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INVESTIGATING THE DUAL QUATERNION EXTENSION OF THE $\mathcal{DGC}$ LEONARDO SEQUENCE

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Abstract. In this study, we introduce a new generalization of the Leonardo sequence, dual quaternions with the $\mathcal{DGC}$ Leonardo sequence coefficients, depending on the parameter $\mathfrak{p} \in \mathbb{R}$. This generalization gives dual quaternions with the dual-complex Leonardo sequence for $\mathfrak{p} = -1$, dual quaternions with the hyper-dual Leonardo sequence for $\mathfrak{p} = 0$, and dual quaternions with the dual-hyperbolic Leonardo sequence for $\mathfrak{p} = 1$. The basic algebraic structures and some special characteristic relations are presented, as well as the Binet's formula, generating function, d'Ocagne's, Catalan's, Cassini's, and Tagiuri's identities.

1. Introduction

Hypercomplex numbers, such as quaternions, tessarines, coquaternions, biquaternions, and octonions, find wide application in fields such as physics, geometry, robotics, quantum mechanics, and computer graphics due to their ability to represent rotations and transformations in higher-dimensional spaces. In mathematics, generalized complex numbers $\mathbb{C}_\mathfrak{p}$ are well-known extensions of complex numbers within the realm of hypercomplex numbers, [22, 24]. These numbers are defined in the form $z = a_1 + a_2 J$, where $a_1, a_2 \in \mathbb{R}$. Here, J denotes the generalized complex unit satisfying $J^2 = \mathfrak{p}$, $J \notin \mathbb{R}$, $\mathfrak{p} \in \mathbb{R}$. The special cases of generalized complex numbers $\mathbb{C}_\mathfrak{p}$ include:

- complex numbers $\mathbb{C}$ for $\mathfrak{p} = -1$ (see [49]),
- hyperbolic numbers (double, binary, split-complex, perplex numbers) $\mathbb{H}$ for $\mathfrak{p} = 1$ (see [9, 15, 42]),
- dual numbers $\mathbb{D}$ for $\mathfrak{p} = 0$ (see [38, 44]).

The dual-generalized complex numbers ($\mathcal{DGC}$) appear as a generalization of both generalized complex numbers and dual numbers, [17]. The set of $\mathcal{DGC}$ numbers is defined in [17] by the following way:

$$\mathbb{DC}_\mathfrak{p} = \left\{ \tilde{a} = z_1 + z_2\varepsilon : \ z_1, z_2 \in \mathbb{C}_\mathfrak{p}, \ \varepsilon^2 = 0, \varepsilon \neq 0, \varepsilon \notin \mathbb{R} \right\}.$$

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Each $\mathcal{DGC}$ number is of the form $\tilde{a} = z_1 + z_2\varepsilon = a_1 + a_2J + a_3\varepsilon + a_4J\varepsilon$, where a_1, a_2, a_3, a_4 are real components, and the $\mathcal{DGC}$ units $\{J, \varepsilon, J\varepsilon\}$ satisfy the following conditions:

$$\varepsilon^2 = 0,\ J^2 = \mathfrak{p},\ J\varepsilon = \varepsilon J,\ (J\varepsilon)^2 = 0. \tag{1}$$

The set $\mathbb{DC}_{\mathfrak{p}}$ forms a commutative ring with unity and a vector space over real numbers, [17]. For special values of $\mathfrak{p}$, $\mathcal{DGC}$ numbers correspond to the following well-known 4-component numbers:

- dual-complex numbers (complex-dual numbers) for $\mathfrak{p} = -1$ (see [6, 7, 30, 35]),
- dual-hyperbolic numbers (hyperbolic-dual numbers) for $\mathfrak{p} = 1$ (see [1, 30]),
- hyper-dual numbers for $\mathfrak{p} = 0$ (see [10, 13, 14]).

Furthermore, real quaternions extend complex numbers into 4-dimensions, defined as $q = q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k}$, where q_0, q_1, q_2, q_3 are real components, and $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ are non-real quaternionic units with the following multiplication schema (see [19–21]):

$$\begin{array}{ccc} \mathbf{i}^2 = -1, & \mathbf{j}^2 = -1, & \mathbf{k}^2 = -1, \\ \mathbf{ij} = -\mathbf{ji} = \mathbf{k}, & \mathbf{jk} = -\mathbf{kj} = \mathbf{i}, & \mathbf{ki} = -\mathbf{ik} = \mathbf{j}. \end{array}$$

Changing conditions in the multiplication schema gives various type of quaternions. The most well-known generalization of dual numbers is dual quaternions, [8, 11, 12, 28, 29, 31, 50]. It is in the same form but with different multiplication conditions for quaternionic units as

$$\begin{array}{ccc} i^2 = 0, & j^2 = 0, & k^2 = 0, \\ ij = ji = 0, & jk = kj = 0, & ki = ik = 0. \end{array} \tag{2}$$

In addition to these subjects, sequences play a crucial role in mathematics, particularly in number theory. Sequences like the Fibonacci, Lucas, Leonardo, Pell, Narayana's cows, Horadam, and Oresme sequences are extensively studied. The Leonardo sequence is defined by second-order linear recurrence relations and explored in various contexts. The Leonardo sequence is defined recursively by the following non-homogeneous recurrence relation:

$$Le_n = Le_{n-1} + Le_{n-2} + 1,\ (n \geq 2)$$

with the initial values $Le_0 = Le_1 = 1$, or the homogeneous recurrence relation:

$$Le_{n+1} = 2Le_n - Le_{n-2},\ (n \geq 2)$$

with the initial values $Le_0 = Le_1 = 1$ and $Le_2 = 3$. Here, Le_n is the n-th Leonardo number, [4]. Defining sequences recursively with two or more components is also popular in the literature. Many studies of Leonardo numbers have presented in [2–5, 23, 25, 26, 32–34, 36, 37, 39–41, 43, 45–48, 51, 52].

Recently, the $\mathcal{DGC}$ Leonardo sequence has been introduced, characterized by 4-component terms and defined recursively [45,52]. The n-th $\mathcal{DGC}$ Leonardo number is expressed as:

$$\tilde{\mathcal{L}}e_n = Le_n + Le_{n+1}J + Le_{n+2}\varepsilon + Le_{n+3}J\varepsilon.$$

The $\mathcal{DGC}$ Leonardo sequence satisfies the following second-order non-homogeneous relation:

$$\tilde{\mathcal{L}}e_n = \tilde{\mathcal{L}}e_{n-1} + \tilde{\mathcal{L}}e_{n-2} + \tilde{1}, \quad (n \geq 2), \tag{3}$$

where $\tilde{1} = 1 + J + \varepsilon + J\varepsilon$ with the initial values $\tilde{\mathcal{L}}e_0 = 1 + J + 3\varepsilon + 5J\varepsilon$, $\tilde{\mathcal{L}}e_1 = 1 + 3J + 5\varepsilon + 9J\varepsilon$. The homogeneous recurrence relation of the $\mathcal{DGC}$ Leonardo sequence is

$$\tilde{\mathcal{L}}e_{n+1} = 2\tilde{\mathcal{L}}e_n - \tilde{\mathcal{L}}e_{n-2}$$

with the initial values $\tilde{\mathcal{L}}e_0 = 1 + J + 3\varepsilon + 5J\varepsilon$, $\tilde{\mathcal{L}}e_1 = 1 + 3J + 5\varepsilon + 9J\varepsilon$, and $\tilde{\mathcal{L}}e_2 = 3+5J+9\varepsilon+15J\varepsilon$. For further information we refer the reader to [45,52].

This paper focuses on studying dual quaternions with the $\mathcal{DGC}$ Leonardo numbers coefficients for $\mathfrak{p} \in \mathbb{R}$. It explores special relations including the Binet's formula, generating function, and various identities such as Catalan's, Cassini's, d'Ocagne's, and Tagiuri's identities.

2. Preliminaries

In this section, we recall some basic notations and results related to the $\mathcal{DGC}$ Fibonacci and $\mathcal{DGC}$ Lucas sequences (see [18]), as well as the $\mathcal{DGC}$ Leonardo sequence (see [45, 52]).

Definition 2.1. *The n-th $\mathcal{DGC}$ Fibonacci number is of the form*

$$\tilde{\mathcal{F}}_n = F_n + F_{n+1}J + F_{n+2}\varepsilon + F_{n+3}J\varepsilon,$$

where F_n is the n-th Fibonacci number and satisfies the following second-order linear recurrence relation:

$$\tilde{\mathcal{F}}_n = \tilde{\mathcal{F}}_{n-1} + \tilde{\mathcal{F}}_{n-2}, \quad (n \geq 2)$$

with the initial values $\tilde{\mathcal{F}}_0 = J + \varepsilon + 2J\varepsilon$ and $\tilde{\mathcal{F}}_1 = 1 + J + 2\varepsilon + 3J\varepsilon$. Similarly, the n-th $\mathcal{DGC}$ Lucas number is of the form

$$\tilde{\mathcal{L}}_n = L_n + L_{n+1}J + L_{n+2}\varepsilon + L_{n+3}J\varepsilon,$$

where L_n is the n-th Lucas number and satisfies the following second-order linear recurrence relation:

$$\tilde{\mathcal{L}}_n = \tilde{\mathcal{L}}_{n-1} + \tilde{\mathcal{L}}_{n-2}, \quad (n \geq 2)$$

with the initial values $\tilde{\mathcal{L}}_0 = 2 + J + 3\varepsilon + 4J\varepsilon$ and $\tilde{\mathcal{L}}_1 = 1 + 3J + 4\varepsilon + 7J\varepsilon$, [18].

