Communications for Statistical Applications and Methods
2024, Vol. 31, No. 6, 661–675

https://doi.org/10.29220/CSAM.2024.31.6.661
Print ISSN 2287-7843 / Online ISSN 2383-4757

Estimation of length biased exponential distribution based on progressive hybrid censoring

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Abstract

The concept of length-biased distribution find various applications in biomedical area such as family history and disease, survival and intermediate events and latency period of AIDS due to blood transfusion. Also, there are many situations in biomedical analysis in which units are removed or lost from experimentation before observed. In this paper, therefore, we consider the maximum likelihood estimator (MLE) and Bayesian estimators of the unknown parameter, reliability and hazard functions of the length biased exponential distribution (LBED) under progressive hybrid censoring (PHC) scheme. We derive the Bayesian estimators of the unknown parameter, reliability and hazard functions based on flexible loss functions. Additionally, we derive the Bayesian estimators using the Lindley's approximation and Markov chain Mote Carlo (MCMC) methods. In particular, the MCMC method is used to obtain the credible interval. To compare the proposed estimators, the Monte Carlo simulation method is conducted. Finally, the leukemia patients dataset based on PHC scheme is analyzed.

Keywords: Bayesian estimation, exponential distribution, length biased exponential distribution, progressive hybrid censoring

1. Introduction

In epidemiological studies, a common objective is to estimate the distribution of intervals from initiation to endpoints or to compare the distributions of these survival times across two or more well-defined groups. When it is feasible to follow all subjects in a group prospectively, standard techniques of survival analysis can be applied. However, subjects are often identified as having experienced initiation through a cross-sectional study at a specific time point. Consequently, only those who have survived until that time are recruited into the study, whereas those who have not will not be included in this initial recruitment phase and, indeed, will not even be identified.

The intervals from initiation to failure or censoring are well known to be length-biased, meaning that the observed time intervals tend to be longer than those arising from the true underlying failure or censoring distributions. The concept of length-biased distribution finds various applications in the biomedical field, such as in family history and disease, survival and intermediate events, and the latency period of AIDS due to blood transfusion (Gupta and Akman, 1998). Significant work has been done to characterize the relationships between original distributions and their length-biased versions.

If a random variable X follows any distribution with probability density function $f(x)$ then the probability density function of length biased distribution of X is defined as $g(x) = xf(x)/E(X)$. Then,

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Published 30 November 2024 / journal homepage: http://csam.or.kr

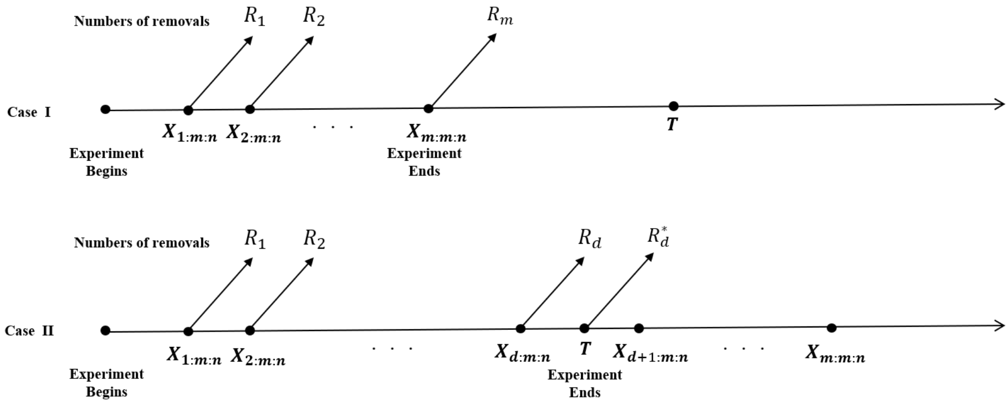

Figure 1: *Schematic representation of PHC scheme.*

if a random variable X follows an exponential distribution, X is said to have the LBED if its probability density function (pdf) and cumulative distribution function (cdf) are given by

$$\begin{aligned} g(x;\theta) &= \frac{x}{\theta^2}e^{-\frac{x}{\theta}}, \quad x>0,\ \theta>0, \\ G(x;\theta) &= 1-\left[\frac{x}{\theta}e^{-\frac{x}{\theta}}+e^{-\frac{x}{\theta}}\right], \quad x>0,\ \theta>0. \end{aligned} \tag{1.1}$$

The reliability and hazard function of the LBED are given by:

$$\mathrm{Re}(t) = e^{-\frac{t}{\theta}}\left(1+\frac{t}{\theta}\right), \tag{1.2}$$

$$H(t) = -\frac{1}{\theta^2}+\frac{1}{(\theta+t)^2}. \tag{1.3}$$

In reliability and survival analysis, there are many situations in which units are removed or lost from experimentation before they are observed, either carelessly or unconsciously. This scenario is referred to as censoring. Recently, work on progressive hybrid censoring (PHC) has become quite popular in life-testing and reliability studies. The PHC scheme can be described as follows. If the first failure is observed ($X_{1:m:n}$), the R_1 survival units are removed randomly from the test. And, if the second failure is observed ($X_{2:m:n}$), the R_2 survival units are removed randomly from the test. Finally, survival test is stopped at a mimimum time of $X_{m:m:n}$ and T, where $T \in (0,\infty)$ and integer m are pre-assigned. In PHC scheme, the total time required to terminate the experiment does not exceed T (Figure 1).

The estimation of length-biased distributions under censoring schemes has been extensively studied by many researchers. Alemdjrodo and Zhao (2020) considered empirical likelihood inference for the mean residual life with length-biased and right-censored data. Shen *et al.* (2022) provided maximum likelihood estimation for length-biased and interval-censored data with a nonsusceptible fraction. Alotaibi *et al.* (2022) investigated inference for a Kavya-Manoharan inverse length-biased exponential distribution under a progressive-stress model based on progressive type-II censoring. Chen and Qiu (2023) analyzed length-biased and partly interval-censored survival data with mismeasured covariates. Li *et al*. (2024) conducted a change point test for length-biased lognormal distributions under random right censoring.

The objective of this paper is to derive the MLE of the unknown parameter, reliability, and hazard function of a LBED based on PHC data. Furthermore, we consider the Bayesian estimators of the unknown parameter, reliability, and hazard function of LBED using flexible priors under PHC data. As Bayesian estimators cannot be obtained in closed form, we provide Bayesian estimates using Lindley's approximation method. Monte Carlo simulations are conducted to compare the performances among different methods.

The paper is organized as follows. In Section 2, we derive the MLE and confidence interval for the unknown parameter, reliability, and hazard functions under PHC data. In Section 3, we obtain the posterior densities of unknown parameter for this model and derive Bayesian estimators for the unknown parameter, reliability, and hazard functions under the squared error loss function (SELF) and linex loss function (LLF). A numerical study is presented in Section 4. Finally, Section 5 concludes the paper.

2. Maximum likelihood estimation

2.1. Maximum likelihood estimator

In this subsection, we provide the MLE of the unknown parameter, reliability and hazard functions. Consider a randomly selected sample of n units are put on test. The integer $m < n$ is pre-fixed and progressive censoring scheme $\boldsymbol{R} = (R_1, R_2, \ldots, R_m)$ are m pre-fixed integers satisfying $R_1 + \cdots + R_m = n$. At the time of first failure $X_{1:m:n}$, R_1 of the remaining units are randomly removed. If the m^{th} failure $X_{m:m:n}$ occurs before the time point T, the experiment stops at the time point $X_{m:m:n}$ (Case I). On the other hand suppose the m^{th} failure does not occur before time point T and only d failures occur before the time point T, where $0 < d < m$, then at the time point T all the remaining R_d^* units are removed and the experiment terminates at the time point T (Case II). Note that $R_d^* = n - (R_1 + \cdots + R_d) - d$. Based on the observed data, the likelihood function is

$$L(\theta) = K \prod_{i=1}^{\tau} \left[g\left(x_{i:m:n}\right) \left[1 - G\left(x_{i:m:n}\right)\right]^{R_i} \right] U_1\left(T, \theta\right), \tag{2.1}$$

where $\tau = m$ and $U_1(T, \theta) = 1$ for Case I, $\tau = d$ and $U_1(T, \theta) = [1 - G(T)]^{R_d^*}$ for Case II. The logarithm of (2.1), can be written (without the constant terms) as

$$\log L(\theta) = -2\tau \log \theta + \sum_{i=1}^{\tau} \left[\log x_{i:m:n} - \frac{x_{i:m:n}}{\theta} (1 + R_i) + R_i \log \left(1 + \frac{x_{i:m:n}}{\theta}\right) \right] + U_2(T, \theta), \tag{2.2}$$

where $U_2(T, \theta) = 0$ for Case I and $\tau = m$, $U_2(T, \theta) = -(1/\theta) T R_d^* + R_d^* \log\left(1 + (T/\theta)\right)$ for Case II. It is assumed that $d > 0$, otherwise the MLE does not exist. Taking derivative with respect to θ of (2.2) and equating them to zero we obtain

$$\frac{\partial \log L(\theta)}{\partial \theta} = -\frac{2\tau}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{\tau} x_{i:m:n} (1 + R_i) - \sum_{i=1}^{\tau} R_i x_{i:m:n} \frac{1}{\theta\left(x_{i:m:n} + \theta\right)} + U_3(T, \theta),$$

where $U_3(T,\theta) = 0$ for Case I and $U_3(T,\theta) = \frac{1}{\theta^2}TR_d^* - TR_d^*\frac{1}{\theta(T+\theta)}$ for Case II. Then, the MLE of θ can be obtained

$$\hat{\theta} = \frac{\sum_{i=1}^{\tau} x_{i:m:n}(1+R_i) + U_4(T,\theta)}{2\tau + \sum_{i=1}^{\tau}\frac{R_i x_{i:m:n}}{x_{i:m:n}+\theta} + \frac{U_4(T,\theta)}{T+\theta}}, \tag{2.3}$$

where $U_4(T,\theta) = 0$ for Case I and $U_4(T,\theta) = TR_d^*$ for Case II. Using the MLE of θ, the MLE of reliability and hazard functions are obtained as

$$\hat{Re}(t) = e^{-\frac{t}{\hat{\theta}}}\left(1+\frac{t}{\hat{\theta}}\right) \quad \text{and} \quad \hat{H}(t) = -\frac{1}{\hat{\theta}^2} + \frac{1}{\left(\hat{\theta}+t\right)^2}. \tag{2.4}$$

2.2. Approximate confidence interval

In this subsection, we obtain the $100(1-\alpha)\%$ approximate CI of the unknown parameter, reliability and hazard functions. By using equation (2.2), the second derivative of equation (2.1) with respect to unknown parameter is given by

$$\frac{\partial^2 \log L(\theta)}{\partial\theta^2} = \frac{2\tau}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{\tau} x_{i:m:n}(1+R_i) + \frac{1}{\theta^2}\sum_{i=1}^{\tau}\frac{R_i x_{i:m:n}(x_{i:m:n}+2\theta)}{(x_{i:m:n}+\theta)^2} + U_5(T,\theta), \tag{2.5}$$

where $U_5(T,\theta) = 0$ for Case I and $U_5(T,\theta) = -\frac{2}{\theta^3}TR_d^* + TR_d^*\left\{\frac{1}{\theta^2(T+\theta)} + \frac{1}{\theta(T+\theta)^2}\right\}$ for Case II. Under some mild regularity conditions, MLE of θ is approximately normal with mean θ and $\widehat{\text{Var}}(\hat{\theta})$. A simpler and equally valid procedure is to use the approximation

$$\hat{\theta} \sim N\left(\theta, \widehat{\text{Var}}(\hat{\theta})\right),$$

where

$$\widehat{\text{Var}}(\hat{\theta}) = \left[-\frac{2\tau}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{\tau} x_{i:m:n}(1+R_i) - \frac{1}{\theta^2}\sum_{i=1}^{\tau}\frac{R_i x_{i:m:n}(x_{i:m:n}+2\theta)}{(x_{i:m:n}+\theta)^2} - U_5(T,\theta)\right]^{-1}.$$

These results yield the $100(1-\alpha)\%$ approximate CI for the unknown parameter given by

$$\left(\hat{\theta} - Z_{\alpha/2}\sqrt{\widehat{\text{Var}}\left(\hat{\theta}\right)},\ \hat{\theta} + Z_{\alpha/2}\sqrt{\widehat{\text{Var}}\left(\hat{\theta}\right)}\right),$$

where $Z_{\alpha/2}$ is the percentile of the standard normal distribution with right-tail probability $\alpha/2$.

