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SEVERAL NEWTON-COTES TYPE INEQUALITIES FOR FUNCTIONS WHOSE DERIVATIVES BELONG TO L^p SPACES WITH $p \in [1, \infty]$

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Abstract. In this study, we introduce a new bi-parameterized integral identity involving at most five points. Using this identity, we establish various integral inequalities for functions whose first derivatives belong to the spaces L^p with $1 \leq p \leq \infty$. Several known results are recaptured. Applications are provided.

1. Introduction

Over the past decades, many researchers have given considerable attention in the study of error estimates of different quadrature formulas for several classes of functions, for convex mappings [19–25], Lipchitzian mappings [4, 13, 14], mappings with bounded variations [2, 16, 17] and mappings belonging to L^p spaces.

We first recall that the usual Lebesgue norms on $L^p[\sigma_1, \sigma_2]$ noted $\|.\|_p$ are defined as follows:

$$\|\mathcal{S}\|_p = \left(\int_{\sigma_1}^{\sigma_2} |\mathcal{S}(t)|^p \, dt \right)^{\frac{1}{p}}, 1 \leq p < \infty$$

and

$$\|\mathcal{S}\|_\infty = \underset{t \in [\sigma_1, \sigma_2]}{ess \sup} |\mathcal{S}(t)|.$$

Here, we cite some known results for an absolutely continuous mapping whose derivatives belong to $L^p[\sigma_1, \sigma_2]$ for $p \geq 1$ of which we will adopt an increasing sorting according to the number of points occurring in the quadrature.

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Midpoint type inequalities

$$\left|\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$
$$\leq\begin{cases}\frac{1}{2}\left(\frac{\sigma_2-\sigma_1}{q+1}\right)^{\frac{1}{q}}\|\mathcal{S}'\|_p & \text{if } \mathcal{S}'\in L^p[\sigma_1,\sigma_2], \quad \text{(see [1,3,15])},\\ \frac{\sigma_2-\sigma_1}{4}\|\mathcal{S}'\|_\infty & \text{if } \mathcal{S}'\in L_\infty[\sigma_1,\sigma_2], \quad \text{(see [1,5,15])},\\ \frac{1}{2}\|\mathcal{S}'\|_1 & \text{if } \mathcal{S}'\in L^1[\sigma_1,\sigma_2], \quad \text{(see [8,15])}.\end{cases}$$

Trapezium type inequalities

$$\left|\frac{\mathcal{S}(\sigma_1)+\mathcal{S}(\sigma_2)}{2}-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$
$$\leq\begin{cases}\frac{1}{2}\left(\frac{\sigma_2-\sigma_1}{q+1}\right)^{\frac{1}{q}}\|\mathcal{S}'\|_p & \text{if } \mathcal{S}'\in L^p[\sigma_1,\sigma_2], \quad \text{(see [1,3,6,7,15])},\\ \frac{\sigma_2-\sigma_1}{4}\|\mathcal{S}'\|_\infty & \text{if } \mathcal{S}'\in L_\infty[\sigma_1,\sigma_2], \quad \text{(see [1,6,7,11,12])},\\ \frac{1}{2}\|\mathcal{S}'\|_1 & \text{if } \mathcal{S}'\in L^1[\sigma_1,\sigma_2], \quad \text{(see [6,7,15])}.\end{cases}$$

Companion Ostrowski type inequalities

$$\left|\frac{\mathcal{S}(x)+\mathcal{S}(\sigma_1+\sigma_2-x)}{2}-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$
$$\leq\begin{cases}\left(\frac{2(\sigma_2-\sigma_1)}{q+1}\right)^{\frac{1}{q}}\left(\left(\frac{x-\sigma_1}{\sigma_2-\sigma_1}\right)^{q+1}+\left(\frac{\frac{\sigma_1+\sigma_2}{2}-x}{\sigma_2-\sigma_1}\right)^{q+1}\right)^{\frac{1}{q}}\|\mathcal{S}'\|_p\\ \text{if } \mathcal{S}'\in L^p[\sigma_1,\sigma_2], \ \text{(see [1,3,15])},\\ \frac{1}{\sigma_2-\sigma_1}\left((x-\sigma_1)^2+\left(\frac{\sigma_1+\sigma_2}{2}-x\right)^2\right)\|\mathcal{S}'\|_\infty\\ \text{if } \mathcal{S}'\in L_\infty[\sigma_1,\sigma_2], \ \text{(see [1,15])},\\ \left[\frac{1}{4}+\left|\frac{x-\frac{3\sigma_1+\sigma_2}{4}}{\sigma_2-\sigma_1}\right|\right]\|\mathcal{S}'\|_1\\ \text{if } \mathcal{S}'\in L^1[\sigma_1,\sigma_2], \ \text{(see [15])}.\end{cases}$$

