 and hence

$$\left(\int_{M\backslash B(p,2R_0)}u^\nu\right)^{2/\nu} \le C_4\int_{B(p,2R_0)\backslash B(p,R_0)}u^2.$$

Notice that the Cauchy-Schwarz inequality implies that

$$\int_{B(p,2R_0)\backslash B(p,R_0)}u^2 \le V_f(p,2R_0)^{2/\nu}\left(\int_{B(p,2R_0)\backslash B(p,R_0)}u^\nu\right)^{2/\nu}.$$

Together these yields that

$$\int_{B(p,2R_0)}u^2\le C_5\int_{B(p,R_0)}u^2 \tag{31}$$

for some constant $C_5 > 0$ depending on $V_f(p,2R_0)$ and ν.

On the other hand, since $\mathrm{Ric}_f \ge -\Lambda$, where $\Lambda := \sup_{B(p,2R_0)}|\mathrm{Ric}_f|$ on $B(p,2R_0)$, then

$$\Delta_f u^2 = 2\mathrm{Ric}_f(\omega,\omega)+2|\nabla\omega|^2\ge -2\Lambda u^2$$

on $B(p, 2R_0)$. By the f-mean value inequality, i.e., Proposition 3.3, for any $x \in B(p, R_0)$,

$$u^2(x) \le \frac{C_6}{V_f(x, R_0)} \int_{B(x,R_0)} u^2 \le \frac{C_6}{V_f(x, R_0)} \int_{B(p,2R_0)} u^2,$$

where C_6 is a constant depending on n, R_0, $A'(2R_0)$ and Λ. Combining this with (31) yields $\sup_{B(p,R_0)} u^2 \le C_7 \int_{B(p,R_0)} u^2$, where C_7 is a constant depending on n, R_0, $A'(2R_0)$, Λ, and $\inf_{x \in B(p,R_0)} V_f(x, R_0)$. This inequality is the same as (25). So, we can follow the argument of Theorem 1.4 and finally prove $\dim H^1(L^2_f(M)) < \infty$. □

Finally we give another version of Theorem 1.5 when only Ric_f^m ($m < \infty$) is bounded below (without any assumption on f). Its proof is similar to the case of Ric_f so we omit the repeated proof here.

Theorem 8.4. *Let $(M, g, e^{-f} dv)$ be an n-dimensional SMMS with a Sobolev inequality*

$$\left(\int_M |\phi|^\nu e^{-f} dv \right)^{2/\nu} \le C_s \int_M |\nabla \phi|^2 e^{-f} dv$$

for all $\phi \in C_0^\infty(M)$, where $\nu > 2$ and $C_s > 0$ are constants. If $\mathrm{Ric}_f^m \ge -\tau(x)$ for some non-negative function $\tau \in C^\infty(M)$ with $\int_M \tau^{\frac{\nu}{\nu-2}} e^{-f} dv < \infty$, then M has finitely many ends.

References

[1] M. T. Anderson, *The Dirichlet problem at infinity for manifolds of negative curvature*, J. Differential Geom. **18** (1983), no. 4, 701–721 (1984). http://projecteuclid.org/euclid.jdg/1214438178

[2] M. T. Anderson and R. Schoen, *Positive harmonic functions on complete manifolds of negative curvature*, Ann. of Math. (2) **121** (1985), no. 3, 429–461. https://doi.org/10.2307/1971181

[3] D. Bakry and M. Emery, *Diffusion hypercontractivitives*, Séminaire de Probabilités XIX, 1983/1984, Lecture Notes in Math., Vol. 1123, Springer-Verlag, Berlin, 1985, pp. 177–206.