In order to find the approximate estimates of the variance of reliability and hazard functions, we use the delta method. Then, asymptotically,

$$\frac{\hat{\text{Re}}(t) - \text{Re}(t)}{\sqrt{\widehat{\text{Var}}\left(\hat{\text{Re}}(t)\right)}} \sim N(0,1) \quad \text{and} \quad \frac{\hat{H}(t) - H(t)}{\sqrt{\widehat{\text{Var}}\left(\hat{H}(t)\right)}} \sim N(0,1), \tag{2.6}$$

where

$$\widehat{\text{Var}}\left(\hat{\text{Re}}(t)\right) = \frac{t^4}{\theta^6}e^{-\frac{2t}{\theta}}\widehat{\text{Var}}\left(\hat{\theta}\right) \quad \text{and} \quad \widehat{\text{Var}}\left(\hat{H}(t)\right) = \left(\frac{1}{\theta^2} - \frac{1}{(\theta+t)^2}\right)^2\widehat{\text{Var}}\left(\hat{\theta}\right).$$

These results yield the $100(1-\alpha)\%$ approximate CIs for the reliability and hazard functions given by

$$\left(\hat{\text{Re}}(t) - Z_{\alpha/2}\sqrt{\widehat{\text{Var}}\left(\hat{\text{Re}}(t)\right)},\ \hat{\text{Re}}(t) + Z_{\alpha/2}\sqrt{\widehat{\text{Var}}\left(\hat{\text{Re}}(t)\right)}\right),$$
$$\left(\hat{H}(t) - Z_{\alpha/2}\sqrt{\widehat{\text{Var}}\left(\hat{H}(t)\right)},\ \hat{H}(t) + Z_{\alpha/2}\sqrt{\widehat{\text{Var}}\left(\hat{H}(t)\right)}\right).$$

3. Bayesian estimation

In this Section, we consider the Bayesian estimation for the parameter, reliability and hazard functions of the LBED under PHC. We obtain the Bayesian estimators under SELF and LLF. Additionally, we assume that the prior distribution of θ is exponential prior distribution with pdf given by

$$\pi(\theta \mid \beta) = \frac{1}{\beta}\exp\left(-\frac{\theta}{\beta}\right), \quad \theta > 0,\ \beta > 0.$$

Thus, the joint density function of θ and data is given by

$$\pi(\theta, X) \propto \theta^{-2\tau}\exp\left[-\frac{1}{\theta}\sum_{i=1}^{\tau} x_{i:m:n}(1+R_i) - \frac{\theta}{\beta}\right]\prod_{i=1}^{\tau}\left(1+\frac{x_{i:m:n}}{\theta}\right)U_6(T,\theta),$$

where $U_6(T,\theta) = 1$ for Case I and $\tau = d,\ U_6(T,\theta) = \left(1+\frac{T}{\theta}\right)^{R_d^*}\exp\left(-\frac{1}{\theta}TR_d^*\right)$ for Case II. Then, under a SELF, the Bayesian estimator of any function of θ, say $\mu(\theta)$, is

$$E(\mu(\theta) \mid X) = \frac{\int_0^\infty \mu(\theta)\pi(\theta,X)d\theta}{\int_0^\infty \pi(\theta,X)d\theta}. \tag{3.1}$$

Note that (3.1) cannot be obtained analytically. Therefore, we adopt Lindley's approximation (Lee, 2024).

Using the Lindley's approximation method, Bayesian estimators of $\mu(\theta)$ can be obtained as

$$E\left[\mu\left(\hat{\theta}\right) \mid X\right] = \mu\left(\hat{\theta}\right) + \frac{1}{2}\left[\left(\hat{\mu}_{\theta\theta} + 2\hat{\mu}_\theta\hat{\rho}_\theta\right)\hat{\sigma}_{\theta\theta}\right] + \frac{1}{2}\left[\left(\hat{\mu}_\theta\hat{\sigma}_{\theta\theta}\right)\left(\hat{L}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta}\right)\right], \tag{3.2}$$

where

$$\hat{\mu}_\theta = \frac{\partial\mu(\hat{\theta})}{\partial\hat{\theta}}, \quad \hat{\mu}_{\theta\theta} = \frac{\partial^2\mu(\hat{\theta})}{\partial\hat{\theta}^2}, \quad \hat{\rho}_\theta = \frac{\partial\pi(\hat{\theta})}{\partial\hat{\theta}}, \quad \hat{\sigma}_{\theta\theta} = -\frac{1}{\hat{L}_{\theta\theta}},$$
$$\hat{L}_{\theta\theta} = \frac{\partial^2\log L(\hat{\theta})}{\partial\hat{\theta}^2}, \quad \hat{L}_{\theta\theta\theta} = \frac{\partial^3\log L(\hat{\theta})}{\partial\hat{\theta}^3}.$$

For our problem, we have

$$\hat{L}_{\theta\theta} = \frac{2\tau}{\theta^2} - \frac{2}{\theta^3}\sum_{i=1}^{\tau} x_{i:m:n}(1+R_i) + \frac{1}{\theta^2}\sum_{i=1}^{\tau}\frac{R_i x_{i:m:n}(x_{i:m:n}+2\theta)}{(x_{i:m:n}+\theta)^2} + U_5(T,\theta),\ \hat{\rho}_\theta = -\frac{1}{\beta},$$
$$\hat{L}_{\theta\theta\theta} = -\frac{4m}{\hat{\theta}^3} + \frac{6}{\hat{\theta}^4}\sum_{i=1}^{\tau} x_{i:m:n}(1+R_i) - 2\sum_{i=1}^{\tau} R_i x_{i:m:n}\left[\frac{1}{(x_{i:m:n}+\hat{\theta})\hat{\theta}^3} + \frac{1}{\hat{\theta}^2(x_{i:m:n}+\hat{\theta})^2} + \frac{1}{\hat{\theta}(x_{i:m:n}+\hat{\theta})^3}\right]$$
$$+ U_7(T,\theta),$$

where $U_7(T,\theta) = 0$ for Case I and $U_7(T,\theta) = \frac{6}{\hat{\theta}^4} TR_d^* + TR_d^* \left[\frac{1}{\hat{\theta}^2 (T+\hat{\theta})} + \frac{1}{\hat{\theta}(T+\hat{\theta})^2} + \frac{1}{\hat{\theta}(T+\hat{\theta})^3} \right]$ for Case II.

Now, we compute the Bayesian estimator using Lindley approximation method. First, we estimate θ under LF, we observe that $\mu(\hat{\theta}) = e^{-h\hat{\theta}}$, $\hat{\mu}_\theta = -he^{-h\hat{\theta}}$, $\hat{\mu}_{\theta\theta} = h^2 e^{-h\hat{\theta}}$. Using (3.2), the Bayesian estimator of θ under LLF is obtained as

$$\hat{\theta}_{LL} = -\frac{1}{h} \log\left[e^{-h\hat{\theta}} + \frac{1}{2}\left\{ \left(h^2 e^{-h\hat{\theta}} - 2he^{-h\hat{\theta}} \hat{\rho}_\theta \right) \hat{\sigma}_{\theta\theta} \right\} - \frac{1}{2} he^{-h\hat{\theta}} \hat{L}_{\theta\theta\theta} \hat{\sigma}_{\theta\theta}^2 \right]. \tag{3.3}$$

The Bayesian estimator of θ under SELF is computed likewise. Under SELF, we observe that $\mu(\hat{\theta}) = \hat{\theta}$, $\hat{\mu}_\theta = \hat{\mu}_{\theta\theta} = 0$. Using (3.2), the Bayesian estimator of θ under SELF is obtained as

$$\hat{\theta}_{SL} = \hat{\theta} \hat{\rho}_\theta \hat{\sigma}_{\theta\theta} + \frac{1}{2} \hat{L}_{\theta\theta\theta} \hat{\sigma}_{\theta\theta}^2. \tag{3.4}$$

Next, we compute the Bayesian estimator of reliability function under LLF. Here, we observe that $\mu(\hat{\theta}) = e^{-h\hat{R}e(t)}$, $\hat{\mu}_\theta = -\frac{ht^2}{\hat{\theta}^3} e^{-\left(\frac{t}{\hat{\theta}} + h\hat{R}e(t)\right)}$, $\hat{\mu}_{\theta\theta} = he^{-h\hat{R}e(t)} \left[h\left(\frac{t^2}{\hat{\theta}^3} e^{\frac{-t}{\hat{\theta}}} \right)^2 + \frac{t^2}{\hat{\theta}^4} \left(3 - \frac{t}{\hat{\theta}} \right) e^{-\frac{t}{\hat{\theta}}} \right]$. Using (3.2), the Bayesian estimator of reliability function under LLF is obtained as

$$\begin{aligned} \hat{\mathrm{R}}\mathrm{e}(t)_{LL} = &-\frac{1}{h} \log\left[e^{-h\hat{R}e(t)} + \frac{1}{2}\left\{ \left(he^{-h\hat{\mathrm{R}}\mathrm{e}(t)} \left(h\left(\frac{t^2}{\hat{\theta}^3} e^{\frac{-t}{\hat{\theta}}} \right)^2 + \frac{t^2}{\hat{\theta}^4} \left(3 - \frac{t}{\hat{\theta}} \right) e^{-\frac{t}{\hat{\theta}}} \right) - 2\frac{ht^2}{\hat{\theta}^3} e^{-\left(\frac{t}{\hat{\theta}} + h\hat{R}e(t)\right)} \hat{\rho}_\theta \right) \hat{\sigma}_{\theta\theta} \right\} \right. \\ &\left. -\frac{ht^2}{2\hat{\theta}^3} e^{-\left(\frac{t}{\hat{\theta}} + h\hat{R}e(t)\right)} \hat{L}_{\theta\theta\theta} \hat{\sigma}_{\theta\theta}^2 \right]. \end{aligned} \tag{3.5}$$

The Bayesian estimators of reliability function under SELF is computed likewise. Under SELF, we observe that $\mu(\hat{\theta}) = \hat{R}e(t)$, $\hat{\mu}_\theta = \frac{t^2}{\hat{\theta}^3} e^{-\frac{t}{\hat{\theta}}}$, $\hat{\mu}_{\theta\theta} = -\frac{t^2}{\hat{\theta}^4} \left(3 - \frac{t}{\hat{\theta}} \right) e^{-\frac{t}{\hat{\theta}}}$. Using (3.2), the Bayesian estimator of reliability function under SELF is obtained as

$$\hat{R}e(t)_{SL} = \hat{R}e(t) - \frac{1}{2}\left[\left\{ \frac{t^2}{\hat{\theta}^4} \left(3 - \frac{t}{\hat{\theta}} \right) e^{-\frac{t}{\hat{\theta}}} - 2\frac{t^2}{\hat{\theta}^3} e^{-\frac{t}{\hat{\theta}}} \hat{\rho}_\theta \right\} \hat{\sigma}_{\theta\theta} \right] + \frac{t^2}{2\hat{\theta}^3} e^{-\frac{t}{\hat{\theta}}} \hat{L}_{\theta\theta\theta} \hat{\sigma}_{\theta\theta}^2. \tag{3.6}$$