Theorem 2.2. *For a positive integer n, the fundamental relations between the $\mathcal{DGC}$ Fibonacci, $\mathcal{DGC}$ Lucas, and $\mathcal{DGC}$ Leonardo sequences are as follows:*

$$\begin{cases} \tilde{\mathcal{L}}e_n = & 2\tilde{\mathcal{F}}_{n+1} - \tilde{1}, \\ \tilde{\mathcal{L}}e_n = & 2\left(\dfrac{\tilde{\mathcal{L}}_n + \tilde{\mathcal{L}}_{n+2}}{5}\right) - \tilde{1}, \\ \tilde{\mathcal{L}}e_{n+3} = & \dfrac{\tilde{\mathcal{L}}_{n+1} + \tilde{\mathcal{L}}_{n+7}}{5} - \tilde{1}, \\ \tilde{\mathcal{L}}e_n = & \tilde{\mathcal{L}}_{n+2} - \tilde{\mathcal{F}}_{n+2} - \tilde{1}, \end{cases}$$

where $\tilde{1} = 1 + J + \varepsilon + J\varepsilon$, [52].

Theorem 2.3. *For a positive integer n, the summation formulas related to the $\mathcal{DGC}$ Leonardo sequence are:*

$$\begin{cases} \displaystyle\sum_{j=0}^{n} \tilde{\mathcal{L}}e_j & = \tilde{\mathcal{L}}e_{n+2} - \tilde{\mathcal{N}}_{n+2} - (J + 2\varepsilon + 5J\varepsilon), \\ \displaystyle\sum_{j=0}^{n} \tilde{\mathcal{L}}e_{2j} & = \tilde{\mathcal{L}}e_{2n+1} - \tilde{\mathcal{N}}_n - (J + J\varepsilon), \\ \displaystyle\sum_{j=0}^{n} \tilde{\mathcal{L}}e_{2j+1} & = \tilde{\mathcal{L}}e_{2n+2} - \tilde{\mathcal{N}}_{n+2} + (J - J\varepsilon), \end{cases}$$

where $\tilde{\mathcal{N}}_n = n + (n+1)J + (n+2)\varepsilon + (n+3)J\varepsilon$, [45, 52].

3. Dual quaternion generalization of the $\mathcal{DGC}$ Leonardo sequence

In the following definition, we determine a dual quaternionic sequence with the $\mathcal{DGC}$ Leonardo number components depending on parameter $\mathfrak{p}$.

Definition 3.1. *The n-th dual quaternion with the $\mathcal{DGC}$ Leonardo sequence coefficients (the n-th $\mathcal{DGC}$ Leonardo dual quaternion), denoted by $\tilde{\mathsf{Q}}\mathsf{L}e_n$, is defined in the following way:*

$$\tilde{\mathsf{Q}}\mathsf{L}e_n = \tilde{\mathcal{L}}e_n + \tilde{\mathcal{L}}e_{n+1}i + \tilde{\mathcal{L}}e_{n+2}j + \tilde{\mathcal{L}}e_{n+3}k, \tag{4}$$

where $\{i, j, k\}$ are the dual quaternionic units that satisfy the conditions given in equation (2) and $\tilde{\mathcal{L}}e_n$ is the n-th $\mathcal{DGC}$ Leonardo number.

This dual quaternion is extended through the $\mathcal{DGC}$ Leonardo sequence to obtain dual quaternion with dual-complex, dual-hyperbolic, and hyper-dual Leonardo sequences coefficients for special cases.

It should be noted that the $\mathcal{DGC}$ units $\{J, \varepsilon, J\varepsilon\}$ which satisfy the conditions given in equation (1) commute with the dual quaternionic units $\{i, j, k\}$. That is, $iJ = Ji$, $i\varepsilon = \varepsilon i$ and $iJ\varepsilon = J\varepsilon i$. This condition also holds for the other quaternionic units. Also, the dual quaternionic units $\{i, j, k\}$ are distinct from the usual dual unit for $\mathfrak{p} = 0$.

Writing the coefficients more clearly gives that

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_n = &\ \tilde{\mathcal{L}}e_n + \tilde{\mathcal{L}}e_{n+1}i + \tilde{\mathcal{L}}e_{n+2}j + \tilde{\mathcal{L}}e_{n+3}k \\ = &\ (Le_n + Le_{n+1}J + Le_{n+2}\varepsilon + Le_{n+3}J\varepsilon) \\ &+(Le_{n+1} + Le_{n+2}J + Le_{n+3}\varepsilon + Le_{n+4}J\varepsilon)i \\ &+(Le_{n+2} + Le_{n+3}J + Le_{n+4}\varepsilon + Le_{n+5}J\varepsilon)j \\ &+(Le_{n+3} + Le_{n+4}J + Le_{n+5}\varepsilon + Le_{n+6}J\varepsilon)k \\ = &\ (Le_n + Le_{n+1}i + Le_{n+2}j + Le_{n+3}k) \\ &+(Le_{n+1} + Le_{n+2}i + Le_{n+3}j + Le_{n+4}k)J \\ &+(Le_{n+2} + Le_{n+3}i + Le_{n+4}j + Le_{n+5}k)\varepsilon \\ &+(Le_{n+3} + Le_{n+4}i + Le_{n+5}j + Le_{n+6}k)J\varepsilon \\ = &\ \hat{\mathcal{L}}e_n + \hat{\mathcal{L}}e_{n+1}J + \hat{\mathcal{L}}e_{n+2}\varepsilon + \hat{\mathcal{L}}e_{n+3}J\varepsilon.\end{aligned}$$

Here, $\hat{\mathcal{L}}e_n$ is referred to as the n-th dual Leonardo quaternion (seen as an ordered Leonardo quadruple number in [36]). It is clear that there is no difference between the dual quaternionic sequence with $\mathcal{DGC}$ Leonardo number coefficients and the $\mathcal{DGC}$ sequence with dual Leonardo quaternions coefficients. We denote the set of all these dual quaternions with the $\mathcal{DGC}$ Leonardo sequence coefficients briefly by $\mathcal{QL}_e$.

Using operations within dual quaternion algebra allows for defining basic arithmetic operations. Let $\tilde{\mathsf{Q}}\mathsf{L}e_n, \tilde{\mathsf{Q}}\mathsf{L}e_m \in \mathcal{QL}_e$. In general, a dual quaternion $\tilde{\mathsf{Q}}\mathsf{L}e_n$ has two parts, a scalar part $S_{\tilde{\mathsf{Q}}\mathsf{L}e_n} = \tilde{\mathcal{L}}e_n$, and a vector part $V_{\tilde{\mathsf{Q}}\mathsf{L}e_n} = \tilde{\mathcal{L}}e_{n+1}i + \tilde{\mathcal{L}}e_{n+2}j + \tilde{\mathcal{L}}e_{n+3}k$. An element $\tilde{\mathsf{Q}}\mathsf{L}e_n^{\dagger}$ is called the conjugate of $\tilde{\mathsf{Q}}\mathsf{L}e_n$ is defined by $\tilde{\mathsf{Q}}\mathsf{L}e_n^{\dagger} = S_{\tilde{\mathsf{Q}}\mathsf{L}e_n} - V_{\tilde{\mathsf{Q}}\mathsf{L}e_n}$. Equality and addition (and hence subtraction) are component-wise defined respectively as follows:

$$\tilde{\mathsf{Q}}\mathsf{L}e_n = \tilde{\mathsf{Q}}\mathsf{L}e_m \leftrightarrow \quad S_{\tilde{\mathsf{Q}}\mathsf{L}e_n} = S_{\tilde{\mathsf{Q}}\mathsf{L}e_m},\ V_{\tilde{\mathsf{Q}}\mathsf{L}e_n} = V_{\tilde{\mathsf{Q}}\mathsf{L}e_m}$$

and

$$\tilde{\mathsf{Q}}\mathsf{L}e_n \pm \tilde{\mathsf{Q}}\mathsf{L}e_m = \left(S_{\tilde{\mathsf{Q}}\mathsf{L}e_n} \pm S_{\tilde{\mathsf{Q}}\mathsf{L}e_m}\right) + \left(V_{\tilde{\mathsf{Q}}\mathsf{L}e_n} \pm V_{\tilde{\mathsf{Q}}\mathsf{L}e_m}\right).$$

The addition is commutative and associative. Also, the multiplication of $\tilde{\mathsf{Q}}\mathsf{L}e_n$ and $\tilde{\mathsf{Q}}\mathsf{L}e_m$ is calculated as:

$$\tilde{\mathsf{Q}}\mathsf{L}e_n\tilde{\mathsf{Q}}\mathsf{L}e_m = S_{\tilde{\mathsf{Q}}\mathsf{L}e_n}S_{\tilde{\mathsf{Q}}\mathsf{L}e_m} + S_{\tilde{\mathsf{Q}}\mathsf{L}e_n}V_{\tilde{\mathsf{Q}}\mathsf{L}e_m} + S_{\tilde{\mathsf{Q}}\mathsf{L}e_m}V_{\tilde{\mathsf{Q}}\mathsf{L}e_n}.$$

The multiplication is commutative, associative, and distributive over addition.

Theorem 3.2. *The n-th $\mathcal{DGC}$ Leonardo dual quaternion satisfies the following non-homogeneous recurrence:*

$$\tilde{\mathsf{Q}}\mathsf{L}e_n = \tilde{\mathsf{Q}}\mathsf{L}e_{n-1} + \tilde{\mathsf{Q}}\mathsf{L}e_{n-2} + \tilde{A},\ \ (n \geq 2) \tag{5}$$

with initial conditions $\tilde{\mathsf{Q}}\mathsf{L}e_0$ and $\tilde{\mathsf{Q}}\mathsf{L}e_1$, where $\tilde{A} = (1+J+\varepsilon+J\varepsilon)(1+i+j+k)$.