[4] K. Brighton, *A Liouville-type theorem for smooth metric measure spaces*, J. Geom. Anal. **23** (2013), no. 2, 562–570. https://doi.org/10.1007/s12220-011-9253-5

[5] J. Case, Y.-J. Shu, and G. Wei, *Rigidity of quasi-Einstein metrics*, Differential Geom. Appl. **29** (2011), no. 1, 93–100. https://doi.org/10.1016/j.difgeo.2010.11.003

[6] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Differential Geometry **6** (1971/72), 119–128. http://projecteuclid.org/euclid.jdg/1214430220

[7] X. Cheng and D. Zhou, *Eigenvalues of the drifted Laplacian on complete metric measure spaces*, Commun. Contemp. Math. **19** (2017), no. 1, 1650001, 17 pp. https://doi.org/10.1142/S0219199716500012

[8] N. T. Dung, N. N. Khanh, and T. C. Son, *The number of cusps of complete Riemannian manifolds with finite volume*, Taiwanese J. Math. **22** (2018), no. 6, 1403–1425. https://doi.org/10.11650/tjm/180604

[9] N. T. Dung and C.-J. A. Sung, *Smooth metric measure spaces with weighted Poincaré inequality*, Math. Z. **273** (2013), no. 3-4, 613–632. https://doi.org/10.1007/s00209-012-1023-y
[10] F. Fang, X. D. Li, and Z. Zhang, *Two generalizations of Cheeger-Gromoll splitting theorem via Bakry-Emery Ricci curvature*, Ann. Inst. Fourier (Grenoble) **59** (2009), no. 2, 563–573. https://doi.org/10.5802/aif.2440
[11] R. S. Hamilton, *The formation of singularities in the Ricci flow*, Surveys in Differential Geometry, Vol. II (Cambridge, MA, 1993), 7–136, Int. Press, Cambridge, MA, 1995.
[12] B. Hua and Z. Lu, *Graphs with positive spectrum*, J. Lond. Math. Soc. (2) **107** (2023), no. 6, 2054–2078. https://doi.org/10.1112/jlms.12733
[13] B. Hua and J.-Y. Wu, *Gap theorems for ends of smooth metric measure spaces*, Proc. Amer. Math. Soc. **150** (2022), no. 11, 4947–4957. https://doi.org/10.1090/proc/16022
[14] P. Li, *On the Sobolev constant and the p-spectrum of a compact Riemannian manifold*, Ann. Sci. École Norm. Sup. (4) **13** (1980), no. 4, 451–468. http://www.numdam.org/item?id=ASENS_1980_4_13_4_451_0
[15] P. Li, *Geometric Analysis*, Cambridge Studies in Advanced Mathematics, 134, Cambridge Univ. Press, Cambridge, 2012. https://doi.org/10.1017/CBO9781139105798
[16] P. Li and L.-F. Tam, *Symmetric Green's functions on complete manifolds*, Amer. J. Math. **109** (1987), no. 6, 1129–1154. https://doi.org/10.2307/2374588
[17] P. Li and L.-F. Tam, *Harmonic functions and the structure of complete manifolds*, J. Differential Geom. **35** (1992), no. 2, 359–383. http://projecteuclid.org/euclid.jdg/1214448079
[18] P. Li and J. Wang, *Complete manifolds with positive spectrum*, J. Differential Geom. **58** (2001), no. 3, 501–534. http://projecteuclid.org/euclid.jdg/1090348357
[19] P. Li and J. Wang, *Weighted Poincaré inequality and rigidity of complete manifolds*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 6, 921–982. https://doi.org/10.1016/j.ansens.2006.11.001
[20] P. Li and J. Wang, *Counting cusps on complete manifolds of finite volume*, Math. Res. Lett. **17** (2010), no. 4, 675–688.
[21] A. Lichnerowicz, *Variétés riemanniennes à tenseur C non négatif*, C. R. Acad. Sci. Paris Sér. A-B **271** (1970), A650–A653.
[22] J. Lott, *Some geometric properties of the Bakry-Émery-Ricci tensor*, Comment. Math. Helv. **78** (2003), no. 4, 865–883. https://doi.org/10.1007/s00014-003-0775-8
[23] O. Munteanu and J. Wang, *Smooth metric measure spaces with non-negative curvature*, Comm. Anal. Geom. **19** (2011), no. 3, 451–486. https://doi.org/10.4310/CAG.2011.v19.n3.a1
[24] O. Munteanu and J. Wang, *Analysis of weighted Laplacian and applications to Ricci solitons*, Comm. Anal. Geom. **20** (2012), no. 1, 55–94. https://doi.org/10.4310/CAG.2012.v20.n1.a3
[25] O. Munteanu and J. Wang, *Geometry of manifolds with densities*, Adv. Math. **259** (2014), 269–305. https://doi.org/10.1016/j.aim.2014.03.023
[26] S. Pigola, M. Rigoli, and A. G. Setti, *Vanishing theorems on Riemannian manifolds, and geometric applications*, J. Funct. Anal. **229** (2005), no. 2, 424–461. https://doi.org/10.1016/j.jfa.2005.05.007
[27] Z. Qian, *Estimates for weighted volumes and applications*, Quart. J. Math. Oxford Ser. (2) **48** (1997), no. 190, 235–242. https://doi.org/10.1093/qmath/48.2.235
[28] R. M. Schoen and S.-T. Yau, *Lectures on Differential Geometry*, translated from the Chinese by Ding and S. Y. Cheng, with a preface translated from the Chinese by Kaising Tso, Conference Proceedings and Lecture Notes in Geometry and Topology, I, Int. Press, Cambridge, MA, 1994.