Bullen-type inequalities

$$\left|\frac{1}{4}\left(\mathcal{S}(\sigma_1)+2\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+\mathcal{S}(\sigma_2)\right)-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$
$$\leq\begin{cases}\frac{1}{4}\left(\frac{\sigma_2-\sigma_1}{1+q}\right)^{\frac{1}{q}}\|\mathcal{S}'\|_p & \text{if } \mathcal{S}'\in L^p[\sigma_1,\sigma_2], \quad \text{(see [1])},\\ \frac{\sigma_2-\sigma_1}{8}\|\mathcal{S}'\|_\infty & \text{if } \mathcal{S}'\in L_\infty[\sigma_1,\sigma_2], \quad \text{(see [1,9,10])}.\end{cases}$$

Simpson type inequalities

$$\left|\frac{1}{6}\left(\mathcal{S}\left(\sigma_1\right)+4\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+\mathcal{S}\left(\sigma_2\right)\right)-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}\left(u\right)du\right|$$

$$\leq\begin{cases}\left(\frac{1+2^{q+1}}{6^{q+1}}\right)^{\frac{1}{q}}\left(\frac{2(\sigma_2-\sigma_1)}{1+q}\right)^{\frac{1}{q}}\left\|\mathcal{S}'\right\|_p & \text{if } \mathcal{S}'\in L^p\left[\sigma_1,\sigma_2\right], \quad (\text{see } [1,3,9,11]),\\ \frac{5(\sigma_2-\sigma_1)}{36}\left\|\mathcal{S}'\right\|_\infty & \text{if } \mathcal{S}'\in L_\infty\left[\sigma_1,\sigma_2\right], \quad (\text{see } [1,11,12]),\\ \frac{1}{3}\left\|\mathcal{S}'\right\|_1 & \text{if } \mathcal{S}'\in L^1\left[\sigma_1,\sigma_2\right], \quad (\text{see } [9,10]).\end{cases}$$

Maclaurin type inequalities (see [3])

$$\left|\frac{1}{8}\left(3\mathcal{S}\left(\frac{5\sigma_1+\sigma_2}{6}\right)+2\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+3\mathcal{S}\left(\frac{\sigma_1+5\sigma_2}{6}\right)\right)-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}\left(u\right)du\right|$$

$$\leq\left(\frac{3^{q+1}+4^{q+1}+5^{q+1}}{24^{q+1}}\right)^{\frac{1}{q}}\left(\frac{2(\sigma_2-\sigma_1)}{1+q}\right)^{\frac{1}{q}}\left\|\mathcal{S}'\right\|_p.$$

$\frac{3}{8}$-Simpson type inequalities

$$\left|\frac{1}{8}\left(\mathcal{S}\left(\sigma_1\right)+3\mathcal{S}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+3\mathcal{S}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+\mathcal{S}\left(\sigma_2\right)\right)-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}\left(u\right)du\right|$$

$$\leq\begin{cases}\left(\frac{4^{q+1}+3^{q+1}+5^{q+1}}{24^{q+1}}\right)^{\frac{1}{q}}\left(\frac{2(\sigma_2-\sigma_1)}{1+q}\right)^{\frac{1}{q}}\left\|\mathcal{S}'\right\|_p\\ \text{if } \mathcal{S}'\in L^p\left[\sigma_1,\sigma_2\right], \ (\text{see } [1,3,18]),\\ \frac{25(\sigma_2-\sigma_1)}{288}\left\|\mathcal{S}'\right\|_\infty\\ \text{if } \mathcal{S}'\in L_\infty\left[\sigma_1,\sigma_2\right], \ (\text{see } [1,18]),\\ \frac{5}{24}\left\|\mathcal{S}'\right\|_1\\ \text{if } \mathcal{S}'\in L^1\left[\sigma_1,\sigma_2\right], \ (\text{see } [18]).\end{cases}$$