Also, we compute the Bayesian estimator of hazard function under LLF. Here, we observe that $\mu(\hat{\theta}) = e^{-h\hat{H}(t)}$, $\hat{\mu}_\theta = -h\left(\frac{2}{\hat{\theta}^3} - \frac{2}{(\hat{\theta}+t)^3} \right) e^{-h\hat{H}(t)}$, $\hat{\mu}_{\theta\theta} = he^{-h\hat{H}(t)} \left[h\left(\frac{2}{\hat{\theta}^3} - \frac{2}{(\hat{\theta}+t)^3} \right)^2 + \left(\frac{6}{\hat{\theta}^4} - \frac{6}{(\hat{\theta}+t)^4} \right) \right]$. Using (3.2), the Bayesian estimator of reliability function under LLF is obtained as

$$\begin{aligned} \hat{H}(t)_{LL} = &-\frac{1}{h} \log\left[e^{-h\hat{H}(t)} + \frac{he^{-h\hat{H}(t)} \hat{\sigma}_{\theta\theta}}{2} \left\{ \left[h\left(\frac{2}{\hat{\theta}^3} - \frac{2}{(\hat{\theta}+t)^3} \right)^2 + \left(\frac{6}{\hat{\theta}^4} - \frac{6}{(\hat{\theta}+t)^4} \right) \right] - \left(\frac{4}{\hat{\theta}^3} - \frac{4}{(\hat{\theta}+t)^3} \right) \hat{\rho}_\theta \right\} \right. \\ &\left. -h\left(\frac{1}{\hat{\theta}^3} - \frac{1}{(\hat{\theta}+t)^3} \right) e^{-h\hat{H}(t)} \hat{L}_{\theta\theta\theta} \hat{\sigma}_{\theta\theta}^2 \right]. \end{aligned} \tag{3.7}$$

The Bayesian estimator of hazard function under SELF is computed likewise. Under SELF, we observe that $\mu(\hat{\theta}) = \hat{H}(t)$, $\hat{\mu}_\theta = \frac{2}{\hat{\theta}^3} - \frac{2}{(\hat{\theta}+t)^3}$, $\hat{\mu}_{\theta\theta} = -\frac{6}{\hat{\theta}^4} + \frac{6}{(\hat{\theta}+t)^4}$. Using (3.2), the Bayesian estimator of hazard function under SELF is obtained as

Table 1: The relative MSEs and biases of parameter estimators with MLE and Bayesian estimators

				MSE(bias)						
T	n	m	$\boldsymbol{R}$	$\hat{\theta}$	$\hat{\theta}_{SL}$	$\hat{\theta}_{LL}(h=-.5)$	$\hat{\theta}_{LL}(h=.5)$	$\hat{\theta}_{MS}$	$\hat{\theta}_{ML}(h=-.5)$	$\hat{\theta}_{ML}(h=.5)$
1.5	20	14	(6,0*13)	.0826(.0404)	.0670(.0048)	.0597(-.0629)	.0652(.0338)	.0880(.0949)	.0614(-.0647)	.0783(.0262)
			(0*6,6,0*7)	.0602(.0176)	.0528(.0030)	.0499(-.0582)	.0510(.0198)	.0640(.0678)	.0519(-.0587)	.0617(.0145)
			(0*13,6)	.0572(.0300)	.0491(.0025)	.0449(-.0446)	.0495(.0273)	.0625(.0726)	.0455(-.0456)	.0575(.0237)
			(3,0*12,3)	.0664(.0328)	.0567(.0034)	.0507(-.0536)	.0558(.0288)	.0723(.0807)	.0515(-.0549)	.0654(.0235)
		12	(8,0*11)	.1183(.0651)	.0738(.0084)	.0742(-.0585)	.0816(.0507)	.1174(.1225)	.0808(-.0590)	.1040(.0440)
			(0*5,8,0*6)	.0703(.0246)	.0579(.0035)	.0587(-.0626)	.0594(.0227)	.0738(.0743)	.0597(-.0633)	.0708(.0162)
			(0*11,8)	.0666(.0448)	.0495(.0026)	.0495(-.0325)	.0563(.0406)	.0721(.0873)	.0509(-.0329)	.0666(.0386)
			(4,0*10,4)	.0846(.0535)	.0597(.0039)	.0588(-.0416)	.0671(.0461)	.0899(.1026)	.0613(-.0422)	.0816(.0425)
		10	(10,0*9)	.1421(.0726)	.0837(.0142)	.0826(-.0725)	.0865(.0522)	.1327(.1354)	.0945(-.0731)	.1159(.0422)
			(0*4,10,0*5)	.0755(.0250)	.0658(.0045)	.0642(-.0725)	.0631(.0239)	.0790(.0807)	.0654(-.0737)	.0760(.0145)
			(0*9,10)	.0726(.0406)	.0504(.0026)	.0556(-.0386)	.0610(.0363)	.0770(.0838)	.0570(-.0386)	.0729(.0341)
			(5,0*8,5)	.0994(.0539)	.0605(.0078)	.0644(-.0479)	.0706(.0437)	.0978(.1039)	.0724(-.0472)	.0912(.0404)
	30	22	(8,0*21)	.0489(.0305)	.0444(.0020)	.0392(-.0362)	.0436(.0292)	.0540(.0698)	.0396(-.0371)	.0500(.0265)
			(0*10,8,0*11)	.0368(.0179)	.0354(.0013)	.0325(-.0345)	.0341(.0182)	.0398(.0504)	.0326(-.0350)	.0382(.0160)
			(0*21,8)	.0358(.0227)	.0331(.0011)	.0303(-.0268)	.0332(.0225)	.0393(.0531)	.0305(-.0273)	.0371(.0212)
			(4,0*20,4)	.0399(.0244)	.0379(.0015)	.0332(-.0322)	.0364(.0239)	.0440(.0587)	.0334(-.0329)	.0412(.0219)
		18	(12,0*17)	.0696(.0464)	.0532(.0030)	.0515(-.0358)	.0586(.0425)	.0756(.0917)	.0528(-.0367)	.0695(.0396)
			(0*8,12,0*9)	.0422(.0205)	.0395(.0016)	.0376(-.0377)	.0388(.0212)	.0454(.0565)	.0377(-.0383)	.0439(.0182)
			(0*17,12)	.0393(.0305)	.0333(.0011)	.0324(-.0199)	.0361(.0298)	.0431(.0609)	.0327(-.0203)	.0408(.0291)
			(6,0*16,6)	.0493(.0381)	.0411(.0017)	.0389(-.0244)	.0442(.0365)	.0545(.0747)	.0394(-.0250)	.0507(.0352)
		14	(16,0*13)	.0737(.0334)	.0662(.0046)	.0561(-.0659)	.0605(.0295)	.0801(.0880)	.0567(-.0681)	.0712(.0211)
			(0*6,16,0*7)	.0428(-.0006)	.0466(.0022)	.0418(-.0672)	.0393(.0020)	.0450(.0415)	.0415(-.0682)	.0445(-.0044)
			(0*13,16)	.0381(.0119)	.0342(.0012)	.0344(-.0388)	.0353(.0122)	.0406(.0433)	.0343(-.0391)	.0395(.0101)
			(8,0*12,8)	.0495(.0248)	.0441(.0020)	.0404(-.0415)	.0440(.0237)	.0541(.0639)	.0408(-.0422)	.0507(.0207)
	40	30	(10,0*29)	.0374(.0258)	.0328(.0011)	.0315(-.0234)	.0347(.0256)	.0410(.0558)	.0317(-.0238)	.0389(.0246)
			(0*14,10,0*15)	.0278(.0133)	.0262(.0007)	.0253(-.0254)	.0263(.0139)	.0297(.0379)	.0254(-.0257)	.0289(.0129)
			(0*29,10)	.0266(.0189)	.0249(.0006)	.0234(-.0182)	.0252(.0191)	.0288(.0422)	.0234(-.0184)	.0276(.0186)
			(5,0*28,5)	.0313(.0226)	.0283(.0008)	.0269(-.0197)	.0293(.0227)	.0341(.0489)	.0270(-.0200)	.0326(.0221)
		26	(14,0*25)	.0426(.0288)	.0375(.0015)	.0349(-.0275)	.0387(.0283)	.0467(.0627)	.0353(-.0281)	.0442(.0268)
			(0,12,14,0*13)	.0294(.0141)	.0282(.0008)	.0270(-.0273)	.0278(.0150)	.0314(.0405)	.0271(-.0276)	.0307(.0136)
			(0*25,14)	.0274(.0223)	.0250(.0006)	.0238(-.0150)	.0260(.0225)	.0298(.0458)	.0239(-.0152)	.0286(.0222)
			(7,0*24,7)	.0351(.0283)	.0301(.0009)	.0294(-.0169)	.0327(.0281)	.0385(.0561)	.0297(-.0172)	.0366(.0278)
		20	(20,0*19)	.0639(.0500)	.0478(.0025)	.0467(-.0243)	.0540(.0468)	.0692(.0914)	.0486(-.0248)	.0645(.0454)
			(0*9,20,0*10)	.0326(.0154)	.0336(.0011)	.0303(-.0334)	.0308(.0171)	.0350(.0466)	.0302(-.0339)	.0340(.0146)
			(0*19,20)	.0287(.0205)	.0252(.0006)	.0254(-.0172)	.0272(.0208)	.0310(.0442)	.0254(-.0174)	.0299(.0203)
			(10,0*18,10)	.0386(.0342)	.0330(.0011)	.0318(-.0155)	.0358(.0337)	.0428(.0644)	.0320(-.0160)	.0401(.0332)
1.75	20	14	(6,0*13)	.0684(.0306)	.0583(.0036)	.0536(-.0583)	.0570(.0261)	.0736(.0798)	.0545(-.0595)	.0672(.0200)
			(0*6,6,0*7)	.0520(.0094)	.0482(.0024)	.0469(-.0620)	.0461(.0089)	.0548(.0522)	.0472(-.0627)	.0534(.0033)
			(0*13,6)	.0488(.0225)	.0422(.0018)	.0406(-.0416)	.0433(.0206)	.0526(.0602)	.0411(-.0421)	.0499(.0178)
			(3,0*12,3)	.0549(.0235)	.0489(.0025)	.0451(-.0504)	.0479(.0210)	.0595(.0663)	.0456(-.0513)	.0554(.0167)
		12	(8,0*11)	.0944(.0495)	.0659(.0047)	.0671(-.0558)	.0729(.0405)	.0981(.1029)	.0701(-.0563)	.0896(.0346)
			(0*5,8,0*6)	.0563(.0162)	.0537(.0029)	.0508(-.0627)	.0499(.0160)	.0601(.0634)	.0508(-.0638)	.0576(.0091)
			(0*11,8)	.0542(.0291)	.0430(.0019)	.0450(-.0368)	.0482(.0267)	.0585(.0673)	.0454(-.0371)	.0555(.0244)
			(4,0*10,4)	.0669(.0404)	.0519(.0029)	.0513(-.0401)	.0565(.0359)	.0721(.0850)	.0526(-.0407)	.0667(.0327)
		10	(10,0*9)	.1273(.0595)	.0718(.0153)	.0783(-.0669)	.0799(.0431)	.1174(.1171)	.0925(-.0656)	.1083(.0362)
			(0*4,10,0*5)	.0665(.0203)	.0615(.0039)	.0590(-.0700)	.0573(.0200)	.0703(.0733)	.0596(-.0712)	.0678(.0112)
			(0*9,10)	.0605(.0210)	.0453(.0021)	.0521(-.0484)	.0527(.0191)	.0628(.0613)	.0531(-.0482)	.0621(.0158)
			(5,0*8,5)	.0779(.0456)	.0553(.0032)	.0584(-.0416)	.0642(.0397)	.0827(.0923)	.0603(-.0418)	.0772(.0364)
	30	22	(8,0*21)	.0402(.0213)	.0381(.0015)	.0343(-.0358)	.0366(.0205)	.0438(.0558)	.0346(-.0364)	.0415(.0181)
			(0*10,8,0*11)	.0320(.0111)	.0318(.0010)	.0294(-.0358)	.0298(.0115)	.0340(.0406)	.0295(-.0362)	.0332(.0094)
			(0*21,8)	.0304(.0182)	.0284(.0008)	.0267(-.0244)	.0284(.0179)	.0328(.0447)	.0268(-.0247)	.0315(.0169)
			(4,0*20,4)	.0356(.0205)	.0326(.0011)	.0307(-.0283)	.0330(.0200)	.0387(.0505)	.0309(-.0288)	.0368(.0186)
		18	(12,0*17)	.0541(.0332)	.0460(.0022)	.0437(-.0366)	.0478(.0310)	.0591(.0737)	.0442(-.0373)	.0552(.0282)
			(0*8,12,0*9)	.0379(.0161)	.0363(.0013)	.0347(-.0373)	.0351(.0168)	.0404(.0495)	.0348(-.0378)	.0394(.0140)