[29] Y.-H. Su and H.-C. Zhang, *Rigidity of manifolds with Bakry-Émery Ricci curvature bounded below*, Geom. Dedicata **160** (2012), 321–331. https://doi.org/10.1007/s10711-011-9685-x
[30] D. Sullivan, *The Dirichlet problem at infinity for a negatively curved manifold*, J. Differential Geom. **18** (1983), no. 4, 723–732 (1984). http://projecteuclid.org/euclid.jdg/1214438179
[31] J. Tang and J.-Y. Wu, *Cheeger-Gromoll splitting theorem for the Bakry-Emery Ricci tensor*, Arch. Math. (Basel) **117** (2021), no. 6, 697–708. https://doi.org/10.1007/s00013-021-01658-1
[32] M. Vieira, *Harmonic forms on manifolds with non-negative Bakry-Émery-Ricci curvature*, Arch. Math. (Basel) **101** (2013), no. 6, 581–590. https://doi.org/10.1007/s00013-013-0594-0
[33] G. Wei and W. Wylie, *Comparison geometry for the Bakry-Emery Ricci tensor*, J. Differential Geom. **83** (2009), no. 2, 377–405. https://doi.org/10.4310/jdg/1261495336
[34] J.-Y. Wu, *Counting ends on complete smooth metric measure spaces*, Proc. Amer. Math. Soc. **144** (2016), no. 5, 2231–2239. https://doi.org/10.1090/proc/12982
[35] J.-Y. Wu and P. Wu, *Heat kernel on smooth metric measure spaces with nonnegative curvature*, Math. Ann. **362** (2015), no. 3-4, 717–742. https://doi.org/10.1007/s00208-014-1146-z
[36] J.-Y. Wu and P. Wu, *Heat kernel on smooth metric measure spaces and applications*, Math. Ann. **365** (2016), no. 1-2, 309–344. https://doi.org/10.1007/s00208-015-1289-6
[37] W. Wylie, *A warped product version of the Cheeger-Gromoll splitting theorem*, Trans. Amer. Math. Soc. **369** (2017), no. 9, 6661–6681. https://doi.org/10.1090/tran/7003
[38] S.-T. Yau, *Harmonic functions on complete Riemannian manifolds*, Comm. Pure Appl. Math. **28** (1975), 201–228. https://doi.org/10.1002/cpa.3160280203

Xuenan Fu
Department of Mathematics
Shanghai University
Shanghai 200444, P. R. China
Email address: xuenanfu97@163.com

Jia-Yong Wu
Department of Mathematics and Newtouch Center for Mathematics,
Shanghai University
Shanghai 200444, P. R. China
Email address: wujiayong@shu.edu.cn