Boole type inequalities (see [3])

$$\left|\frac{7\mathcal{S}(\sigma_1)+32\mathcal{S}\left(\frac{\sigma_1+3\sigma_2}{4}\right)+12\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+32\mathcal{S}\left(\frac{3\sigma_1+\sigma_2}{4}\right)+7\mathcal{S}(\sigma_2)}{90}-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}\left(u\right)du\right|$$

$$\leq\left(\frac{14^{q+1}+31^{q+1}+12^{q+1}+33^{q+1}}{180^{q+1}}\right)^{\frac{1}{q}}\left(\frac{2(\sigma_2-\sigma_1)}{1+q}\right)^{\frac{1}{q}}\left\|\mathcal{S}'\right\|_p.$$

Inspired and motivated by the above results, in this study, we introduce a new bi-parameterized integral identity involving at most five points. Using this identity, we establish various integral inequalities for functions whose first derivatives belong to the spaces L^p with $1\leq p\leq\infty$. Several known results are recaptured. Some applications to special means and random variables are provided.

2. Main results

Lemma 2.1. *Let $\mathcal{S} : I \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, $\sigma_1, \sigma_2 \in I^\circ$ with $\sigma_1 < \sigma_2$, and $\mathcal{S}' \in L^1[\sigma_1, \sigma_2]$, then the following equality holds for all real number $\lambda, \gamma \in [0,1]$ and $x \in \left[\sigma_1, \frac{\sigma_1+\sigma_2}{2}\right]$*

$$\mathcal{Q}_{\lambda,\gamma}(\sigma_1, x, \sigma_2; \mathcal{S}) - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} \mathcal{S}(u)\,du = \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} \mathcal{K}(u,x)\,\mathcal{S}(u)\,du, \tag{1}$$

where

$$\begin{aligned}&\mathcal{Q}_{\lambda,\gamma}(\sigma_1, x, \sigma_2; \mathcal{S}) \\ &= \tfrac{\lambda(x-\sigma_1)}{(1+\gamma)(\sigma_2-\sigma_1)}\mathcal{S}(\sigma_1) + \tfrac{(\sigma_2-\sigma_1)-2\lambda(x-\sigma_1)}{2(1+\gamma)(\sigma_2-\sigma_1)}\mathcal{S}(x) + \tfrac{\gamma}{1+\gamma}\mathcal{S}\left(\tfrac{\sigma_1+\sigma_2}{2}\right) \\ &\quad + \tfrac{(\sigma_2-\sigma_1)-2\lambda(x-\sigma_1)}{2(1+\gamma)(\sigma_2-\sigma_1)}\mathcal{S}(\sigma_1+\sigma_2-x) + \tfrac{\lambda(x-\sigma_1)}{(1+\gamma)(\sigma_2-\sigma_1)}\mathcal{S}(\sigma_2)\end{aligned} \tag{2}$$

and

$$\mathcal{K}(u,x) = \begin{cases} u - \frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma} & \text{if } t \in [\sigma_1, x], \\ u - \frac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)} & \text{if } t \in \left[x, \frac{\sigma_1+\sigma_2}{2}\right], \\ u - \frac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)} & \text{if } t \in \left[\frac{\sigma_1+\sigma_2}{2}, \sigma_1+\sigma_2-x\right], \\ u - \frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma} & \text{if } t \in [\sigma_1+\sigma_2-x, \sigma_2]. \end{cases} \tag{3}$$

Proof. Integrating by part the first integral on the right side of (1), we get

$$\begin{aligned}&\int_{\sigma_1}^{x}\left(u - \tfrac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}\right)\mathcal{S}'(u)\,du \\ &= \left(u - \tfrac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}\right)\mathcal{S}(u)\Big|_{\sigma_1}^{x} - \int_{\sigma_1}^{x}\mathcal{S}(u)\,du \\ &= \tfrac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}\mathcal{S}(x) + \tfrac{\lambda(x-\sigma_1)}{1+\gamma}\mathcal{S}(\sigma_1) - \int_{\sigma_1}^{x}\mathcal{S}(u)\,du.\end{aligned} \tag{4}$$