				MSE(bias)						
T	n	m	$\boldsymbol{R}$	$\hat{\theta}$	$\hat{\theta}_{SL}$	$\hat{\theta}_{LL}(h=-.5)$	$\hat{\theta}_{LL}(h=.5)$	$\hat{\theta}_{MS}$	$\hat{\theta}_{ML}(h=-.5)$	$\hat{\theta}_{ML}(h=.5)$
1.75	30	18	(0*17,12)	.0335(.0217)	.0288(.0008)	.0293(-.0216)	.0312(.0213)	.0360(.0485)	.0295(-.0218)	.0348(.0206)
			(6,0*16,6)	.0421(.0293)	.0352(.0013)	.0351(-.0242)	.0384(.0281)	.0459(.0613)	.0355(-.0245)	.0437(.0270)
		14	(16,0*13)	.0658(.0270)	.0574(.0035)	.0525(-.0598)	.0555(.0238)	.0708(.0759)	.0532(-.0610)	.0652(.0176)
			(0*6,16,0*7)	.0405(-.0001)	.0438(.0019)	.0396(-.0628)	.0374(.0021)	.0426(.0397)	.0393(-.0637)	.0421(-.0036)
			(0*13,16)	.0326(.0006)	.0315(.0010)	.0319(-.0456)	.0305(.0016)	.0336(.0301)	.0317(-.0458)	.0338(-.0013)
			(8,0*12,8)	.0427(.0180)	.0383(.0015)	.0370(-.0395)	.0388(.0173)	.0460(.0528)	.0372(-.0399)	.0441(.0148)
	40	30	(10,0*29)	.0304(.0190)	.0282(.0008)	.0268(-.0231)	.0286(.0189)	.0329(.0453)	.0269(-.0234)	.0316(.0180)
			(0*14,10,0*15)	.0235(.0097)	.0236(.0006)	.0221(-.0251)	.0224(.0103)	.0248(.0321)	.0221(-.0253)	.0243(.0092)
			(0*29,10)	.0222(.0149)	.0214(.0005)	.0201(-.0170)	.0212(.0150)	.0238(.0352)	.0201(-.0172)	.0230(.0145)
			(5,0*28,5)	.0246(.0160)	.0243(.0006)	.0221(-.0202)	.0234(.0161)	.0265(.0389)	.0221(-.0205)	.0254(.0153)
		26	(14,0*25)	.0379(.0264)	.0324(.0011)	.0321(-.0225)	.0350(.0258)	.0412(.0562)	.0325(-.0228)	.0394(.0249)
			(0,12,14,0*13)	.0260(.0120)	.0259(.0007)	.0243(-.0260)	.0248(.0128)	.0276(.0364)	.0244(-.0263)	.0271(.0115)
			(0*25,14)	.0247(.0192)	.0215(.0005)	.0221(-.0130)	.0236(.0192)	.0265(.0396)	.0222(-.0131)	.0257(.0190)
			(7,0*24,7)	.0282(.0203)	.0258(.0007)	.0248(-.0183)	.0266(.0202)	.0305(.0445)	.0250(-.0185)	.0293(.0197)
		20	(20,0*19)	.0501(.0392)	.0416(.0018)	.0402(-.0241)	.0451(.0374)	.0553(.0763)	.0406(-.0246)	.0515(.0359)
			(0*9,20,0*10)	.0307(.0143)	.0314(.0010)	.0287(-.0314)	.0290(.0156)	.0328(.0435)	.0286(-.0319)	.0319(.0135)
			(0*19,20)	.0251(.0096)	.0227(.0005)	.0238(-.0240)	.0240(.0101)	.0263(.0312)	.0238(-.0241)	.0261(.0092)
			(10,0*18,10)	.0311(.0242)	.0283(.0008)	.0271(-.0182)	.0292(.0240)	.0338(.0505)	.0272(-.0185)	.0323(.0232)

$$\hat{H}(t)_{SL} = \hat{H}(t) + \hat{\sigma}_{\theta\theta}\left[\left(\frac{3}{(\hat{\theta}+t)^4} - \frac{3}{\hat{\theta}^4}\right) + \left(\frac{2}{\hat{\theta}^3} - \frac{2}{(\hat{\theta}+t)^3}\right)\hat{\rho}_\theta + \left(\frac{1}{\hat{\theta}^3} - \frac{1}{(\hat{\theta}+t)^3}\right)\hat{L}_{\theta\theta\theta}\hat{\sigma}_{\theta\theta}\right]. \tag{3.8}$$

Although Lindley approximation method gives Bayesian estimators of the unknown parameter, reliability and hazard functions, it cannot be used to get HPD credible intervals. Therefore, we propose to use the Markov chain Monte Carlo (MCMC) method to obtain Bayesian estimators and also to get HPD credible intervals. To obtain the MCMC samples from the posterior distribution, we use the adaptive Metropolis-Hastings method and consider the normal distribution as the proposal distribution. Let $\tilde{\theta}_i$ denotes the ith the MCMC sample obtained by the posterior distribution. The range of the random variable of the normal distribution is $(-\infty, \infty)$. Therefore, the MCMC samples are obtained as the $\log\tilde{\theta}_i$. The $\log\hat{\theta}$ as initial value is used.

Then, we can obtain the Bayesian estimators by the MCMC samples as follows.

$$\hat{\theta}_{MS} = \frac{1}{N}\sum_{i=B+1}^{N+B}\tilde{\theta}_i, \quad \hat{\theta}_{ML} = -\frac{1}{h}\log\left[\frac{1}{N}\sum_{i=B+1}^{N+B}e^{-h\tilde{\theta}_i}\right], \tag{3.9}$$

where N+B is the number of the MCMC samples and B is the burn-in period of the Markov chain. Furthermore, we compute $(R(t)_1, \ldots, R(t)_N)$ and $(H(t)_1, \ldots, H(t)_N)$. Rearrange $(\theta_1, \ldots, \theta_N)$, $(R(t)_1, \ldots, R(t)_N)$ and $(H(t)_1, \ldots, H(t)_N)$ into $(\theta_{(1)}, \ldots, \theta_{(N)})$, $(R(t)_{(1)}, \ldots, R(t)_{(N)})$ and $(H(t)_{(1)}, \ldots, H(t)_{(N)})$, where $(\theta_{(1)} < \cdots < \theta_{(N)})$, $(R(t)_{(1)} < \cdots < R(t)_{(N)})$ and $(H(t)_{(1)} < \cdots < H(t)_{(N)})$. Then, the $100(1-\alpha)\%$ Bayesian credible interval of the unknown parameter, reliability and hazard functions are given by $(\theta_{(N\alpha/2)}, \theta_{(N(1-\alpha/2))})$, $(R(t)_{(N\alpha/2)}, R(t)_{(N(1-\alpha/2))})$ and $(H(t)_{(N\alpha/2)}, H(t)_{(N(1-\alpha/2))})$.

4. Simulation and example

4.1. Simulation

In this subsection, we use Monte Carlo simulations to compare the proposed estimators. First of all, we consider various n, m, T, and four different progressive censoring schemes; (a) $R_1 = n - m$ and

Table 2: The relative MSEs and biases of reliability estimators with MLE and Bayesian estimators

				MSE(bias)						
T	n	m	$\boldsymbol{R}$	$\hat{Re}$	$\hat{Re}_{SL}$	$\hat{Re}_{LL}(h=-.5)$	$\hat{Re}_{LL}(h=.5)$	$\hat{Re}_{MS}$	$\hat{Re}_{ML}(h=-.5)$	$\hat{Re}_{ML}(h=.5)$
1.5	20	14	(6,0*13)	.0173(.0025)	.0118(-.0471)	.0087(-.0097)	.0075(.0104)	.0115(-.0477)	.0091(-.0148)	.0077(-.0085)
			(0*6,6,0*7)	.0143(-.0051)	.0092(-.0409)	.0074(-.0131)	.0058(.0066)	.0090(-.0412)	.0071(-.0139)	.0061(-.0092)
			(0*13,6)	.0130(.0023)	.0078(-.0337)	.0063(-.0068)	.0057(.0085)	.0077(-.0341)	.0066(-.0095)	.0058(-.0060)
			(3,0*12,3)	.0148(.0017)	.0094(-.0398)	.0073(-.0085)	.0064(.0089)	.0092(-.0402)	.0076(-.0121)	.0066(-.0075)
		12	(8,0*11)	.0212(.0091)	.0145(-.0505)	.0105(-.0077)	.0089(.0143)	.0140(-.0512)	.0109(-.0141)	.0091(-.0066)
			(0*5,8,0*6)	.0160(-.0034)	.0120(-.0469)	.0085(-.0133)	.0071(.0057)	.0116(-.0470)	.0090(-.0170)	.0076(-.0113)
			(0*11,8)	.0142(.0083)	.0083(-.0306)	.0068(-.0035)	.0062(.0117)	.0082(-.0308)	.0071(-.0061)	.0063(-.0028)
			(4,0*10,4)	.0167(.0092)	.0101(-.0374)	.0080(-.0045)	.0071(.0134)	.0099(-.0378)	.0083(-.0084)	.0072(-.0036)
		10	(10,0*9)	.0231(.0092)	.0168(-.0593)	.0114(-.0088)	.0094(.0163)	.0162(-.0605)	.0119(-.0174)	.0096(-.0074)
			(0*4,10,0*5)	.0170(-.0043)	.0137(-.0531)	.0092(-.0147)	.0075(.0068)	.0132(-.0533)	.0098(-.0192)	.0081(-.0120)
			(0*9,10)	.0156(.0048)	.0101(-.0356)	.0079(-.0071)	.0069(.0089)	.0098(-.0355)	.0082(-.0099)	.0071(-.0061)
			(5,0*8,5)	.0176(.0074)	.0112(-.0416)	.0086(-.0064)	.0075(.0125)	.0110(-.0420)	.0089(-.0108)	.0076(-.0054)
	30	22	(8,0*21)	.0112(.0044)	.0064(-.0281)	.0053(-.0041)	.0049(.0098)	.0063(-.0284)	.0055(-.0061)	.0049(-.0033)
			(0*10,8,0*11)	.0092(.0004)	.0054(-.0251)	.0044(-.0057)	.0041(.0061)	.0053(-.0253)	.0046(-.0068)	.0042(-.0047)
			(0*21,8)	.0088(.0032)	.0047(-.0212)	.0041(-.0034)	.0039(.0074)	.0047(-.0214)	.0043(-.0043)	.0039(-.0027)
			(4,0*20,4)	.0096(.0032)	.0053(-.0244)	.0045(-.0039)	.0042(.0082)	.0053(-.0246)	.0047(-.0053)	.0042(-.0032)
		18	(12,0*17)	.0148(.0085)	.0088(-.0327)	.0072(-.0038)	.0065(.0124)	.0087(-.0330)	.0075(-.0066)	.0065(-.0028)
			(0*8,12,0*9)	.0105(.0006)	.0067(-.0286)	.0053(-.0066)	.0048(.0066)	.0066(-.0287)	.0056(-.0080)	.0049(-.0053)
			(0*17,12)	.0092(.0067)	.0049(-.0192)	.0043(-.0013)	.0041(.0094)	.0049(-.0193)	.0045(-.0023)	.0041(-.0007)
			(6,0*16,6)	.0112(.0084)	.0061(-.0234)	.0053(-.0013)	.0049(.0115)	.0061(-.0236)	.0055(-.0029)	.0049(-.0007)
		14	(16,0*13)	.0164(.0005)	.0115(-.0472)	.0084(-.0106)	.0072(.0094)	.0111(-.0479)	.0088(-.0152)	.0075(-.0090)
			(0*6,16,0*7)	.0115(-.0111)	.0087(-.0424)	.0063(-.0156)	.0053(.0005)	.0085(-.0425)	.0066(-.0176)	.0057(-.0134)
			(0*13,16)	.0098(-.0032)	.0062(-.0279)	.0050(-.0087)	.0045(.0029)	.0061(-.0279)	.0053(-.0099)	.0047(-.0078)
			(8,0*12,8)	.0115(.0012)	.0069(-.0308)	.0056(-.0067)	.0051(.0074)	.0068(-.0310)	.0059(-.0086)	.0052(-.0057)
	40	30	(10,0*29)	.0091(.0045)	.0049(-.0203)	.0043(-.0027)	.0040(.0079)	.0048(-.0205)	.0044(-.0036)	.0040(-.0020)
			(0*14,10,0*15)	.0072(.0001)	.0040(-.0189)	.0035(-.0046)	.0032(.0044)	.0040(-.0190)	.0036(-.0051)	.0033(-.0039)
			(0*29,10)	.0067(.0034)	.0034(-.0153)	.0031(-.0019)	.0030(.0064)	.0034(-.0154)	.0032(-.0023)	.0030(-.0014)
			(5,0*28,5)	.0076(.0043)	.0040(-.0171)	.0035(-.0019)	.0034(.0074)	.0039(-.0172)	.0037(-.0025)	.0034(-.0013)
		26	(14,0*25)	.0098(.0050)	.0053(-.0229)	.0046(-.0028)	.0043(.0092)	.0053(-.0232)	.0048(-.0040)	.0043(-.0020)
			(0,12,14,0*13)	.0075(.0001)	.0043(-.0202)	.0036(-.0048)	.0034(.0049)	.0042(-.0203)	.0038(-.0053)	.0034(-.0039)
			(0*25,14)	.0067(.0051)	.0034(-.0141)	.0031(-.0008)	.0030(.0075)	.0034(-.0142)	.0032(-.0011)	.0030(-.0003)
			(7,0*24,7)	.0083(.0065)	.0042(-.0168)	.0038(-.0007)	.0036(.0090)	.0042(-.0169)	.0039(-.0014)	.0036(-.0002)