Proof. Substituting the non-homogeneous recurrence relation of the $\mathcal{DGC}$ Leonardo sequence in equation (3) into equation (4) yields that

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_n = \;& (\tilde{\mathcal{L}}e_{n-1} + \tilde{\mathcal{L}}e_{n-2} + \tilde{1}) + (\tilde{\mathcal{L}}e_n + \tilde{\mathcal{L}}e_{n-1} + \tilde{1})i \\ & +(\tilde{\mathcal{L}}e_{n+1} + \tilde{\mathcal{L}}e_n + \tilde{1})j + (\tilde{\mathcal{L}}e_{n+2} + \tilde{\mathcal{L}}e_{n+1} + \tilde{1})k \\ = \;& (\tilde{\mathcal{L}}e_{n-1} + \tilde{\mathcal{L}}e_n i + \tilde{\mathcal{L}}e_{n+1} j + \tilde{\mathcal{L}}e_{n+2} k) \\ & +(\tilde{\mathcal{L}}e_{n-2} + \tilde{\mathcal{L}}e_{n-1} i + \tilde{\mathcal{L}}e_n j + \tilde{\mathcal{L}}e_{n+1} k) \\ & +\tilde{1}(1+i+j+k) \\ = \;& \tilde{\mathsf{Q}}\mathsf{L}e_{n-1} + \tilde{\mathsf{Q}}\mathsf{L}e_{n-2} + \tilde{A}.\end{aligned}$$

□

Theorem 3.3. *The homogeneous recurrence relation of the $\mathcal{DGC}$ Leonardo dual quaternion sequence is*

$$\tilde{\mathsf{Q}}\mathsf{L}e_{n+1} = 2\tilde{\mathsf{Q}}\mathsf{L}e_n - \tilde{\mathsf{Q}}\mathsf{L}e_{n-2}, \quad (n \geq 2) \tag{6}$$

with initial conditions $\tilde{\mathsf{Q}}\mathsf{L}e_0$, $\tilde{\mathsf{Q}}\mathsf{L}e_1$ *and* $\tilde{\mathsf{Q}}\mathsf{L}e_2$.

Proof. From equation (5) we see that

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_{n+1} = \;& \tilde{\mathsf{Q}}\mathsf{L}e_n + \tilde{\mathsf{Q}}\mathsf{L}e_{n-1} + \tilde{A} \\ = \;& \tilde{\mathsf{Q}}\mathsf{L}e_n + (\tilde{\mathsf{Q}}\mathsf{L}e_n - \tilde{\mathsf{Q}}\mathsf{L}e_{n-2} - \tilde{A}) + \tilde{A} \\ = \;& 2\tilde{\mathsf{Q}}\mathsf{L}e_n - \tilde{\mathsf{Q}}\mathsf{L}e_{n-2}.\end{aligned}$$

□

In what follows, $\tilde{\alpha} = 1 + \alpha J + \alpha^2\varepsilon + \alpha^3 J\varepsilon$, $\tilde{\beta} = 1 + \beta J + \beta^2\varepsilon + \beta^3 J\varepsilon$, $\tilde{A} = \tilde{1}(1+i+j+k)$, $\tilde{\alpha}^* = \tilde{\alpha}(1+\alpha i+\alpha^2 j+\alpha^3 k)$, and $\tilde{\beta}^* = \tilde{\beta}(1+\beta i+\beta^2 j+\beta^3 k)$.

Theorem 3.4. *For a positive integer n, the Binet's formula of the n-th $\mathcal{DGC}$ Leonardo dual quaternion $\tilde{\mathsf{Q}}\mathsf{L}e_n$ is as follows:*

$$\tilde{\mathsf{Q}}\mathsf{L}e_n = \frac{2\tilde{\alpha}^*\alpha^{n+1} - 2\tilde{\beta}^*\beta^{n+1}}{\alpha - \beta} - \tilde{A}. \tag{7}$$

Proof. We begin by recalling the Binet's formula of the $\mathcal{DGC}$ Leonardo sequence (see [45])

$$\tilde{\mathcal{L}}e_n = 2\left(\frac{\tilde{\alpha}\alpha^{n+1} - \tilde{\beta}\beta^{n+1}}{\alpha - \beta}\right) - \tilde{1}.$$

Substituting this into equation (4) we see that

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_n = \;& \frac{2\tilde{\alpha}(1+\alpha i+\alpha^2 j+\alpha^3 k)\alpha^{n+1} - 2\tilde{\beta}(1+\beta i+\beta^2 j+\beta^3 k)\beta^{n+1}}{\alpha - \beta} \\ & -\tilde{1}(1+i+j+k) \\ = \;& \frac{2\tilde{\alpha}^*\alpha^{n+1} - 2\tilde{\beta}^*\beta^{n+1}}{\alpha - \beta} - \tilde{A}.\end{aligned}$$

□

Theorem 3.5. *For a positive integer n, the generating function for the $\mathcal{DGC}$ Leonardo dual quaternionic sequence is as follows:*

$$\mathsf{G}(x) = \sum_{n=0}^{\infty} \tilde{\mathsf{Q}}\mathsf{L}e_n x^n = \frac{\tilde{\mathsf{Q}}\mathsf{L}e_0 + (\tilde{\mathsf{Q}}\mathsf{L}e_1 - 2\tilde{\mathsf{Q}}\mathsf{L}e_0)x + (\tilde{\mathsf{Q}}\mathsf{L}e_2 - 2\tilde{\mathsf{Q}}\mathsf{L}e_1)x^2}{1 - 2x + x^3},$$

where $1 - 2x + x^3 \neq 0$.

Proof. Let us assume that $\mathsf{G}(x) = \sum_{n=0}^{\infty} \tilde{\mathsf{Q}}\mathsf{L}e_n x^n$ be a generating function of $\tilde{\mathsf{Q}}\mathsf{L}e_n$. This gives

$$\mathsf{G}(x) = \tilde{\mathsf{Q}}\mathsf{L}e_0 + \tilde{\mathsf{Q}}\mathsf{L}e_1 x + \tilde{\mathsf{Q}}\mathsf{L}e_2 x^2 + \sum_{n=3}^{\infty} \tilde{\mathsf{Q}}\mathsf{L}e_n x^n.$$

From equation (6) it follows that

$$\begin{aligned} \mathsf{G}(x) = &\ \tilde{\mathsf{Q}}\mathsf{L}e_0 + \tilde{\mathsf{Q}}\mathsf{L}e_1 x + \tilde{\mathsf{Q}}\mathsf{L}e_2 x^2 + \sum_{n=3}^{\infty} (2\tilde{\mathsf{Q}}\mathsf{L}e_{n-1} - \tilde{\mathsf{Q}}\mathsf{L}e_{n-3}) x^n \\ = &\ \tilde{\mathsf{Q}}\mathsf{L}e_0 + \tilde{\mathsf{Q}}\mathsf{L}e_1 x + \tilde{\mathsf{Q}}\mathsf{L}e_2 x^2 + 2x \left(\left(\sum_{n=0}^{\infty} \tilde{\mathsf{Q}}\mathsf{L}e_n x^n \right) - \tilde{\mathsf{Q}}\mathsf{L}e_0 - \tilde{\mathsf{Q}}\mathsf{L}e_1 x \right) \\ &- x^3 \sum_{n=0}^{\infty} \tilde{\mathsf{Q}}\mathsf{L}e_n x^n. \end{aligned}$$

The proof is completed by showing that

$$(1 - 2x + x^3)\mathsf{G}(x) = \tilde{\mathsf{Q}}\mathsf{L}e_0 + \left(\tilde{\mathsf{Q}}\mathsf{L}e_1 - 2\tilde{\mathsf{Q}}\mathsf{L}e_0\right) x + \left(\tilde{\mathsf{Q}}\mathsf{L}e_2 - 2\tilde{\mathsf{Q}}\mathsf{L}e_1\right) x^2,$$

with $1 - 2x + x^3 \neq 0$. □

3.1. Auxiliary Results

In this subsection, we formulate the key concepts of the $\mathcal{DGC}$ Leonardo dual quaternionic sequence. Particularly, the Catalan's, Cassini's, d'Ocagne's, and Tagiuri's identities are important in sequences because they establish relations between their numbers.

Theorem 3.6. *Let $\tilde{\mathsf{Q}}\mathsf{L}e_n$ be the n-th $\mathcal{DGC}$ Leonardo dual quaternion, $\tilde{\mathcal{Q}}_n$ be the n-th $\mathcal{DGC}$ Fibonacci dual quaternion, and $\tilde{\mathcal{K}}_n$ be the n-th $\mathcal{DGC}$ Lucas dual quaternion. Then the following relations hold:*

1) $\tilde{\mathsf{Q}}\mathsf{L}e_n = 2\tilde{\mathcal{Q}}_{n+1} - \tilde{A}$,
2) $\tilde{\mathsf{Q}}\mathsf{L}e_n = \tilde{\mathcal{Q}}_n + \tilde{\mathcal{K}}_n - \tilde{A}$,

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$\tilde{\mathcal{Q}}_n$ is of the form $\tilde{\mathcal{Q}}_n = \tilde{\mathcal{F}}_n + \tilde{\mathcal{F}}_{n+1} i + \tilde{\mathcal{F}}_{n+2} j + \tilde{\mathcal{F}}_{n+3} k$, where $\{i, j, k\}$ are the dual quaternionic units and $\tilde{\mathcal{F}}_n$ is the n-th $\mathcal{DGC}$ Fibonacci number (see [16]).

$\tilde{\mathcal{K}}_n$ is of the form $\tilde{\mathcal{K}}_n = \tilde{\mathcal{L}}_n + \tilde{\mathcal{L}}_{n+1} i + \tilde{\mathcal{L}}_{n+2} j + \tilde{\mathcal{L}}_{n+3} k$, where $\{i, j, k\}$ are the dual quaternionic units and $\tilde{\mathcal{L}}_n$ is the n-th $\mathcal{DGC}$ Lucas number (see [16]).