Similarly, we have

$$\begin{aligned}&\int_{x}^{\frac{\sigma_1+\sigma_2}{2}}\left(u - \tfrac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)}\right)\mathcal{S}'(u)\,du \\ &= \tfrac{\gamma(\sigma_2-\sigma_1)}{2(1+\gamma)}\mathcal{S}\left(\tfrac{\sigma_1+\sigma_2}{2}\right) + \tfrac{(\sigma_2-x)-(1+2\gamma)(x-\sigma_1)}{2(1+\gamma)}\mathcal{S}(x) - \int_{x}^{\frac{\sigma_1+\sigma_2}{2}}\mathcal{S}(u)\,du,\end{aligned} \tag{5}$$

$$(6)\quad \int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_1+\sigma_2-x} \left(u-\frac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)}\right)\mathcal{S}'(u)\,du$$

$$=\frac{\sigma_2-x-(1+2\gamma)(x-\sigma_1)}{2(1+\gamma)}\mathcal{S}(\sigma_1+\sigma_2-x)+\frac{\gamma(\sigma_2-\sigma_1)}{2(1+\gamma)}\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)-\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_1+\sigma_2-x}\mathcal{S}(u)\,du$$

and

$$(7)\quad \int_{\sigma_1+\sigma_2-x}^{\sigma_2} \left(u-\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}\right)\mathcal{S}'(u)\,du$$

$$=\frac{\lambda(x-\sigma_1)}{1+\gamma}\mathcal{S}(\sigma_2)+\frac{(1+\gamma-\lambda)(x-\sigma_1)}{1+\gamma}\mathcal{S}(\sigma_1+\sigma_2-x)-\int_{\sigma_1+\sigma_2-x}^{\sigma_2}\mathcal{S}(u)\,du.$$

The required outcome may be obtained by summing the equalities (4)-(7) and multiplying the resultant equality by $\frac{1}{\sigma_2-\sigma_1}$. □

Remark 2.2. *Note that the kernel $\mathcal{K}(u,x)$ will be reduced for $x=\sigma_1$ and $x=\frac{\sigma_1+\sigma_2}{2}$, respectively to*

$$\mathcal{K}(u,\sigma_1)=\begin{cases} u-\frac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)} & \text{if } t\in\left[\sigma_1,\frac{\sigma_1+\sigma_2}{2}\right],\\ u-\frac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)} & \text{if } t\in\left[\frac{\sigma_1+\sigma_2}{2},\sigma_2\right],\end{cases}$$

and

$$\mathcal{K}\left(u,\frac{\sigma_1+\sigma_2}{2}\right)=\begin{cases} u-\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma} & \text{if } t\in\left[\sigma_1,\frac{\sigma_1+\sigma_2}{2}\right],\\ u-\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma} & \text{if } t\in\left[\frac{\sigma_1+\sigma_2}{2},\sigma_2\right].\end{cases}$$

Therefore, in the calculation, the first and fourth integrals will be worth zero and will not be taken into consideration in the case where $x=\sigma_1$. However, for $x=\frac{\sigma_1+\sigma_2}{2}$, the second and third integrals will be zero and will not be taken into consideration.

Theorem 2.3. *Let $\mathcal{S}:[\sigma_1,\sigma_2]\to\mathbb{R}$ be an absolutely continuous mapping $[\sigma_1,\sigma_2]$. If $\mathcal{S}'$ belongs to $L^p[\sigma_1,\sigma_2]$, then for all $\lambda,\gamma\in[0,1], p>1$ with $\frac{1}{p}+\frac{1}{q}=1$ and $x\in\left[\sigma_1,\frac{\sigma_1+\sigma_2}{2}\right]$, we have*

$$\left|\mathcal{Q}_{\lambda,\gamma}(\sigma_1,x,\sigma_2;\mathcal{S})-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$

$$\leq\begin{cases} \frac{2^{\frac{1}{q}}}{(1+q)^{\frac{1}{q}}(\sigma_2-\sigma_1)}\left(\Omega-\left(\frac{(1+2\gamma)(x-\sigma_1)-(\sigma_2-x)}{2(1+\gamma)}\right)^{q+1}\right)^{\frac{1}{q}}\|\mathcal{S}'\|_p & \text{if } \gamma>\frac{\frac{\sigma_1+\sigma_2}{2}-x}{x-\sigma_1},\\ \frac{2^{\frac{1}{q}}}{(1+q)^{\frac{1}{q}}(\sigma_2-\sigma_1)}\left(\Omega+\left(\frac{\sigma_2-x-(1+2\gamma)(x-\sigma_1)}{2(1+\gamma)}\right)^{q+1}\right)^{\frac{1}{q}}\|\mathcal{S}'\|_p & \text{if } \gamma\leq\frac{\frac{\sigma_1+\sigma_2}{2}-x}{x-\sigma_1},\end{cases}$$

where

$$\Omega=\left(\tfrac{\lambda(x-\sigma_1)}{1+\gamma}\right)^{q+1}+\left(\tfrac{(1+\gamma-\lambda)(x-\sigma_1)}{1+\gamma}\right)^{q+1}+\left(\tfrac{\gamma(\sigma_2-\sigma_1)}{2(1+\gamma)}\right)^{q+1}.$$