$R_i = 0$ for $i \neq 1$, (b) $R_{m/2} = n - m$ and $R_i = 0$ for $i \neq m/2$, (c) $R_m = n - m$ and $R_i = 0$ for $i \neq m$, (d) $R_1 = R_m = (n-m)/2$ and $R_i = 0$ for $i \neq 1, m$.

In each case, we set $\theta = 1$, Re $= 0.7538$ and $H = -0.7500$, and replicate the process 1,000 times. Bayesian estimators for the unknown parameter, reliability, and hazard functions are obtained with respect to the SELF and LLF. Bayesian estimators under LLF are computed for two distinct values of $h = -0.5$ and $h = 0.5$. Various schemes are considered to calculate the MSE, bias, coverage length (CL), and coverage probability (CP) of the proposed MLE and Bayesian estimators, as detailed in Tables 1–4.

In Tables 1–3, the MSEs and biases of all estimators for the unknown parameter, reliability, and hazard functions are presented for various values of n, m, T, and progressive censoring schemes. The MSE and bias of the respective MLE are shown in the fifth column of Tables 1–3. Additionally, the MSE and bias of the respective Bayesian estimates are presented in the other columns. Specifically, the MSE and bias of the Bayesian estimates obtained using Lindley's approximation method are tabulated in the sixth to eighth columns, while the MSE and bias of the Bayesian estimates obtained using the MCMC method are provided in the remaining columns.

T	n	m	$\boldsymbol{R}$	MSE(bias) $\hat{Re}$	$\hat{Re}_{SL}$	$\hat{Re}_{LL}(h=-.5)$	$\hat{Re}_{LL}(h=.5)$	$\hat{Re}_{MS}$	$\hat{Re}_{ML}(h=-.5)$	$\hat{Re}_{ML}(h=.5)$
1.5	40	20	(20,0*19)	.0131(.0118)	.0071(-.0258)	.0060(-.0001)	.0056(.0144)	.0070(-.0261)	.0062(-.0022)	.0055(.0007)
			(0*9,20,0*10)	.0086(.0000)	.0053(-.0242)	.0043(-.0057)	.0039(.0057)	.0052(-.0243)	.0045(-.0064)	.0040(-.0044)
			(0*19,20)	.0073(.0037)	.0039(-.0158)	.0035(-.0021)	.0033(.0064)	.0039(-.0158)	.0036(-.0025)	.0033(-.0016)
			(10,0*18,10)	.0092(.0087)	.0048(-.0174)	.0043(.0002)	.0041(.0107)	.0047(-.0175)	.0044(-.0006)	.0040(.0008)
1.75	20	14	(6,0*13)	.0154(.0001)	.0104(-.0430)	.0078(-.0102)	.0067(.0080)	.0102(-.0434)	.0082(-.0143)	.0070(-.0091)
			(0*6,6,0*7)	.0129(-.0077)	.0093(-.0419)	.0068(-.0140)	.0058(.0022)	.0091(-.0420)	.0072(-.0169)	.0062(-.0126)
			(0*13,6)	.0114(.0002)	.0070(-.0309)	.0057(-.0074)	.0051(.0063)	.0069(-.0311)	.0059(-.0096)	.0052(-.0067)
			(3,0*12,3)	.0129(-.0007)	.0083(-.0364)	.0065(-.0090)	.0057(.0066)	.0082(-.0366)	.0068(-.0120)	.0059(-.0082)
		12	(8,0*11)	.0188(.0050)	.0129(-.0468)	.0095(-.0090)	.0081(.0111)	.0125(-.0471)	.0099(-.0143)	.0084(-.0080)
			(0*5,8,0*6)	.0141(-.0050)	.0107(-.0443)	.0077(-.0133)	.0064(.0046)	.0104(-.0444)	.0081(-.0166)	.0069(-.0115)
			(0*11,8)	.0129(.0024)	.0081(-.0311)	.0065(-.0069)	.0058(.0071)	.0080(-.0311)	.0068(-.0092)	.0060(-.0062)
			(4,0*10,4)	.0146(.0058)	.0091(-.0345)	.0072(-.0056)	.0064(.0105)	.0090(-.0347)	.0076(-.0088)	.0066(-.0049)
		10	(10,0*9)	.0212(.0054)	.0154(-.0550)	.0107(-.0103)	.0088(.0128)	.0149(-.0556)	.0112(-.0174)	.0092(-.0091)
			(0*4,10,0*5)	.0158(-.0050)	.0127(-.0503)	.0086(-.0144)	.0071(.0059)	.0122(-.0505)	.0092(-.0186)	.0077(-.0120)
			(0*9,10)	.0141(-.0032)	.0101(-.0385)	.0075(-.0118)	.0063(.0037)	.0098(-.0383)	.0079(-.0144)	.0068(-.0105)
			(5,0*8,5)	.0163(.0064)	.0107(-.0377)	.0082(-.0065)	.0072(.0108)	.0104(-.0378)	.0086(-.0101)	.0074(-.0056)
	30	22	(8,0*21)	.0097(.0014)	.0056(-.0262)	.0047(-.0053)	.0043(.0071)	.0056(-.0264)	.0049(-.0069)	.0044(-.0046)
			(0*10,8,0*11)	.0082(-.0021)	.0049(-.0245)	.0041(-.0068)	.0037(.0040)	.0049(-.0246)	.0042(-.0079)	.0038(-.0060)
			(0*21,8)	.0076(.0021)	.0041(-.0189)	.0036(-.0034)	.0034(.0061)	.0041(-.0190)	.0037(-.0042)	.0034(-.0029)
			(4,0*20,4)	.0088(.0021)	.0049(-.0220)	.0042(-.0042)	.0039(.0065)	.0048(-.0222)	.0044(-.0053)	.0040(-.0037)
		18	(12,0*17)	.0124(.0047)	.0075(-.0302)	.0061(-.0049)	.0055(.0097)	.0074(-.0304)	.0064(-.0072)	.0056(-.0041)
			(0*8,12,0*9)	.0096(-.0008)	.0061(-.0273)	.0049(-.0070)	.0044(.0053)	.0060(-.0274)	.0051(-.0083)	.0046(-.0059)
			(0*17,12)	.0083(.0033)	.0047(-.0189)	.0040(-.0031)	.0038(.0065)	.0046(-.0190)	.0042(-.0040)	.0038(-.0026)
			(6,0*16,6)	.0099(.0054)	.0055(-.0219)	.0047(-.0027)	.0044(.0087)	.0054(-.0220)	.0049(-.0040)	.0044(-.0021)
		14	(16,0*13)	.0150(-.0013)	.0103(-.0432)	.0077(-.0109)	.0066(.0071)	.0101(-.0436)	.0081(-.0147)	.0069(-.0096)
			(0*6,16,0*7)	.0110(-.0102)	.0081(-.0397)	.0059(-.0146)	.0050(.0006)	.0079(-.0398)	.0062(-.0165)	.0054(-.0127)
			(0*13,16)	.0090(-.0079)	.0061(-.0298)	.0048(-.0115)	.0041(-.0003)	.0060(-.0297)	.0050(-.0127)	.0045(-.0105)
			(8,0*12,8)	.0105(-.0009)	.0066(-.0290)	.0053(-.0076)	.0048(.0051)	.0065(-.0291)	.0056(-.0093)	.0050(-.0068)
	40	30	(10,0*29)	.0077(.0025)	.0041(-.0185)	.0036(-.0031)	.0034(.0062)	.0041(-.0186)	.0038(-.0039)	.0034(-.0027)
			(0*14,10,0*15)	.0063(-.0008)	.0035(-.0176)	.0030(-.0046)	.0028(.0036)	.0035(-.0177)	.0031(-.0051)	.0029(-.0040)
			(0*29,10)	.0058(.0023)	.0030(-.0137)	.0027(-.0021)	.0026(.0052)	.0030(-.0138)	.0028(-.0025)	.0026(-.0017)
			(5,0*28,5)	.0064(.0023)	.0034(-.0158)	.0031(-.0026)	.0029(.0057)	.0034(-.0158)	.0032(-.0031)	.0029(-.0021)
		26	(14,0*25)	.0089(.0048)	.0048(-.0200)	.0042(-.0024)	.0039(.0082)	.0048(-.0201)	.0043(-.0034)	.0039(-.0018)
			(0,12,14,0*13)	.0068(-.0002)	.0039(-.0188)	.0033(-.0045)	.0031(.0044)	.0038(-.0188)	.0034(-.0050)	.0031(-.0038)
			(0*25,14)	.0062(.0040)	.0032(-.0128)	.0029(-.0012)	.0028(.0061)	.0031(-.0129)	.0030(-.0015)	.0028(-.0008)
			(7,0*24,7)	.0070(.0038)	.0036(-.0158)	.0032(-.0018)	.0031(.0068)	.0036(-.0159)	.0034(-.0024)	.0031(-.0014)
		20	(20,0*19)	.0115(.0087)	.0063(-.0238)	.0054(-.0013)	.0050(.0117)	.0063(-.0240)	.0056(-.0030)	.0050(-.0007)
			(0*9,20,0*10)	.0082(-.0001)	.0050(-.0228)	.0041(-.0055)	.0037(.0052)	.0049(-.0229)	.0042(-.0062)	.0038(-.0044)
			(0*19,20)	.0067(-.0013)	.0038(-.0179)	.0033(-.0052)	.0030(.0028)	.0038(-.0179)	.0034(-.0057)	.0031(-.0046)
			(10,0*18,10)	.0078(.0052)	.0042(-.0168)	.0037(-.0014)	.0035(.0079)	.0041(-.0169)	.0038(-.0022)	.0035(-.0010)

Table 3: The relative MSEs and biases of hazard estimators with MLE and Bayesian estimators