3) $\tilde{\mathsf{Q}}\mathsf{L}e_n = \dfrac{2\left(\tilde{\mathcal{K}}_n + \tilde{\mathcal{K}}_{n+2}\right)}{5} - \tilde{A}$,

4) $\tilde{\mathsf{Q}}\mathsf{L}e_{n+3} = \dfrac{\tilde{\mathcal{K}}_{n+1} + \tilde{\mathcal{K}}_{n+7}}{5} - \tilde{A}$,

5) $\tilde{\mathsf{Q}}\mathsf{L}e_n = \tilde{\mathcal{K}}_{n+2} - \tilde{\mathcal{Q}}_{n+2} - \tilde{A}$.

Proof. If we use the corresponding relations between the $\mathcal{DGC}$ Leonardo, $\mathcal{DGC}$ Fibonacci, and $\mathcal{DGC}$ Lucas sequences in Theorem 2.2, then we get the desired results.

1) Applying equation (4) and $\tilde{\mathcal{L}}e_n = 2\tilde{\mathcal{F}}_{n+1} - \tilde{1}$ (see Theorem 2.2), we can see that

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_n &= \tilde{\mathcal{L}}e_n + \tilde{\mathcal{L}}e_{n+1}i + \tilde{\mathcal{L}}e_{n+2}j + \tilde{\mathcal{L}}e_{n+3}k \\ &= (2\tilde{\mathcal{F}}_{n+1} - \tilde{1}) + (2\tilde{\mathcal{F}}_{n+2} - \tilde{1})i + (2\tilde{\mathcal{F}}_{n+3} - \tilde{1})j + (2\tilde{\mathcal{F}}_{n+4} - \tilde{1})k \\ &= 2\tilde{Q}_{n+1} - \tilde{A}.\end{aligned}$$

2) From equation (4) and $\tilde{\mathcal{L}}e_n = \tilde{\mathcal{F}}_n + \tilde{\mathcal{L}}_n - \tilde{1}$ (see [45]), we obtain

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_n &= (\tilde{\mathcal{F}}_n + \tilde{\mathcal{F}}_{n+1}i + \tilde{\mathcal{F}}_{n+2}j + \tilde{\mathcal{F}}_{n+3}k) + (\tilde{\mathcal{L}}_n + \tilde{\mathcal{L}}_{n+1}i + \tilde{\mathcal{L}}_{n+2}j + \tilde{\mathcal{L}}_{n+3}k) \\ &\quad -\tilde{A} \\ &= \tilde{\mathcal{Q}}_n + \tilde{\mathcal{K}}_n - \tilde{A}.\end{aligned}$$

3) By equation (4) and $\tilde{\mathcal{L}}e_n = \frac{2\left(\tilde{\mathcal{L}}_n + \tilde{\mathcal{L}}_{n+2}\right)}{5} - \tilde{1}$ (see Theorem 2.2), we can see that

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_n &= \tfrac{2\left(\tilde{\mathcal{L}}_n + \tilde{\mathcal{L}}_{n+1}i + \tilde{\mathcal{L}}_{n+2}j + \tilde{\mathcal{L}}_{n+3}k\right) + 2\left(\tilde{\mathcal{L}}_{n+2} + \tilde{\mathcal{L}}_{n+3}i + \tilde{\mathcal{L}}_{n+4}j + \tilde{\mathcal{L}}_{n+5}k\right)}{5} \\ &\quad -\tilde{1}\,(1 + i + j + k) \\ &= \tfrac{2\left(\tilde{\mathcal{K}}_n + \tilde{\mathcal{K}}_{n+2}\right)}{5} - \tilde{A}.\end{aligned}$$

The other parts can be easily proved by using same methods. □

Theorem 3.7. *For a positive integer n, the following summation formulas related to the $\mathcal{DGC}$ Leonardo dual quaternionic sequence hold:*

1) $\displaystyle\sum_{j=0}^{n} \tilde{\mathsf{Q}}\mathsf{L}e_j = \tilde{\mathsf{Q}}\mathsf{L}e_{n+2} - \tilde{\mathsf{Q}}\mathsf{N}_{n+2} - (J + 2\varepsilon + 5J\varepsilon)(1 + i + j + k) - \tilde{\mathcal{L}}e_0 i - (\tilde{\mathcal{L}}e_2 - \tilde{1})j - (2\tilde{\mathcal{L}}e_2 - \tilde{1})k$,

2) $\displaystyle\sum_{j=0}^{n} \tilde{\mathsf{Q}}\mathsf{L}e_{2j} = \tilde{\mathsf{Q}}\mathsf{L}e_{2n+1} - \tilde{\mathsf{Q}}\mathsf{N}_n - ((J + J\varepsilon) + (1 + \varepsilon + 2J\varepsilon)\,i + (J + 2\varepsilon + 5J\varepsilon)\,j + (1 + 2J + 5\varepsilon + 10J\varepsilon)\,k)$,

3) $\displaystyle\sum_{j=0}^{n} \tilde{\mathsf{Q}}\mathsf{L}e_{2j+1} = \tilde{\mathsf{Q}}\mathsf{L}e_{2n+2} - \tilde{\mathsf{Q}}\mathsf{N}_{n+2} + ((J - J\varepsilon) + (1 - \varepsilon - 4J\varepsilon)\,i - (J + 4\varepsilon + 9J\varepsilon)\,j - (1 + 4J + 9\varepsilon + 18J\varepsilon)\,k)$,

where $\tilde{\mathsf{Q}}\mathsf{N}_n = \tilde{\mathcal{N}}_n + \tilde{\mathcal{N}}_{n+1}i + \tilde{\mathcal{N}}_{n+2}j + \tilde{\mathcal{N}}_{n+3}k$ *with* $\tilde{\mathcal{N}}_n = n + (n+1)J + (n+2)\varepsilon + (n+3)J\varepsilon$.

Proof. 1) According to equation (4), we have:

$$\begin{aligned}\sum_{j=0}^{n}\tilde{\mathsf{Q}}\mathsf{L}e_j =& \ \tilde{\mathsf{Q}}\mathsf{L}e_0+\tilde{\mathsf{Q}}\mathsf{L}e_1+\tilde{\mathsf{Q}}\mathsf{L}e_2+\cdots+\tilde{\mathsf{Q}}\mathsf{L}e_n\\ =& \ \sum_{j=0}^{n}\tilde{\mathcal{L}}e_j+\left(\sum_{j=0}^{n+1}\tilde{\mathcal{L}}e_j-\tilde{\mathcal{L}}e_0\right)i+\left(\sum_{j=0}^{n+2}\tilde{\mathcal{L}}e_j-\tilde{\mathcal{L}}e_0-\tilde{\mathcal{L}}e_1\right)j\\ &+\left(\sum_{j=0}^{n+3}\tilde{\mathcal{L}}e_j-\tilde{\mathcal{L}}e_0-\tilde{\mathcal{L}}e_1-\tilde{\mathcal{L}}e_2\right)k.\end{aligned}$$

Then, from Theorem 2.3 we get:

$$\begin{aligned}\sum_{j=0}^{n}\tilde{\mathsf{Q}}\mathsf{L}e_j =& \ \left(\tilde{\mathcal{L}}e_{n+2}+\tilde{\mathcal{L}}e_{n+3}i+\tilde{\mathcal{L}}e_{n+4}j+\tilde{\mathcal{L}}e_{n+5}k\right)\\ &-\left(\tilde{\mathcal{N}}_{n+2}+\tilde{\mathcal{N}}_{n+3}i+\tilde{\mathcal{N}}_{n+4}j+\tilde{\mathcal{N}}_{n+5}k\right)\\ &-(J+2\varepsilon+5J\varepsilon)(1+i+j+k)-\tilde{\mathcal{L}}e_0 i-(\tilde{\mathcal{L}}e_0+\tilde{\mathcal{L}}e_1)j\\ &-(\tilde{\mathcal{L}}e_0+\tilde{\mathcal{L}}e_1+\tilde{\mathcal{L}}e_2)k\\ =& \ \tilde{\mathsf{Q}}\mathsf{L}e_{n+2}-\tilde{\mathsf{Q}}\tilde{\mathsf{N}}_{n+2}-(J+2\varepsilon+5J\varepsilon)(1+i+j+k)-\tilde{\mathcal{L}}e_0 i\\ &-(\tilde{\mathcal{L}}e_2-\tilde{1})j-(2\tilde{\mathcal{L}}e_2-\tilde{1})k.\end{aligned}$$

2) By equation (4) and Theorem 2.3, we get that

$$\begin{aligned}\sum_{j=0}^{n}\tilde{\mathsf{Q}}\mathsf{L}e_{2j} =& \ \tilde{\mathsf{Q}}\mathsf{L}e_0+\tilde{\mathsf{Q}}\mathsf{L}e_2+\tilde{\mathsf{Q}}\mathsf{L}e_4+\cdots+\tilde{\mathsf{Q}}\mathsf{L}e_{2n}\\ =& \ \sum_{j=0}^{n}\tilde{\mathcal{L}}e_{2j}+\sum_{j=0}^{n}\tilde{\mathcal{L}}e_{2j+1}i+\left(\sum_{j=0}^{n+1}\tilde{\mathcal{L}}e_{2j}-\tilde{\mathcal{L}}e_0\right)j\\ &+\left(\sum_{j=0}^{n+1}\tilde{\mathcal{L}}e_{2j+1}-\tilde{\mathcal{L}}e_1\right)k\\ =& \ \left(\tilde{\mathcal{L}}e_{2n+1}-\tilde{\mathcal{N}}_n-(J+J\varepsilon)\right)+\left(\tilde{\mathcal{L}}e_{2n+2}-\tilde{\mathcal{N}}_{n+2}+(J-J\varepsilon)\right)i\\ &+\left(\tilde{\mathcal{L}}e_{2n+3}-\tilde{\mathcal{N}}_{n+1}-(J+J\varepsilon)-\tilde{\mathcal{L}}e_0\right)j\\ &+\left(\tilde{\mathcal{L}}e_{2n+4}-\tilde{\mathcal{N}}_{n+3}+(J-J\varepsilon)-\tilde{\mathcal{L}}e_1\right)k\\ =& \ \tilde{\mathsf{Q}}\mathsf{L}e_{2n+1}-\tilde{\mathsf{Q}}\tilde{\mathsf{N}}_n-((J+J\varepsilon)+(1+\varepsilon+2J\varepsilon)\,i\\ &+(J+2\varepsilon+5J\varepsilon)\,j\ +(1+2J+5\varepsilon+10J\varepsilon)\,k).\end{aligned}$$