Proof. By taking the absolute value on both sides of (1) and then using Hölder's inequality, then we obtain

$$\begin{aligned}
&\left|\mathcal{Q}_{\lambda,\gamma}(\sigma_1,x,\sigma_2;\mathcal{S})-\tfrac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|\\
&\leq\tfrac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}|\mathcal{K}(t,x)|\,|\mathcal{S}'(t)|\,dt\\
&\leq\tfrac{1}{\sigma_2-\sigma_1}\left(\int_{\sigma_1}^{\sigma_2}|\mathcal{K}(t,x)|^q\,dt\right)^{\frac{1}{q}}\left(\int_{\sigma_1}^{\sigma_2}|\mathcal{S}'(t)|^p\,dt\right)^{\frac{1}{p}}\\
&\leq\tfrac{1}{\sigma_2-\sigma_1}\left(\int_{\sigma_1}^{x}\left|t-\tfrac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}\right|^q dt+\int_{x}^{\frac{\sigma_1+\sigma_2}{2}}\left|t-\tfrac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)}\right|^q dt\right.\\
&\quad\left.+\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_1+\sigma_2-x}\left|t-\tfrac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)}\right|^q dt+\int_{\sigma_1+\sigma_2-x}^{\sigma_2}\left|t-\tfrac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}\right|^q dt\right)^{\frac{1}{q}}\|\mathcal{S}'\|_p.
\end{aligned}\tag{8}$$

Two cases arise. For $\gamma>\frac{\frac{\sigma_1+\sigma_2}{2}-x}{x-\sigma_1}$, (8) gives

$$\begin{aligned}
&\left|\mathcal{Q}_{\lambda,\gamma}(\sigma_1,x,\sigma_2;\mathcal{S})-\tfrac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|\\
&\leq\tfrac{1}{\sigma_2-\sigma_1}\left(\int_{\sigma_1}^{\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}}\left(\tfrac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}-t\right)^q dt\right.\\
&\quad+\int_{\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}}^{x}\left(t-\tfrac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}\right)^q dt\\
&\quad+\int_{x}^{\frac{\sigma_1+\sigma_2}{2}}\left(t-\tfrac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)}\right)^q dt+\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_1+\sigma_2-x}\left(\tfrac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)}-t\right)^q dt
\end{aligned}$$

$$+\int_{\sigma_1+\sigma_2-x}^{\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}}\left(\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}-t\right)^q dt$$

$$\left.+\int_{\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}}^{\sigma_2}\left(t-\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}\right)^q dt\right)^{\frac{1}{q}}\left\|\mathcal{S}'\right\|_p$$

$$=\frac{2^{\frac{1}{q}}}{(1+q)^{\frac{1}{q}}(\sigma_2-\sigma_1)}\left(\left(\frac{\lambda(x-\sigma_1)}{1+\gamma}\right)^{q+1}+\left(\frac{(1+\gamma-\lambda)(x-\sigma_1)}{1+\gamma}\right)^{q+1}\right.$$

$$\left.+\left(\frac{\gamma(\sigma_2-\sigma_1)}{2(1+\gamma)}\right)^{q+1}-\left(\frac{(1+2\gamma)(x-\sigma_1)-(\sigma_2-x)}{2(1+\gamma)}\right)^{q+1}\right)^{\frac{1}{q}}\left\|\mathcal{S}'\right\|_p. \tag{9}$$

For $\gamma \leq \frac{\frac{\sigma_1+\sigma_2}{2}-x}{x-\sigma_1}$, (8) gives

$$\left|\mathcal{Q}_{\lambda,\gamma}(\sigma_1,x,\sigma_2;\mathcal{S})-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$

$$\leq\frac{1}{\sigma_2-\sigma_1}\left(\int_{\sigma_1}^{\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}}\left(\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}-t\right)^q dt\right.$$

$$+\int_{\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}}^{x}\left(t-\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}\right)^q dt$$

$$+\int_{x}^{\frac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)}}\left(\frac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)}-t\right)^q dt+\int_{\frac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)}}^{\frac{\sigma_1+\sigma_2}{2}}\left(t-\frac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)}\right)^q dt$$

$$+\int_{\frac{\sigma_1+\sigma_2}{2}}^{\frac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)}}\left(\frac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)}-t\right)^q dt+\int_{\frac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)}}^{\sigma_1+\sigma_2-x}\left(t-\frac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)}\right)^q dt$$