T	n	m	$\boldsymbol{R}$	MSE(bias) $\hat{H}$	$\hat{H}_{SL}$	$\hat{H}_{LL}(h=-.5)$	$\hat{H}_{LL}(h=.5)$	$\hat{H}_{MS}(h=-.5)$	$\hat{H}_{ML}(h=-.5)$	$\hat{H}_{ML}(h=.5)$
1.5	20	14	(6,0*13)	.0598(.1098)	.0547(.0294)	.0324(-.0140)	.0399(.0288)	.0576(.1105)	.0353(.0252)	.0424(.0396)
			(0*6,6,0*7)	.0460(.0939)	.0465(.0367)	.0253(-.0079)	.0340(.0344)	.0445(.0941)	.0279(.0250)	.0331(.0357)
			(0*13,6)	.0376(.0774)	.0389(.0205)	.0243(-.0119)	.0282(.0203)	.0368(.0779)	.0257(.0180)	.0296(.0260)
			(3,0*12,3)	.0461(.0916)	.0453(.0253)	.0276(-.0120)	.0329(.0247)	.0449(.0922)	.0297(.0219)	.0348(.0325)
		12	(8,0*11)	.0768(.1206)	.0675(.0258)	.0391(-.0206)	.0493(.0267)	.0729(.1210)	.0427(.0230)	.0524(.0405)
			(0*5,8,0*6)	.0625(.1100)	.0547(.0383)	.0318(-.0045)	.0403(.0362)	.0594(.1094)	.0358(.0312)	.0432(.0445)
			(0*11,8)	.0409(.0716)	.0427(.0128)	.0268(-.0180)	.0309(.0140)	.0397(.0718)	.0282(.0120)	.0325(.0197)
			(4,0*10,4)	.0503(.0878)	.0500(.0158)	.0305(-.0206)	.0362(.0171)	.0488(.0883)	.0324(.0146)	.0381(.0256)
		10	(10,0*9)	.0913(.1419)	.0734(.0291)	.0409(-.0244)	.0537(.0300)	.0859(.1430)	.0452(.0252)	.0575(.0483)
			(0*4,10,0*5)	.0732(.1251)	.0593(.0422)	.0334(-.0064)	.0439(.0398)	.0689(.1246)	.0383(.0331)	.0474(.0499)
			(0*9,10)	.0517(.0844)	.0500(.0225)	.0303(-.0115)	.0366(.0227)	.0494(.0837)	.0330(.0198)	.0388(.0289)
			(5,0*8,5)	.0577(.0979)	.0546(.0211)	.0327(-.0184)	.0397(.0219)	.0554(.0982)	.0352(.0189)	.0420(.0315)

				MSE(bias)						
T	n	m	$\boldsymbol{R}$	$\hat{H}$	$\hat{H}_{SL}$	$\hat{H}_{LL}(h=-.5)$	$\hat{H}_{LL}(h=.5)$	$\hat{H}_{MS}$	$\hat{H}_{ML}(h=-.5)$	$\hat{H}_{ML}(h=.5)$
1.5	30	22	(8,0*21)	.0297(.0640)	.0321(.0132)	.0205(-.0155)	.0231(.0136)	.0293(.0646)	.0213(.0114)	.0242(.0177)
			(0*10,8,0*11)	.0250(.0570)	.0270(.0165)	.0173(-.0086)	.0195(.0159)	.0246(.0572)	.0182(.0137)	.0204(.0185)
			(0*21,8)	.0216(.0481)	.0250(.0106)	.0165(-.0115)	.0180(.0108)	.0214(.0484)	.0169(.0093)	.0187(.0130)
			(4,0*20,4)	.0244(.0552)	.0273(.0121)	.0177(-.0129)	.0196(.0123)	.0241(.0556)	.0183(.0105)	.0205(.0153)
		18	(12,0*17)	.0431(.0764)	.0445(.0137)	.0279(-.0191)	.0322(.0149)	.0419(.0769)	.0293(.0123)	.0339(.0210)
			(0*8,12,0*9)	.0322(.0656)	.0330(.0194)	.0207(-.0088)	.0241(.0188)	.0314(.0656)	.0223(.0159)	.0253(.0220)
			(0*17,12)	.0228(.0443)	.0265(.0058)	.0175(-.0153)	.0191(.0069)	.0225(.0445)	.0179(.0055)	.0198(.0090)
			(6,0*16,6)	.0287(.0541)	.0322(.0064)	.0210(-.0188)	.0232(.0079)	.0282(.0545)	.0216(.0061)	.0243(.0113)
		14	(16,0*13)	.0582(.1097)	.0528(.0314)	.0313(-.0122)	.0386(.0303)	.0560(.1104)	.0343(.0260)	.0411(.0402)
			(0*6,16,0*7)	.0431(.0963)	.0393(.0420)	.0232(.0041)	.0289(.0383)	.0417(.0962)	.0262(.0332)	.0307(.0429)
			(0*13,16)	.0297(.0637)	.0312(.0245)	.0196(-.0015)	.0228(.0230)	.0290(.0635)	.0212(.0206)	.0239(.0256)
			(8,0*12,8)	.0331(.0703)	.0347(.0196)	.0219(-.0103)	.0252(.0192)	.0324(.0706)	.0232(.0168)	.0264(.0235)
	40	30	(10,0*29)	.0221(.0464)	.0257(.0090)	.0169(-.0124)	.0185(.0096)	.0219(.0466)	.0174(.0080)	.0192(.0115)
			(0*14,10,0*15)	.0180(.0427)	.0208(.0131)	.0136(-.0059)	.0150(.0127)	.0178(.0428)	.0142(.0111)	.0155(.0138)
			(0*29,10)	.0153(.0346)	.0186(.0064)	.0125(-.0102)	.0134(.0069)	.0152(.0348)	.0128(.0057)	.0138(.0078)
			(5,0*28,5)	.0177(.0388)	.0212(.0066)	.0142(-.0120)	.0152(.0072)	.0176(.0390)	.0145(.0059)	.0158(.0085)
		26	(14,0*25)	.0246(.0524)	.0278(.0094)	.0181(-.0148)	.0199(.0101)	.0243(.0527)	.0186(.0082)	.0208(.0128)
			(0,12,14,0*13)	.0195(.0457)	.0220(.0137)	.0143(-.0068)	.0159(.0133)	.0193(.0458)	.0150(.0113)	.0165(.0146)
			(0*25,14)	.0149(.0321)	.0184(.0035)	.0124(-.0126)	.0131(.0044)	.0148(.0323)	.0126(.0033)	.0136(.0053)
			(7,0*24,7)	.0187(.0385)	.0226(.0039)	.0151(-.0151)	.0161(.0050)	.0186(.0387)	.0153(.0037)	.0167(.0065)
		20	(20,0*19)	.0332(.0601)	.0364(.0038)	.0233(-.0244)	.0261(.0060)	.0326(.0606)	.0239(.0038)	.0273(.0105)
			(0*9,20,0*10)	.0247(.0550)	.0263(.0165)	.0168(-.0080)	.0191(.0159)	.0243(.0551)	.0179(.0130)	.0200(.0176)
			(0*19,20)	.0176(.0362)	.0210(.0072)	.0140(-.0099)	.0151(.0077)	.0174(.0362)	.0144(.0064)	.0156(.0086)
			(10,0*18,10)	.0216(.0403)	.0257(.0021)	.0171(-.0180)	.0184(.0037)	.0214(.0405)	.0174(.0023)	.0192(.0055)
1.75	20	14	(6,0*13)	.0522(.0996)	.0490(.0298)	.0293(-.0099)	.0358(.0288)	.0505(.1000)	.0321(.0258)	.0381(.0377)
			(0*6,6,0*7)	.0459(.0960)	.0427(.0386)	.0254(.0011)	.0314(.0357)	.0445(.0959)	.0285(.0321)	.0334(.0420)
			(0*13,6)	.0336(.0707)	.0350(.0214)	.0220(-.0081)	.0254(.0207)	.0329(.0709)	.0235(.0189)	.0268(.0255)
			(3,0*12,3)	.0404(.0834)	.0403(.0261)	.0248(-.0079)	.0293(.0250)	.0394(.0837)	.0268(.0227)	.0310(.0314)
		12	(8,0*11)	.0674(.1109)	.0605(.0283)	.0354(-.0149)	.0443(.0283)	.0643(.1108)	.0390(.0250)	.0472(.0398)
			(0*5,8,0*6)	.0553(.1031)	.0489(.0376)	.0287(-.0030)	.0361(.0352)	.0529(.1027)	.0324(.0307)	.0387(.0426)
			(0*11,8)	.0401(.0726)	.0408(.0211)	.0253(-.0088)	.0297(.0208)	.0389(.0724)	.0273(.0187)	.0314(.0257)
			(4,0*10,4)	.0453(.0807)	.0453(.0183)	.0279(-.0154)	.0330(.0188)	.0439(.0809)	.0299(.0167)	.0348(.0258)
		10	(10,0*9)	.0829(.1312)	.0688(.0323)	.0388(-.0175)	.0504(.0323)	.0782(.1313)	.0433(.0282)	.0541(.0476)
			(0*4,10,0*5)	.0670(.1180)	.0555(.0412)	.0316(-.0049)	.0410(.0387)	.0634(.1175)	.0362(.0327)	.0443(.0479)
			(0*9,10)	.0516(.0903)	.0477(.0339)	.0282(-.0012)	.0351(.0320)	.0493(.0893)	.0317(.0287)	.0375(.0379)
			(5,0*8,5)	.0551(.0895)	.0523(.0212)	.0316(-.0150)	.0383(.0217)	.0526(.0892)	.0343(.0192)	.0407(.0298)
	30	22	(8,0*21)	.0260(.0593)	.0283(.0156)	.0181(-.0106)	.0204(.0153)	.0257(.0597)	.0190(.0136)	.0214(.0188)
			(0*10,8,0*11)	.0228(.0552)	.0247(.0190)	.0158(-.0047)	.0179(.0179)	.0225(.0553)	.0168(.0160)	.0187(.0202)
			(0*21,8)	.0185(.0427)	.0216(.0103)	.0143(-.0093)	.0156(.0104)	.0184(.0429)	.0148(.0093)	.0162(.0122)
			(4,0*20,4)	.0223(.0499)	.0254(.0127)	.0166(-.0097)	.0183(.0126)	.0221(.0502)	.0172(.0113)	.0191(.0151)
		18	(12,0*17)	.0363(.0698)	.0377(.0157)	.0237(-.0145)	.0274(.0161)	.0355(.0701)	.0251(.0140)	.0288(.0212)
			(0*8,12,0*9)	.0292(.0623)	.0303(.0201)	.0191(-.0067)	.0221(.0192)	.0286(.0624)	.0205(.0166)	.0232(.0220)
			(0*17,12)	.0216(.0435)	.0248(.0101)	.0163(-.0096)	.0179(.0104)	.0213(.0435)	.0169(.0092)	.0187(.0122)
			(6,0*16,6)	.0256(.0504)	.0288(.0093)	.0188(-.0137)	.0208(.0101)	.0252(.0506)	.0195(.0087)	.0217(.0129)
		14	(16,0*13)	.0515(.1000)	.0481(.0315)	.0287(-.0082)	.0351(.0302)	.0498(.1003)	.0315(.0267)	.0374(.0384)
			(0*6,16,0*7)	.0398(.0901)	.0371(.0394)	.0221(.0038)	.0273(.0359)	.0386(.0900)	.0249(.0315)	.0289(.0402)
			(0*13,16)	.0292(.0674)	.0296(.0312)	.0182(.0047)	.0217(.0284)	.0285(.0671)	.0202(.0259)	.0228(.0311)
			(8,0*12,8)	.0316(.0663)	.0331(.0219)	.0209(-.0058)	.0241(.0209)	.0309(.0663)	.0224(.0189)	.0253(.0246)
	40	30	(10,0*29)	.0186(.0419)	.0218(.0098)	.0144(-.0096)	.0157(.0099)	.0185(.0421)	.0149(.0087)	.0163(.0116)
			(0*14,10,0*15)	.0158(.0396)	.0183(.0130)	.0120(-.0046)	.0132(.0124)	.0157(.0397)	.0126(.0111)	.0137(.0134)
			(0*29,10)	.0132(.0309)	.0162(.0066)	.0109(-.0082)	.0116(.0068)	.0131(.0310)	.0112(.0060)	.0120(.0077)
			(5,0*28,5)	.0153(.0356)	.0183(.0080)	.0123(-.0088)	.0132(.0081)	.0152(.0357)	.0126(.0072)	.0136(.0093)
		26	(14,0*25)	.0218(.0457)	.0252(.0082)	.0165(-.0131)	.0181(.0089)	.0216(.0459)	.0170(.0075)	.0188(.0111)
			(0,12,14,0*13)	.0175(.0423)	.0199(.0129)	.0130(-.0061)	.0144(.0124)	.0173(.0424)	.0137(.0108)	.0149(.0136)