A similar proof can be given for the other part. □

Theorem 3.8. *For positive integers n and m, with $n \geqslant m$, the following identities hold:*

1) $\tilde{\mathsf{Q}}\mathsf{L}e_{n+m}+(-1)^m\tilde{\mathsf{Q}}\mathsf{L}e_{n-m}=L_m\tilde{\mathsf{Q}}\mathsf{L}e_n+\tilde{A}(L_m-(-1)^m-1)$,
2) $\tilde{\mathsf{Q}}\mathsf{L}e_{n+m}-(-1)^m\tilde{\mathsf{Q}}\mathsf{L}e_{n-m}=2F_m\tilde{\mathcal{K}}_{n+1}+\tilde{A}((-1)^m-1)$,

where F_n is the n-th Fibonacci number, L_n is the n-th Lucas number, and $\tilde{\mathcal{K}}_n$ is the n-th $\mathcal{DGC}$ Lucas dual quaternion.

Proof. 1) By using $\alpha\beta = -1$, the Binet's formula of the $\mathcal{DGC}$ Leonardo dual quaternionic sequence in equation (7), and the Binet's formula of the Lucas sequence $L_m = \alpha^m + \beta^m$ (see [27]), we get

$$\begin{aligned}
\tilde{\mathsf{Q}}\mathsf{L}e_{n+m} + (-1)^m \tilde{\mathsf{Q}}\mathsf{L}e_{n-m} = & \left(\frac{2\tilde{\alpha}^* \alpha^{n+m+1} - 2\tilde{\beta}^* \beta^{n+m+1}}{\alpha - \beta} - \tilde{A} \right) \\
& + (\alpha\beta)^m \left(\frac{2\tilde{\alpha}^* \alpha^{n-m+1} - 2\tilde{\beta}^* \beta^{n-m+1}}{\alpha - \beta} - \tilde{A} \right) \\
= & \frac{2\tilde{\alpha}^* \alpha^{n+1} (\alpha^m + \beta^m) - 2\tilde{\beta}^* \beta^{n+1} (\alpha^m + \beta^m)}{\alpha - \beta} \\
& - \tilde{A} \left((\alpha\beta)^m + 1 \right). \\
= & (\alpha^m + \beta^m) \left(\frac{2\tilde{\alpha}^* \alpha^{n+1} - 2\tilde{\beta}^* \beta^{n+1}}{\alpha - \beta} \right) \\
& - \tilde{A}((-1)^m + 1) \\
= & L_m(\tilde{\mathsf{Q}}\mathsf{L}e_n + \tilde{A}) - \tilde{A} \left((-1)^m + 1 \right) \\
= & L_m \tilde{\mathsf{Q}}\mathsf{L}e_n + \tilde{A} \left(L_m - (-1)^m - 1 \right).
\end{aligned}$$

2) Similarly, applying $\alpha\beta = -1$, the Binet's formula of the $\mathcal{DGC}$ Leonardo dual quaternionic sequence in equation (7), the Binet's formula of the Fibonacci sequence $F_m = \frac{\alpha^m - \beta^m}{\alpha - \beta}$ (see [27]), and the Binet's formula of the $\mathcal{DGC}$ Lucas dual quaternionic sequence $\tilde{\mathcal{K}}_n = \tilde{\alpha}^* \alpha^n + \tilde{\beta}^* \beta^n$ (see [16]), we obtain

$$\begin{aligned}
\tilde{\mathsf{Q}}\mathsf{L}e_{n+m} - (-1)^m \tilde{\mathsf{Q}}\mathsf{L}e_{n-m} = & \left(\frac{2\tilde{\alpha}^* \alpha^{n+m+1} - 2\tilde{\beta}^* \beta^{n+m+1}}{\alpha - \beta} - \tilde{A} \right) \\
& - (\alpha\beta)^m \left(\frac{2\tilde{\alpha}^* \alpha^{n-m+1} - 2\tilde{\beta}^* \beta^{n-m+1}}{\alpha - \beta} - \tilde{A} \right) \\
= & \frac{2\tilde{\alpha}^* \alpha^{n+1} (\alpha^m - \beta^m) + 2\tilde{\beta}^* \beta^{n+1} (\alpha^m - \beta^m)}{\alpha - \beta} \\
& + \tilde{A} \left((\alpha\beta)^m - 1 \right) \\
= & \left(\frac{\alpha^m - \beta^m}{\alpha - \beta} \right) \left(2\tilde{\alpha}^* \alpha^{n+1} + 2\tilde{\beta}^* \beta^{n+1} \right) \\
& + \tilde{A} \left((-1)^m - 1 \right) \\
= & 2F_m \tilde{\mathcal{K}}_{n+1} + \tilde{A} \left((-1)^m - 1 \right).
\end{aligned}$$

□

Theorem 3.9. *For positive integers k, m, s and t, with $k \geq m$, $s \geq t$ and $k+m = s+t$, we have*

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_k\tilde{\mathsf{Q}}\mathsf{L}e_m - \tilde{\mathsf{Q}}\mathsf{L}e_s\tilde{\mathsf{Q}}\mathsf{L}e_t = & \frac{4}{5}\tilde{\alpha}^*\tilde{\beta}^*\left((-1)^m L_{k-m} - (-1)^t L_{s-t}\right)\\ & -\tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_k + \tilde{\mathsf{Q}}\mathsf{L}e_m - \tilde{\mathsf{Q}}\mathsf{L}e_s - \tilde{\mathsf{Q}}\mathsf{L}e_t),\end{aligned}$$

where L_n is the n-th Lucas number.

Proof. Writing $\alpha\beta = -1$, the Binet's formula of the $\mathcal{DGC}$ Leonardo dual quaternionic sequence in equation (7), and the Binet's formula of the Lucas sequence $L_m = \alpha^m + \beta^m$ (see [27]) gives

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_k\tilde{\mathsf{Q}}\mathsf{L}e_m - \tilde{\mathsf{Q}}\mathsf{L}e_s\tilde{\mathsf{Q}}\mathsf{L}e_t = & \frac{4\tilde{\alpha}^*\tilde{\beta}^*\left(-\alpha^{k+1}\beta^{m+1}-\beta^{k+1}\alpha^{m+1}+\alpha^{s+1}\beta^{t+1}+\beta^{s+1}\alpha^{t+1}\right)}{(\alpha-\beta)^2}\\ & -\tilde{A}\left((\tilde{\mathsf{Q}}\mathsf{L}e_k + \tilde{A}) + (\tilde{\mathsf{Q}}\mathsf{L}e_m + \tilde{A}) - (\tilde{\mathsf{Q}}\mathsf{L}e_s + \tilde{A}\right)\\ & -(\tilde{\mathsf{Q}}\mathsf{L}e_t + \tilde{A}))\\ = & \frac{4}{5}\tilde{\alpha}^*\tilde{\beta}^*\left(\alpha^k\beta^m + \alpha^m\beta^k - \alpha^s\beta^t - \alpha^t\beta^s\right)\\ & -\tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_k + \tilde{\mathsf{Q}}\mathsf{L}e_m - \tilde{\mathsf{Q}}\mathsf{L}e_s - \tilde{\mathsf{Q}}\mathsf{L}e_t)\\ = & \frac{4}{5}\tilde{\alpha}^*\tilde{\beta}^*(\alpha\beta)^m\left(\alpha^{k-m} + \beta^{k-m}\right)\\ & -(\alpha\beta)^t\left(\alpha^{s-t} + \beta^{s-t}\right)\\ & -\tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_k + \tilde{\mathsf{Q}}\mathsf{L}e_m - \tilde{\mathsf{Q}}\mathsf{L}e_s - \tilde{\mathsf{Q}}\mathsf{L}e_t)\\ = & \frac{4}{5}\tilde{\alpha}^*\tilde{\beta}^*\left((-1)^m L_{k-m} - (-1)^t L_{s-t}\right)\\ & -\tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_k + \tilde{\mathsf{Q}}\mathsf{L}e_m - \tilde{\mathsf{Q}}\mathsf{L}e_s - \tilde{\mathsf{Q}}\mathsf{L}e_t).\end{aligned}$$

□

Theorem 3.10. *For positive integers n and r, with $n \geq r$, the general Catalan's identity can be obtained as follows:*

$$\tilde{\mathsf{Q}}\mathsf{L}e_n^2 - \tilde{\mathsf{Q}}\mathsf{L}e_{n-r}\tilde{\mathsf{Q}}\mathsf{L}e_{n+r} = 4\tilde{\alpha}^*\tilde{\beta}^*(-1)^{n-r+1}F_r^2 + \tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{n-r} + \tilde{\mathsf{Q}}\mathsf{L}e_{n+r} - 2\tilde{\mathsf{Q}}\mathsf{L}e_n),$$

where F_n is the n-th Fibonacci number.