T	n	m	$\boldsymbol{R}$	MSE(bias) $\hat{H}$	$\hat{H}_{SL}$	$\hat{H}_{LL}(h=-.5)$	$\hat{H}_{LL}(h=.5)$	$\hat{H}_{MS}$	$\hat{H}_{ML}(h=-.5)$	$\hat{H}_{ML}(h=.5)$
1.75	40	26	(0*25,14)	.0140(.0293)	.0173(.0045)	.0117(-.0099)	.0124(.0051)	.0139(.0293)	.0119(.0044)	.0128(.0059)
			(7,0*24,7)	.0162(.0359)	.0195(.0063)	.0130(-.0110)	.0139(.0068)	.0161(.0360)	.0133(.0058)	.0145(.0081)
		20	(20,0*19)	.0295(.0553)	.0326(.0064)	.0210(-.0193)	.0234(.0079)	.0290(.0556)	.0217(.0062)	.0246(.0116)
			(0*9,20,0*10)	.0230(.0517)	.0248(.0158)	.0159(-.0072)	.0180(.0152)	.0226(.0518)	.0169(.0128)	.0188(.0169)
			(0*19,20)	.0174(.0404)	.0198(.0146)	.0129(-.0028)	.0143(.0138)	.0172(.0404)	.0137(.0125)	.0149(.0149)
			(10,0*18,10)	.0189(.0385)	.0223(.0057)	.0148(-.0129)	.0160(.0065)	.0187(.0386)	.0152(.0054)	.0166(.0081)

Table 4: The relative CLs and CPs of interval estimators with MLE and Bayesian estimators

T	n	m	$\boldsymbol{R}$	CL(CP) $\hat{\theta}$	$\hat{Re}$	$\hat{H}$	$\hat{\theta}_{MS}$	$\hat{Re}_{ML}(h=-.5)$	$\hat{H}_{ML}(h=-.5)$	$\hat{Re}_{ML}(h=.5)$	$\hat{H}_{ML}(h=.5)$
1.5	20	14	(6,0*13)	1.1120(96.1)	.3861(95.3)	.8507(95.4)	1.0814(92.5)	.3809(95.3)	.8033(95.7)	.3449(95.4)	.7748(94.9)
			(0*6,6,0*7)	.9740(95.8)	.3476(95.7)	.7485(95.5)	.9230(92.8)	.3446(95.8)	.7074(95.5)	.3103(95.7)	.7102(94.3)
			(0*13,6)	.9595(96.2)	.3269(95.7)	.7027(95.7)	.9016(93.7)	.3248(95.8)	.6813(95.7)	.3030(95.8)	.6526(93.8)
			(3,0*12,3)	1.0233(95.6)	.3513(95.9)	.7582(96.0)	.9766(93.5)	.3466(95.9)	.7292(96.2)	.3201(96.2)	.7042(94.8)
		12	(8,0*11)	1.2655(95.7)	.4364(95.8)	.9709(95.7)	1.2081(92.2)	.4284(95.7)	.9022(95.5)	.3814(95.8)	.8640(94.5)
			(0*5,8,0*6)	1.0991(95.7)	.4017(95.6)	.8836(95.6)	.9695(92.8)	.3949(95.7)	.8268(95.5)	.3518(95.4)	.7737(94.4)
			(0*11,8)	1.0543(95.7)	.3503(95.3)	.7598(95.4)	.9252(93.8)	.3480(95.5)	.7292(95.3)	.3218(95.4)	.6869(95.0)
			(4,0*10,4)	1.1376(95.7)	.3735(95.8)	.8082(95.7)	1.0387(93.3)	.3694(95.8)	.7682(95.6)	.3364(95.8)	.7423(94.8)
		10	(10,0*9)	1.3137(96.3)	.4505(95.7)	1.0068(95.7)	1.3290(92.9)	.4406(95.5)	.9138(95.8)	.3802(95.8)	.9002(94.5)
			(0*4,10,0*5)	1.1301(96.0)	.4101(95.5)	.9017(95.5)	1.0266(93.0)	.4023(95.5)	.8264(95.3)	.3516(95.4)	.8057(94.1)
			(0*9,10)	1.1099(95.3)	.3838(95.4)	.8407(95.4)	.9319(93.5)	.3792(95.4)	.7962(95.5)	.3421(95.4)	.7445(94.7)
			(5,0*8,5)	1.1893(95.5)	.3988(95.7)	.8744(95.7)	1.0763(93.7)	.3928(95.7)	.8249(95.8)	.3562(95.7)	.7764(93.8)
	30	22	(8,0*21)	.8993(95.6)	.2961(95.6)	.6289(95.5)	.8470(95.1)	.2941(95.5)	.6135(95.7)	.2760(95.5)	.5930(95.4)
			(0*10,8,0*11)	.7718(95.8)	.2752(96.0)	.5839(95.8)	.7392(94.9)	.2729(95.8)	.5671(95.8)	.2566(95.8)	.5440(95.0)
			(0*21,8)	.7587(96.0)	.2597(96.1)	.5473(95.9)	.7219(94.7)	.2589(96.1)	.5402(96.1)	.2472(96.1)	.5233(95.1)
			(4,0*20,4)	.7986(95.8)	.2747(95.8)	.5823(96.0)	.7746(94.9)	.2736(95.8)	.5702(95.8)	.2594(95.8)	.5468(95.1)
		18	(12,0*17)	1.0238(96.2)	.3493(95.9)	.7549(96.0)	.9544(93.5)	.3469(95.9)	.7273(96.0)	.3201(96.0)	.7011(94.2)
			(0*8,12,0*9)	.8587(96.0)	.3088(95.7)	.6596(95.5)	.7797(94.0)	.3064(95.7)	.6330(95.6)	.2818(95.5)	.6032(93.8)
			(0*17,12)	.8169(95.4)	.2717(95.5)	.5745(95.5)	.7311(94.8)	.2715(95.5)	.5640(95.4)	.2593(95.5)	.5402(95.2)
			(6,0*16,6)	.9274(95.5)	.3041(95.4)	.6474(95.4)	.8219(95.2)	.3023(95.4)	.6301(95.4)	.2852(95.4)	.5963(94.9)
		14	(16,0*13)	1.0703(95.7)	.3798(95.9)	.8350(95.9)	1.0507(92.5)	.3744(95.7)	.7892(95.8)	.3405(95.8)	.7607(95.0)
			(0*6,16,0*7)	.8364(96.0)	.3292(95.9)	.7143(96.0)	.8220(93.0)	.3243(96.1)	.6737(96.0)	.2935(95.9)	.6491(93.9)
			(0*13,16)	.7889(95.9)	.2970(95.6)	.6349(95.6)	.7241(93.2)	.2931(95.6)	.6125(95.7)	.2728(95.6)	.5842(93.4)
			(8,0*12,8)	.8908(96.0)	.3127(95.8)	.6690(95.6)	.8411(93.5)	.3101(95.8)	.6508(95.8)	.2913(95.8)	.6172(94.3)
	40	30	(10,0*29)	.7852(96.1)	.2638(95.9)	.5546(95.9)	.7184(93.8)	.2632(95.9)	.5455(95.9)	.2503(95.9)	.5311(94.4)
			(0*14,10,0*15)	.6854(95.7)	.2400(95.7)	.5019(95.7)	.6298(94.8)	.2392(95.7)	.4946(95.8)	.2279(95.6)	.4769(95.1)
			(0*29,10)	.6625(95.5)	.2235(95.7)	.4651(95.7)	.6188(95.4)	.2223(95.6)	.4614(95.6)	.2150(95.6)	.4519(96.3)
			(5,0*28,5)	.6970(96.3)	.2392(95.6)	.5024(95.8)	.6639(94.8)	.2384(95.6)	.4969(95.7)	.2288(95.5)	.4824(94.4)
		26	(14,0*25)	.8351(95.6)	.2796(95.6)	.5941(95.7)	.7727(95.1)	.2789(95.5)	.5782(95.4)	.2644(95.6)	.5519(94.3)
			(0,12,14,0*13)	.6965(95.8)	.2465(96.1)	.5173(96.1)	.6510(94.0)	.2449(96.1)	.5026(96.0)	.2320(96.0)	.4908(94.1)
			(0*25,14)	.6685(95.5)	.2246(95.6)	.4680(95.6)	.6223(96.3)	.2242(95.3)	.4645(95.6)	.2166(95.6)	.4487(94.7)
			(7,0*24,7)	.7739(95.5)	.2511(95.7)	.5230(95.5)	.6895(95.4)	.2507(95.7)	.5163(95.5)	.2395(95.6)	.4971(94.5)
		20	(20,0*19)	.9547(96.0)	.3126(95.8)	.6646(95.7)	.9035(95.2)	.3104(96.0)	.6425(95.7)	.2889(95.8)	.6324(95.8)
			(0*9,20,0*10)	.7345(95.7)	.2710(95.8)	.5745(95.9)	.7060(94.3)	.2695(95.9)	.5530(95.8)	.2505(95.7)	.5379(95.2)
			(0*19,20)	.6939(95.6)	.2411(95.6)	.5069(95.6)	.6244(95.1)	.2406(95.6)	.4989(95.7)	.2306(95.6)	.4804(95.1)
			(10,0*18,10)	.7878(95.9)	.2599(95.8)	.5424(95.7)	.7267(94.6)	.2578(95.5)	.5370(95.8)	.2465(95.5)	.5316(96.1)
1.75	20	14	(6,0*13)	1.0451(95.4)	.3678(95.7)	.8021(95.7)	.9937(92.7)	.3636(95.7)	.7593(95.8)	.3307(95.8)	.7323(94.6)
			(0*6,6,0*7)	.9452(95.4)	.3456(95.4)	.7485(95.5)	.8686(93.1)	.3403(95.2)	.7110(95.6)	.3118(95.5)	.6796(93.9)
			(0*13,6)	.9140(95.4)	.3180(95.3)	.6821(95.4)	.8308(93.9)	.3146(95.2)	.6584(95.4)	.2938(95.3)	.6194(93.7)
			(3,0*12,3)	.9428(95.9)	.3343(95.7)	.7250(95.7)	.8968(93.7)	.3314(95.7)	.6980(95.7)	.3074(95.7)	.6638(94.0)
		12	(8,0*11)	1.1992(95.8)	.4162(95.8)	.9267(96.1)	1.0982(93.0)	.4112(95.8)	.8635(96.0)	.3687(96.1)	.8168(94.8)
			(0*5,8,0*6)	.9605(95.6)	.3759(95.8)	.8292(95.8)	.9179(92.5)	.3686(95.8)	.7737(96.0)	.3314(95.9)	.7319(93.9)
			(0*11,8)	.9573(95.9)	.3430(95.6)	.7450(95.6)	.8448(93.6)	.3392(95.6)	.7098(95.6)	.3116(95.6)	.6709(94.8)
			(4,0*10,4)	1.0448(95.5)	.3568(95.3)	.7757(95.3)	.9471(93.3)	.3539(95.3)	.7386(95.4)	.3231(95.1)	.7076(94.9)
		10	(10,0*9)	1.2379(96.4)	.4338(96.4)	.9621(96.2)	1.2197(92.7)	.4256(96.3)	.8875(96.4)	.3724(96.3)	.8705(94.3)
			(0*4,10,0*5)	1.0384(95.9)	.3953(95.5)	.8705(95.5)	.9850(92.7)	.3870(95.5)	.8029(95.5)	.3421(95.4)	.7789(94.1)
			(0*9,10)	.9936(95.6)	.3767(95.9)	.8245(95.9)	.8596(93.5)	.3696(95.7)	.7723(95.9)	.3319(95.8)	.7238(94.5)
			(5,0*8,5)	1.1194(95.6)	.3946(95.9)	.8646(95.8)	.9893(93.3)	.3881(95.8)	.8155(96.0)	.3499(95.9)	.7621(94.7)

				CL(CP)							
T	n	m	$\boldsymbol{R}$	$\hat{\theta}$	$\hat{Re}$	$\hat{H}$	$\hat{\theta}_{MS}$	$\hat{Re}_{ML}(h=-.5)$	$\hat{H}_{ML}(h=-.5)$	$\hat{Re}_{ML}(h=.5)$	$\hat{H}_{ML}(h=.5)$
1.75	30	22	(8,0*21)	.8001(95.7)	.2812(95.7)	.5978(95.7)	.7793(95.0)	.2797(95.7)	.5846(96.0)	.2628(95.9)	.5566(95.1)
			(0*10,8,0*11)	.7154(95.5)	.2615(95.6)	.5547(95.8)	.6972(94.8)	.2610(95.7)	.5377(95.7)	.2454(95.7)	.5191(95.1)
			(0*21,8)	.6956(95.6)	.2434(95.7)	.5104(95.8)	.6686(94.9)	.2428(95.8)	.5040(95.8)	.2319(95.7)	.4871(95.8)
			(4,0*20,4)	.7595(95.8)	.2643(95.3)	.5571(95.1)	.7190(94.7)	.2627(95.3)	.5498(95.4)	.2505(95.3)	.5276(95.4)
		18	(12,0*17)	.9701(95.5)	.3285(95.5)	.7055(95.7)	.8718(94.1)	.3258(95.5)	.6814(95.8)	.3024(95.7)	.6451(94.4)
			(0*8,12,0*9)	.8053(95.2)	.2997(95.6)	.6425(95.7)	.7455(94.4)	.2972(95.6)	.6142(95.5)	.2746(95.4)	.5772(94.4)
			(0*17,12)	.7514(95.6)	.2637(95.5)	.5571(95.5)	.6753(94.5)	.2633(95.5)	.5433(95.5)	.2491(95.5)	.5228(94.9)
			(6,0*16,6)	.8395(95.5)	.2850(95.7)	.6005(95.7)	.7568(94.5)	.2837(95.7)	.5875(95.7)	.2667(95.7)	.5639(94.9)
		14	(16,0*13)	1.0266(95.5)	.3636(95.5)	.7923(95.5)	.9752(92.6)	.3594(95.5)	.7521(95.5)	.3272(95.6)	.7249(94.5)
			(0*6,16,0*7)	.8220(95.7)	.3201(96.4)	.6897(96.5)	.7990(93.3)	.3152(96.3)	.6526(96.5)	.2885(96.5)	.6313(93.5)
			(0*13,16)	.7477(95.9)	.2913(95.8)	.6260(95.7)	.6859(93.2)	.2887(95.8)	.5974(95.8)	.2673(95.8)	.5659(93.3)
			(8,0*12,8)	.8537(95.8)	.3088(95.8)	.6633(95.8)	.7800(93.1)	.3061(95.8)	.6403(95.8)	.2846(95.7)	.6028(93.3)
	40	30	(10,0*29)	.6985(95.8)	.2440(95.6)	.5142(95.8)	.6638(94.8)	.2434(95.7)	.5050(95.7)	.2323(95.6)	.4890(94.8)
			(0*14,10,0*15)	.6269(95.9)	.2239(95.6)	.4692(95.7)	.5972(94.8)	.2231(95.6)	.4604(95.7)	.2136(95.6)	.4476(94.9)
			(0*29,10)	.6148(95.6)	.2100(95.6)	.4374(95.6)	.5741(95.4)	.2094(95.6)	.4344(95.7)	.2033(95.6)	.4214(95.8)
			(5,0*28,5)	.6425(95.7)	.2228(95.8)	.4651(95.8)	.6132(95.5)	.2226(96.0)	.4611(96.0)	.2134(95.8)	.4486(96.2)
		26	(14,0*25)	.8051(95.9)	.2655(95.9)	.5564(95.9)	.7198(94.1)	.2654(96.0)	.5458(95.9)	.2508(96.0)	.5254(94.2)
			(0,12,14,0*13)	.6639(95.5)	.2380(95.7)	.4992(95.6)	.6238(94.7)	.2372(95.7)	.4867(95.8)	.2241(95.7)	.4674(94.9)
			(0*25,14)	.6537(95.5)	.2212(95.5)	.4619(95.6)	.5788(93.9)	.2212(95.4)	.4547(95.6)	.2136(95.6)	.4359(94.7)
			(7,0*24,7)	.6762(95.8)	.2300(95.7)	.4817(96.0)	.6354(95.4)	.2297(95.7)	.4749(95.9)	.2213(95.9)	.4619(94.7)
		20	(20,0*19)	.8905(95.9)	.2970(95.9)	.6311(96.1)	.8292(94.8)	.2962(96.0)	.6117(96.0)	.2765(96.0)	.5988(95.3)
			(0*9,20,0*10)	.7149(95.8)	.2644(95.8)	.5602(95.8)	.6835(94.2)	.2627(95.8)	.5404(95.8)	.2453(95.8)	.5224(94.9)
			(0*19,20)	.6463(95.4)	.2355(95.5)	.4963(95.6)	.5868(94.9)	.2349(95.6)	.4814(95.5)	.2223(95.5)	.4662(95.3)
			(10,0*18,10)	.7234(95.3)	.2475(95.2)	.5232(95.3)	.6684(95.1)	.2469(95.2)	.5139(95.3)	.2350(95.3)	.4950(95.5)

In Table 4, interval estimates for the parameter, reliability, and hazard functions are presented for various n, m, T, and progressive censoring schemes. The approximate CI are computed using the normality property of the MLE. High posterior density (HPD) credible intervals using the MCMC method are also tabulated. Additionally, Table 4 includes the CL and CP for the interval estimates. The CL and CP for the respective MLEs, calculated using the approximation method, are shown in the fifth to seventh columns. The CL and CP for the Bayesian estimates, obtained using the MCMC method, are displayed in the remaining columns.

In general, the MSE and bias decrease as the sample size and progressive censored sample size increase. For fixed sample size and progressive censored sample size, the MSE and bias typically decrease as the time T increases. Additionally, the estimates for progressive censoring scheme (c) behave quite similarly in terms of bias and MSE. The CL of the approximate CI and the HPD credible interval using the MCMC method tends to decrease with an effective increase in sample size and progressive censored sample size.

In Tables 1–3, we observe that the Bayesian estimators under SELF and LLF for $h = 0.5$ behave almost similarly in terms of MSE. However, the Bayesian estimators under LLF are more efficient than those under SELF in terms of MSE. In Table 4, we note that the CL of the approximate CI is wider than the corresponding CL of the HPD credible interval. The CP of the approximate CI is mostly above 95%. In the case of the credible interval, however, the CP in some instances falls below 95%, while in other schemes, it exceeds 95%.

4.2. Example

In order to analyze the real life data set, in this section, we use the proposed estimators in the above section. Lawless (1982) gives the results of a study to investigate the effect of a certain kind of therapy for 30 leukemia patients. After the therapy, patients go into remission for some period of time, the length of which is random. The observed times were 1, 1, 2, 2, 2, 6, 6, 6, 7, 8, 9, 9, 10, 12, 13, 14, 18, 19, 24, 26, 29, 31, 42, 45, 50, 57, 60, 71, 85, 91 weeks. Fattah and Ahmed (2018) indicated that

Table 5: Remission times for patients receiving a particular leukemia therapy

n	m	T	$\boldsymbol{R}$	$\hat{\theta}$	$\hat{\theta}_{SL}$	$\hat{\theta}_{LL}(h=-.5)$	$\hat{\theta}_{LL}(h=.5)$	$\hat{\theta}_{MS}$	$\hat{\theta}_{ML}(h=-.5)$	$\hat{\theta}_{ML}(h=.5)$
30	24	30	(6, 0*23)	7.0845	6.8382	7.3673	6.2261	6.4233	8.7476	6.2968
	14	30	(3, 0*10, 3, 0*2)	8.0966	7.7379	8.5409	7.0116	7.2485	10.4538	7.1166
n	m	T	$\boldsymbol{R}$	$\hat{Re}$	$\hat{Re}_{SL}$	$\hat{Re}_{LL}(h=-.5)$	$\hat{Re}_{LL}(h=.5)$	$\hat{Re}_{MS}$	$\hat{Re}_{ML}(h=-.5)$	$\hat{Re}_{ML}(h=.5)$
30	24	30	(6, 0*23)	.3752	.3561	.3963	.3064	.3228	.4887	.3123
	14	30	(3, 0*10, 3, 0*2)	.4474	.4229	.4760	.3696	.3875	.5798	.3776
n	m	T	$\boldsymbol{R}$	$\hat{H}$	$\hat{H}_{SL}$	$\hat{H}_{LL}(h=-.5)$	$\hat{H}_{LL}(h=.5)$	$\hat{H}_{MS}$	$\hat{H}_{ML}(h=-.5)$	$\hat{H}_{ML}(h=.5)$
30	24	30	(6, 0*23)	.0959	.1004	.1135	.0910	.1090	.1119	.0722
	14	30	(3, 0*10, 3, 0*2)	.0802	.0853	.0972	.0746	.0930	.0953	.0564

the exponential distribution provides a satisfactory fit. From this data, we take PHC scheme such as Scheme 1: $\boldsymbol{R} = (6, 0 * 23)$, $m = 24$ and $T = 30$, and Scheme 2: $\boldsymbol{R} = (3, 0 * 10, 3, 0 * 2)$, $m = 14$ and $T = 30$.

We observed that Bayesian estimates of the parameter, reliability and hazard functions using LF with $h = -.5$ is marginally larger than the corresponding MLE of the parameter, reliability, and hazard functions. Conversely, the Bayesian estimates using the SELF and LLF with $h = .5$ are smaller than the corresponding MLEs.

5. Conclusions

The concept of length-biased distribution finds various applications in biomedical areas such as family history and disease, survival and intermediate events, and the latency period of AIDS due to blood transfusion. Additionally, in biomedical analysis, there are numerous situations where units are removed or lost from experimentation before being observed. Therefore, in this paper, we consider the MLE and Bayesian estimators of the parameter, reliability, and hazard functions of the LBED under the PHC scheme. We derive the Bayesian estimators of the unknown parameter, reliability, and hazard functions based on flexible loss functions. Furthermore, we derive the Bayesian estimators using Lindley's approximation and the MCMC methods. In particular, the MCMC method is used to obtain the credible interval.

In general, the MSE and bias decrease as the sample size and progressive censored sample size increase. For fixed sample size and progressive censored sample size, generally, the MSE and bias decrease as the time T increases. Additionally, the CL of the approximate CI and HPD credible interval using MCMC method tends to decrease with the effective increase in sample size and progressive censored sample size.

Among the proposed estimators, we observed that Bayesian estimators are superior to the respective MLE in terms of MSE and bias. Among the Bayesian estimators, we can observe that the Baysian estimators under LLF is more efficient than the Bayesian estimators under SELF in terms of MSE. Also, we observe that the CL of approximate CI is wider than the corresponding CL of HPD credible interval.

Although we focused on the unknown parameter, reliability and hazard functions estimate of the LBED based on PHC scheme, the Bayesian estimation can be applied to any other length biased distributions. The estimation of the parameter, reliability and hazard functions from other length biased distributions based on PHC scheme is of potential interest in future research.

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Received July 12, 2024; Revised August 20, 2024; Accepted August 24, 2024