Proof. Substituting the Binet's formula of the $\mathcal{DGC}$ Leonardo dual quaternionic sequence in equation (7) into $\tilde{\mathsf{Q}}\mathsf{L}e_n^2 - \tilde{\mathsf{Q}}\mathsf{L}e_{n-r}\tilde{\mathsf{Q}}\mathsf{L}e_{n+r}$ gives

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_n^2 - \tilde{\mathsf{Q}}\mathsf{L}e_{n-r}\tilde{\mathsf{Q}}\mathsf{L}e_{n+r} = & \frac{4\tilde{\alpha}^*\tilde{\beta}^*\left(\alpha^{n-r+1}\beta^{n+r+1}+\alpha^{n+r+1}\beta^{n-r+1}-2\alpha^{n+1}\beta^{n+1}\right)}{(\alpha-\beta)^2}\\ & +\tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{n-r} + \tilde{A} + \tilde{\mathsf{Q}}\mathsf{L}e_{n+r} + \tilde{A} - 2(\tilde{\mathsf{Q}}\mathsf{L}e_n + \tilde{A}))\\ = & \frac{4\tilde{\alpha}^*\tilde{\beta}^*\left(\alpha\beta\right)^{n-r+1}\left(\alpha^{2r} + \beta^{2r} - 2\alpha^r\beta^r\right)}{(\alpha-\beta)^2}\\ & +\tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{n-r} + \tilde{\mathsf{Q}}\mathsf{L}e_{n+r} - 2\tilde{\mathsf{Q}}\mathsf{L}e_n).\end{aligned}$$

Considering $\alpha\beta = -1$ and the Binet's formula of the Fibonacci sequence yields

$$\tilde{\mathsf{Q}}\mathsf{L}e_n^2 - \tilde{\mathsf{Q}}\mathsf{L}e_{n-r}\tilde{\mathsf{Q}}\mathsf{L}e_{n+r} = 4\tilde{\alpha}^*\tilde{\beta}^*(-1)^{n-r+1}F_r^2 + \tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{n-r} + \tilde{\mathsf{Q}}\mathsf{L}e_{n+r} - 2\tilde{\mathsf{Q}}\mathsf{L}e_n).$$

Instead of this approach, by taking $k \to n,\ m \to n,\ s \to n+r,\ t \to n-r$ in Theorem 3.9, and using the Binet's formulas of the Fibonacci and Lucas sequences (see [27]), the general Catalan's identity can be proved. □

Theorem 3.11. *For a positive integer n, the general Cassini's identity (sometimes called Simson's identity) can be given as follows:*

$$\tilde{\mathsf{Q}}\mathsf{L}e_n^2 - \tilde{\mathsf{Q}}\mathsf{L}e_{n-1}\tilde{\mathsf{Q}}\mathsf{L}e_{n+1} = 4\tilde{\alpha}^*\tilde{\beta}^*(-1)^n + \tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{n-1} + \tilde{\mathsf{Q}}\mathsf{L}e_{n+1} - 2\tilde{\mathsf{Q}}\mathsf{L}e_n).$$

Proof. Writing $r \to 1$ in Theorem 3.10, we obtain the general Cassini's identity. □

Theorem 3.12. *For positive integers n and m, with $m \geq n$ the general d'Ocagne's identity can be expressed as follows:*

$$\tilde{\mathsf{Q}}\mathsf{L}e_m\tilde{\mathsf{Q}}\mathsf{L}e_{n+1} - \tilde{\mathsf{Q}}\mathsf{L}e_{m+1}\tilde{\mathsf{Q}}\mathsf{L}e_n = 4\tilde{\alpha}^*\tilde{\beta}^*(-1)^{n+1}F_{m-n} + \tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{m-1} - \tilde{\mathsf{Q}}\mathsf{L}e_{n-1}),$$

where F_n is the n-th Fibonacci number.

Proof. We first write the Binet's formula of the $\mathcal{DGC}$ Leonardo dual quaternionic sequence in equation (7), then rearrange the left-hand side, and see that

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_m\tilde{\mathsf{Q}}\mathsf{L}e_{n+1} - \tilde{\mathsf{Q}}\mathsf{L}e_{m+1}\tilde{\mathsf{Q}}\mathsf{L}e_n = \;& 4\tilde{\alpha}^*\tilde{\beta}^*\frac{\alpha^{n+1}\beta^{n+1}(\alpha^{m-n} - \beta^{m-n})}{\alpha-\beta} \\ & +\tilde{A}\,((\tilde{\mathsf{Q}}\mathsf{L}e_{m+1} - \tilde{\mathsf{Q}}\mathsf{L}e_m) \\ & -(\tilde{\mathsf{Q}}\mathsf{L}e_{n+1} - \tilde{\mathsf{Q}}\mathsf{L}e_n)).\end{aligned}$$

By using $\alpha\beta = -1$ and the Binet's formula of the Fibonacci sequence, we have

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_m\tilde{\mathsf{Q}}\mathsf{L}e_{n+1} - \tilde{\mathsf{Q}}\mathsf{L}e_{m+1}\tilde{\mathsf{Q}}\mathsf{L}e_n = \;& 4\tilde{\alpha}^*\tilde{\beta}^*(-1)^{n+1}F_{m-n} \\ & +\tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{m-1} - \tilde{\mathsf{Q}}\mathsf{L}e_{n-1}).\end{aligned}$$

Instead of this method, Theorem 3.9 can be used. By taking $k \to m,\ m \to n+1$, $s \to m+1$, $t \to n$, and considering the Binet's formulas of the Fibonacci and Lucas sequences (see [27]), the general d'Ocagne's identity can be proved. □

Theorem 3.13. *For positive integers n and m, with $n \geq m$, the following identity holds:*

$$\tilde{\mathcal{Q}}_n\tilde{\mathsf{Q}}\mathsf{L}e_m - \tilde{\mathcal{Q}}_m\tilde{\mathsf{Q}}\mathsf{L}e_n = 2\tilde{\alpha}^*\tilde{\beta}^*(-1)^m F_{n-m} - \tilde{A}(\tilde{\mathcal{Q}}_n - \tilde{\mathcal{Q}}_m),$$

where F_n is the n-th Fibonacci number and $\tilde{\mathcal{Q}}_n$ is the n-th $\mathcal{DGC}$ Fibonacci dual quaternion.

Proof. Applying the Binet's formula of the $\mathcal{DGC}$ Leonardo dual quaternionic sequence in equation (7), the Binet's formula of the Fibonacci sequence, and the Binet's formula of the $\mathcal{DGC}$ Fibonacci dual quaternionic sequence

$\tilde{\mathcal{Q}}_n = \dfrac{\tilde{\alpha}^* \alpha^n - \tilde{\beta}^* \beta^n}{\alpha - \beta}$ (see [16]), we can see that

$$\begin{aligned}
\tilde{\mathcal{Q}}_n \tilde{\mathsf{Q}}\mathsf{L}e_m - \tilde{\mathcal{Q}}_m \tilde{\mathsf{Q}}\mathsf{L}e_n = {} & 2\tilde{\alpha}^* \tilde{\beta}^* \left(\frac{\alpha^n \beta^m (\alpha - \beta) - \alpha^m \beta^n (\alpha - \beta)}{(\alpha - \beta)^2} \right) \\
& - \tilde{A}(\tilde{\mathcal{Q}}_n - \tilde{\mathcal{Q}}_m) \\
= {} & 2\tilde{\alpha}^* \tilde{\beta}^* (\alpha\beta)^m \left(\frac{\alpha^{n-m} - \beta^{n-m}}{\alpha - \beta} \right) - \tilde{A}(\tilde{\mathcal{Q}}_n - \tilde{\mathcal{Q}}_m) \\
= {} & 2\tilde{\alpha}^* \tilde{\beta}^* (-1)^m F_{n-m} - \tilde{A}(\tilde{\mathcal{Q}}_n - \tilde{\mathcal{Q}}_m).
\end{aligned}$$

□

Theorem 3.14. *For positive integers n and m, with $n \geq m$, the following identity holds:*

$$\begin{aligned}
\tilde{\mathcal{Q}}_n \tilde{\mathsf{Q}}\mathsf{L}e_m + \tilde{\mathcal{Q}}_m \tilde{\mathsf{Q}}\mathsf{L}e_n = {} & \frac{4}{5}[(2K_{n+m+1} - L_{n+m+1}) + \mathfrak{p}\,(2K_{n+m+3} - L_{n+m+3}) \\
& + (4K_{n+m+2} - 2L_{n+m+2})\,J \\
& + ((4K_{n+m+3} - 2L_{n+m+3}) \\
& + \mathfrak{p}\,(4K_{n+m+5} - 2L_{n+m+5}))\,\varepsilon \\
& + (8K_{n+m+4} - 4L_{n+m+4})\,J\varepsilon] \\
& - \frac{2\tilde{\alpha}^* \tilde{\beta}^*}{5}(-1)^m L_{n-m} - \tilde{A}(\tilde{\mathcal{Q}}_n + \tilde{\mathcal{Q}}_m),
\end{aligned}$$

where L_n is the n-th Lucas number, K_n is the n-th dual Lucas quaternion and $\tilde{\mathcal{Q}}_n$ is the n-th $\mathcal{DGC}$ Fibonacci dual quaternion.

Proof. We give only the main steps of the proof. Firstly, we write the Binet's formula of the $\mathcal{DGC}$ Leonardo dual quaternionic sequence in equation (7), and then arrange it. Hence

$$\begin{aligned}
\tilde{\mathcal{Q}}_n \tilde{\mathsf{Q}}\mathsf{L}e_m + \tilde{\mathcal{Q}}_m \tilde{\mathsf{Q}}\mathsf{L}e_n = {} & \frac{4}{5}((\tilde{\alpha}^*)^2 \alpha^{n+m+1} + (\tilde{\beta}^*)^2 \beta^{n+m+1}) \\
& - \frac{2\tilde{\alpha}^* \tilde{\beta}^*}{5}(\alpha^m \beta^m (\alpha^{n-m} + \beta^{n-m})) \\
& - \tilde{A}(\tilde{\mathcal{Q}}_n + \tilde{\mathcal{Q}}_m).
\end{aligned}$$

We need to calculate $(\tilde{\alpha}^*)^2$ and $(\tilde{\beta}^*)^2$. The proof is completed by using $\alpha\beta = -1$, $\alpha + \beta = 1$, and the definition of dual Lucas quaternion (see [53] for more details on dual Lucas quaternions). □

Theorem 3.15. *For a positive integer n, we have*

$$\tilde{\mathsf{Q}}\mathsf{L}e_{n+1}^2 - \tilde{\mathsf{Q}}\mathsf{L}e_n^2 = 4\tilde{\mathcal{Q}}_n(\tilde{\mathcal{Q}}_{n+3} - \tilde{A}),$$

where $\tilde{\mathcal{Q}}_n$ is the n-th $\mathcal{DGC}$ Fibonacci dual quaternion.

K_n is of the form $K_n = L_n + L_{n+1}i + L_{n+2}j + L_{n+3}k$, where $\{i, j, k\}$ are the dual quaternionic units and L_n is the n-th Lucas number (see [53]).

Proof. Applying Theorem 3.6, item 1, and the recurrence relation of the $\mathcal{DGC}$ Fibonacci dual quaternion sequence (see [16]), we can write

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_{n+1}^2 - \tilde{\mathsf{Q}}\mathsf{L}e_n^2 &= (2\tilde{\mathcal{Q}}_{n+2} - \tilde{A})^2 - (2\tilde{\mathcal{Q}}_{n+1} - \tilde{A})^2 \\ &= 4((\tilde{\mathcal{Q}}_{n+2} - \tilde{\mathcal{Q}}_{n+1})(\tilde{\mathcal{Q}}_{n+2} + \tilde{\mathcal{Q}}_{n+1}) - \tilde{A}\tilde{\mathcal{Q}}_n) \\ &= 4(\tilde{\mathcal{Q}}_n\tilde{\mathcal{Q}}_{n+3} - \tilde{A}\tilde{\mathcal{Q}}_n) \\ &= 4\tilde{\mathcal{Q}}_n(\tilde{\mathcal{Q}}_{n+3} - \tilde{A}).\end{aligned}$$

□

Theorem 3.16. *For positive integers n and m, with $m \geq n$, the following identity holds:*

$$\tilde{\mathsf{Q}}\mathsf{L}e_{m+n}^2 - \tilde{\mathsf{Q}}\mathsf{L}e_{m-n}^2 = 4F_{2n}\tilde{\mathcal{Q}}_{m+1}\tilde{\mathcal{K}}_{m+1} - 2\tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{m+n} - \tilde{\mathsf{Q}}\mathsf{L}e_{m-n}),$$

where F_n is the n-th Fibonacci number, $\tilde{\mathcal{Q}}_n$ is the n-th $\mathcal{DGC}$ Fibonacci dual quaternion, and $\tilde{\mathcal{K}}_n$ is the n-th $\mathcal{DGC}$ Lucas dual quaternion.

Proof. According to Theorem 3.6, item 1, we have

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_{m+n}^2 - \tilde{\mathsf{Q}}\mathsf{L}e_{m-n}^2 &= (2\tilde{\mathcal{Q}}_{m+n+1} - \tilde{A})^2 - (2\tilde{\mathcal{Q}}_{m-n+1} - \tilde{A})^2 \\ &= 4(\tilde{\mathcal{Q}}_{m+1+n} - \tilde{\mathcal{Q}}_{m+1-n})(\tilde{\mathcal{Q}}_{m+1+n} + \tilde{\mathcal{Q}}_{m+1-n}) \\ &\quad -2\tilde{A}(2\tilde{\mathcal{Q}}_{m+1+n} - \tilde{A} - (2\tilde{\mathcal{Q}}_{m+1-n} - \tilde{A})).\end{aligned}$$

Now, the proof will be divided into two parts. We prove this theorem by considering the case where n is even. From Theorem 2.1, items 2 and 3 in [16], we conclude that $\tilde{\mathcal{Q}}_{m+1+n} + \tilde{\mathcal{Q}}_{m+1-n} = L_n\tilde{\mathcal{Q}}_{m+1}$ and $\tilde{\mathcal{Q}}_{m+1+n} - \tilde{\mathcal{Q}}_{m+1-n} = F_n\tilde{\mathcal{K}}_{m+1}$. A similar proof works for the case where n is odd. Hence, we complete the proof. □

Theorem 3.17. *For a positive integer n, the following identity is satisfied:*

$$\tilde{\mathsf{Q}}\mathsf{L}e_{n+1}\tilde{\mathcal{Q}}_{n+1} - \tilde{\mathsf{Q}}\mathsf{L}e_n\tilde{\mathcal{Q}}_n = 2\tilde{\mathcal{Q}}_{n+1}\tilde{\mathcal{Q}}_n + \tilde{\mathsf{Q}}\mathsf{L}e_n\tilde{\mathcal{Q}}_{n-1},$$

where $\tilde{\mathcal{Q}}_n$ is the n-th $\mathcal{DGC}$ Fibonacci dual quaternion.

Proof. On account of Theorem 3.6, item 1, and the recurrence relation of the $\mathcal{DGC}$ Fibonacci dual quaternion sequence (see [16]), we get the following result:

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_{n+1}\tilde{\mathcal{Q}}_{n+1} - \tilde{\mathsf{Q}}\mathsf{L}e_n\tilde{\mathcal{Q}}_n &= (2\tilde{\mathcal{Q}}_{n+2} - \tilde{A})\tilde{\mathcal{Q}}_{n+1} - (2\tilde{\mathcal{Q}}_{n+1} - \tilde{A})\tilde{\mathcal{Q}}_n \\ &= 2\tilde{\mathcal{Q}}_{n+1}(\tilde{\mathcal{Q}}_{n+2} - \tilde{\mathcal{Q}}_n) - \tilde{A}(\tilde{\mathcal{Q}}_{n+1} - \tilde{\mathcal{Q}}_n) \\ &= 2\tilde{\mathcal{Q}}_{n+1}(\tilde{\mathcal{Q}}_n + \tilde{\mathcal{Q}}_{n-1}) - \tilde{A}\tilde{\mathcal{Q}}_{n-1} \\ &= 2\tilde{\mathcal{Q}}_{n+1}\tilde{\mathcal{Q}}_n + (2\tilde{\mathcal{Q}}_{n+1} - \tilde{A})\tilde{\mathcal{Q}}_{n-1} \\ &= 2\tilde{\mathcal{Q}}_{n+1}\tilde{\mathcal{Q}}_n + \tilde{\mathsf{Q}}\mathsf{L}e_n\tilde{\mathcal{Q}}_{n-1}.\end{aligned}$$

□

Theorem 3.18. *For positive integers n, m, r and s, with $r \geqslant s$, the special case of Tagiuri's identity is as below:*

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_{n+r}\tilde{\mathsf{Q}}\mathsf{L}e_{n+s} - \tilde{\mathsf{Q}}\mathsf{L}e_{n}\tilde{\mathsf{Q}}\mathsf{L}e_{n+r+s} = \ & \frac{4}{5}\,\tilde{\alpha}^*\tilde{\beta}^*(-1)^{n+1}\left(L_{r+s} - (-1)^s L_{r-s}\right) \\ & + \tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{n} + \tilde{\mathsf{Q}}\mathsf{L}e_{n+r+s} \\ & - \tilde{\mathsf{Q}}\mathsf{L}e_{n+r} - \tilde{\mathsf{Q}}\mathsf{L}e_{n+s}),\end{aligned}$$

where L_n is the n-th Lucas number.

Proof. We begin by writing the Binet's formula of the $\mathcal{DGC}$ Leonardo dual quaternionic sequence in equation (7) into left-hand side and rearrange then we see that:

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_{n+r}\tilde{\mathsf{Q}}\mathsf{L}e_{n+s} - \tilde{\mathsf{Q}}\mathsf{L}e_{n}\tilde{\mathsf{Q}}\mathsf{L}e_{n+r+s} = \ & \tfrac{4}{5}\,\tilde{\alpha}^*\tilde{\beta}^*(\alpha\beta)^{n+1}\left(\alpha^{r+s} + \beta^{r+s}\right. \\ & \left. - (\alpha\beta)^s(\alpha^{r-s} + \beta^{r-s})\right) \\ & + \tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{n} + \tilde{\mathsf{Q}}\mathsf{L}e_{n+r+s} \\ & - \tilde{\mathsf{Q}}\mathsf{L}e_{n+r} - \tilde{\mathsf{Q}}\mathsf{L}e_{n+s}).\end{aligned}$$

From $\alpha\beta = -1$ and the Binet's formula of the Lucas sequence we complete the proof.

Instead of this approach, we can prove this identity using Theorem 3.9. By substituting $k \to n+r,\ m \to n+s,\ s \to n+r+s,\ t \to n$, and considering the Binet's formulas of the Fibonacci and Lucas sequences (see [27]), we obtain the special case of Tagiuri's identity. □

Theorem 3.19. *For positive integers k, m and s, with $m \geq k$ and $m \geq s$, the following identity holds:*

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_{m+k}\tilde{\mathsf{Q}}\mathsf{L}e_{m-k} - \tilde{\mathsf{Q}}\mathsf{L}e_{m+s}\tilde{\mathsf{Q}}\mathsf{L}e_{m-s} = \ & 4\tilde{\alpha}^*\tilde{\beta}^*((-1)^{m-k}F_k^2 - (-1)^{m-s}F_s^2) \\ & + \tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{m+s} + \tilde{\mathsf{Q}}\mathsf{L}e_{m-s} \\ & - \tilde{\mathsf{Q}}\mathsf{L}e_{m+k} - \tilde{\mathsf{Q}}\mathsf{L}e_{m-k}).\end{aligned}$$

where F_n is the n-th Fibonacci number.

Proof. We first write the Binet's formula of the $\mathcal{DGC}$ Leonardo dual quaternionic sequence in equation (7), and then rearrange it as follows:

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_{m+k}\tilde{\mathsf{Q}}\mathsf{L}e_{m-k} - \tilde{\mathsf{Q}}\mathsf{L}e_{m+s}\tilde{\mathsf{Q}}\mathsf{L}e_{m-s} = \ & \frac{4}{5}\,\tilde{\alpha}^*\tilde{\beta}^*\left(\alpha^{m+s+1}\beta^{m-s+1}\right. \\ & + \alpha^{m-s+1}\beta^{m+s+1} - \alpha^{m+k+1}\beta^{m-k+1} \\ & \left. - \alpha^{m-k+1}\beta^{m+k+1}\right) \\ & + \tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{m+s} + \tilde{\mathsf{Q}}\mathsf{L}e_{m-s} \\ & - \tilde{\mathsf{Q}}\mathsf{L}e_{m+k} - \tilde{\mathsf{Q}}\mathsf{L}e_{m-k}) \\ = \ & \frac{4}{5}\,\tilde{\alpha}^*\tilde{\beta}^*\left(-(\alpha\beta)^{m-k+1}(\alpha^{2k} + \beta^{2k})\right. \\ & \left. + (\alpha\beta)^{m-s+1}(\alpha^{2s} + \beta^{2s})\right) \\ & + \tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{m+s} + \tilde{\mathsf{Q}}\mathsf{L}e_{m-s} \\ & - \tilde{\mathsf{Q}}\mathsf{L}e_{m+k} - \tilde{\mathsf{Q}}\mathsf{L}e_{m-k}).\end{aligned}$$

From the Binet's formula of the Lucas sequence and $L_{2k} = 5F_k^2 + 2(-1)^k$ (see in [27]), we obtain

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_{m+k}\tilde{\mathsf{Q}}\mathsf{L}e_{m-k} - \tilde{\mathsf{Q}}\mathsf{L}e_{m+s}\tilde{\mathsf{Q}}\mathsf{L}e_{m-s} = {} & \frac{4}{5}\,\tilde{\alpha}^*\tilde{\beta}^*\left((-1)^{m-k}L_{2k} - (-1)^{m-s}L_{2s}\right) \\ & +\tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{m+s} + \tilde{\mathsf{Q}}\mathsf{L}e_{m-s} \\ & -\tilde{\mathsf{Q}}\mathsf{L}e_{m+k} - \tilde{\mathsf{Q}}\mathsf{L}e_{m-k}) \\ = {} & 4\tilde{\alpha}^*\tilde{\beta}^*((-1)^{m-k}F_k^2 - (-1)^{m-s}F_s^2) \\ & +\tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{m+s} + \tilde{\mathsf{Q}}\mathsf{L}e_{m-s} \\ & -\tilde{\mathsf{Q}}\mathsf{L}e_{m+k} - \tilde{\mathsf{Q}}\mathsf{L}e_{m-k}).\end{aligned}$$

□

Theorem 3.20. *For positive integers n and m, the following identity is satisfied:*

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_{m+1}\tilde{\mathsf{Q}}\mathsf{L}e_{n+1} - \tilde{\mathsf{Q}}\mathsf{L}e_{m-1}\tilde{\mathsf{Q}}\mathsf{L}e_{n-1} = {} & 4\left((2Q_{n+m+2} - F_{n+m+2})\right. \\ & +\mathfrak{p}\left(2Q_{n+m+4} - F_{n+m+4}\right) \\ & +\left(4Q_{n+m+3} - 2F_{n+m+3}\right)J \\ & +\left((4Q_{n+m+4} - 2F_{n+m+4})\right. \\ & +\mathfrak{p}\left(4Q_{n+m+6} - 2F_{n+m+6}\right))\,\varepsilon \\ & \left.+\left(8Q_{n+m+5} - 4F_{n+m+5}\right)J\varepsilon\right) \\ & -\tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_n + \tilde{\mathsf{Q}}\mathsf{L}e_m) \\ & -2\tilde{A}^2,\end{aligned}$$

where F_n is the n-th Fibonacci number and Q_n is the n-th dual Fibonacci quaternion.

Proof. We first apply the Binet's formula of the $\mathcal{DGC}$ Leonardo dual quaternionic sequence in equation (7) to the left-hand side. We thus get

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_{m+1}\tilde{\mathsf{Q}}\mathsf{L}e_{n+1} - \tilde{\mathsf{Q}}\mathsf{L}e_{m-1}\tilde{\mathsf{Q}}\mathsf{L}e_{n-1} = {} & \frac{4}{5}((\tilde{\alpha}^*)^2\alpha^{m+n+4} + (\tilde{\beta}^*)^2\beta^{m+n+4} \\ & -(\tilde{\alpha}^*)^2\alpha^{m+n} - (\tilde{\beta}^*)^2\beta^{m+n}) - \frac{4}{5}\,\tilde{\alpha}^*\tilde{\beta}^* \\ & \left(\alpha^m\beta^n\left((\alpha\beta)^2 - 1\right) + \alpha^n\beta^m\left((\alpha\beta)^2 - 1\right)\right) \\ & +\tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{m-1} - \tilde{\mathsf{Q}}\mathsf{L}e_{m+1} + \tilde{\mathsf{Q}}\mathsf{L}e_{n-1} \\ & -\tilde{\mathsf{Q}}\mathsf{L}e_{n+1}).\end{aligned}$$

Here, we need to find $(\tilde{\alpha}^*)^2$ and $(\tilde{\beta}^*)^2$ and substitute them into the above equation. Considering $\alpha\beta = -1$ and referring to the definition of dual Lucas quaternion (see [53] for more details related to dual Lucas quaternions), we

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Q_n is of the form $Q_n = F_n + F_{n+1}i + F_{n+2}j + F_{n+3}k$, where $\{i, j, k\}$ are the dual quaternionic units and F_n is the n-th Fibonacci number (see [53]).

conclude that

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_{m+1}\tilde{\mathsf{Q}}\mathsf{L}e_{n+1}-\tilde{\mathsf{Q}}\mathsf{L}e_{m-1}\tilde{\mathsf{Q}}\mathsf{L}e_{n-1}= \; & \frac{4}{5}\left(2(K_{n+m+4}-K_{n+m})\right.\\ & -\left(L_{n+m+4}-L_{n+m}\right)\\ & +\mathfrak{p}\left(2\left(K_{n+m+6}-K_{n+m+2}\right)\right.\\ & -\left.\left(L_{n+m+6}-L_{n+m+2}\right)\right)\\ & +\left(4\left(K_{n+m+5}-K_{n+m+1}\right)\right.\\ & \left.-2\left(L_{n+m+5}-L_{n+m+1}\right)\right)J\\ & +\left(4\left(K_{n+m+6}-K_{n+m+2}\right)\right.\\ & -2\left(L_{n+m+6}-L_{n+m+2}\right)\\ & +\mathfrak{p}\left(4\left(K_{n+m+8}-K_{n+m+4}\right)\right.\\ & \left.\left.-2\left(L_{n+m+8}-L_{n+m+4}\right)\right)\right)\varepsilon\\ & +\left(8\left(K_{n+m+7}-K_{n+m+3}\right)\right.\\ & \left.\left.-4\left(L_{n+m+7}-L_{n+m+3}\right)\right)J\varepsilon\right)\\ & -\tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{n}+\tilde{\mathsf{Q}}\mathsf{L}e_{m})-2\tilde{A}^{2}.\end{aligned}$$

According to the definitions of dual Fibonacci and dual Lucas quaternions (see [53] for more details related to dual Lucas quaternions) and $L_{n+r}-L_{n-r}=5F_nF_r$ for even integer r (see [27]), we have

$$\begin{aligned}\tilde{\mathsf{Q}}\mathsf{L}e_{m+1}\tilde{\mathsf{Q}}\mathsf{L}e_{n+1}-\tilde{\mathsf{Q}}\mathsf{L}e_{m-1}\tilde{\mathsf{Q}}\mathsf{L}e_{n-1}= \; & 4\left(\left(2Q_{n+m+2}-F_{n+m+2}\right)\right.\\ & +\mathfrak{p}\left(2Q_{n+m+4}-F_{n+m+4}\right)\\ & +\left(4Q_{n+m+3}-2F_{n+m+3}\right)J\\ & +\left(\left(4Q_{n+m+4}-2F_{n+m+4}\right)\right.\\ & \left.+\,\mathfrak{p}\left(4Q_{n+m+6}-2F_{n+m+6}\right)\right)\varepsilon\\ & \left.+\left(8Q_{n+m+5}-4F_{n+m+5}\right)J\varepsilon\right)\\ & -\tilde{A}(\tilde{\mathsf{Q}}\mathsf{L}e_{n}+\tilde{\mathsf{Q}}\mathsf{L}e_{m})-2\tilde{A}^{2}.\end{aligned}$$

□

4. Conclusions

In this paper, we investigate and discuss the dual quaternionic sequence with the $\mathcal{DGC}$ Leonardo number components for $\mathfrak{p}\in\mathbb{R}$ in detail. Within the framework of $\mathcal{DGC}$ number structures, we have

- the dual quaternionic sequence with dual-complex Leonardo for $\mathfrak{p}=-1$,
- the dual quaternionic sequence with hyper-dual Leonardo for $\mathfrak{p}=0$,
- the dual quaternionic sequence with dual-hyperbolic Leonardo for $\mathfrak{p}=1$.

Additionally, we present some characteristic properties of this sequence, including its Binet's formula, generating function, d'Ocagne's, Catalan's, Cassini's, and Tagiuri's identities.

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