$$+\int_{\sigma_1+\sigma_2-x}^{\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}}\left(\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}-t\right)^q dt$$

$$+\left.\int\limits_{\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}}^{\sigma_2}\left(t-\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}\right)^q dt\right)^{\frac{1}{q}}\|\mathcal{S}'\|_p$$

$$=\frac{2^{\frac{1}{q}}}{(1+q)^{\frac{1}{q}}(\sigma_2-\sigma_1)}\left(\left(\frac{\lambda(x-\sigma_1)}{1+\gamma}\right)^{q+1}+\left(\frac{(1+\gamma-\lambda)(x-\sigma_1)}{1+\gamma}\right)^{q+1}\right.$$

$$+\left.\left(\frac{\sigma_2-x-(1+2\gamma)(x-\sigma_1)}{2(1+\gamma)}\right)^{q+1}+\left(\frac{\gamma(\sigma_2-\sigma_1)}{2(1+\gamma)}\right)^{q+1}\right)^{\frac{1}{q}}\|\mathcal{S}'\|_p. \tag{10}$$

The desired result follows from (9) and (10). The proof is completed. □

Corollary 2.4. *In Theorem 2.3, if we take* $x=\frac{\sigma_1+\sigma_2}{2}$*, then we obtain the following Simpson-like type inequality:*

$$\left|\frac{\lambda}{2(1+\gamma)}\mathcal{S}(\sigma_1)+\frac{1+\gamma-\lambda}{1+\gamma}\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{\lambda}{2(1+\gamma)}\mathcal{S}(\sigma_2)-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$

$$\leq\frac{1}{2}\left(\frac{\lambda^{q+1}+(1+\gamma-\lambda)^{q+1}}{(1+\gamma)^{q+1}}\right)^{\frac{1}{q}}\left(\frac{\sigma_2-\sigma_1}{1+q}\right)^{\frac{1}{q}}\|\mathcal{S}'\|_p.$$

Remark 2.5. *In Corollary 2.4, if we take*

- $\gamma=0$*, then we obtain Corollary 7 from [1].*
- $\gamma=\lambda=0$*, then we obtain the inequality (1) of Corollary 8 from [1].*
- $\gamma=0$ *and* $\lambda=1$*, then we obtain the inequality (5.4) from [3].*
- $\gamma=0$ *and* $\lambda=\frac{1}{3}$*, then we obtain Theorem 2.1 from [9].*
- $\gamma=0$ *and* $\lambda=\frac{1}{2}$*, then we obtain the inequality (3) of Corollary 8 from [1].*

Corollary 2.6. *In Corollary 2.4, if we take* $\gamma=0$ *and* $\lambda=\frac{7}{15}$*, then we obtain the following corrected Simpson's inequality:*

$$\left|\frac{1}{30}\left(7\mathcal{S}(\sigma_1)+16\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+7\mathcal{S}(\sigma_2)\right)-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$

$$\leq\frac{1}{2}\left(\left(\frac{7}{15}\right)^{q+1}+\left(\frac{8}{15}\right)^{q+1}\right)^{\frac{1}{q}}\left(\frac{\sigma_2-\sigma_1}{1+q}\right)^{\frac{1}{q}}\|\mathcal{S}'\|_p.$$

Corollary 2.7. *In Corollary 2.4, taking* $\gamma=0$ *and* $\lambda=\frac{3}{8}$*, we get the following spline inequality:*

$$\left|\frac{1}{16}\left(3\mathcal{S}(\sigma_1)+10\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+3\mathcal{S}(\sigma_2)\right)-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$

$$\leq\frac{1}{2}\left(\left(\frac{3}{8}\right)^{q+1}+\left(\frac{5}{8}\right)^{q+1}\right)^{\frac{1}{q}}\left(\frac{\sigma_2-\sigma_1}{1+q}\right)^{\frac{1}{q}}\|\mathcal{S}'\|_p.$$

Corollary 2.8. *In Theorem 2.3, if we take* $x = \frac{3\sigma_1+\sigma_2}{4}$, *then we obtain the following* 5*-point Newton-Cotes type inequality:*

$$\left| \frac{\lambda\mathcal{S}(\sigma_1)+(2-\lambda)\mathcal{S}\left(\frac{3\sigma_1+\sigma_2}{4}\right)+4\gamma\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+(2-\lambda)\mathcal{S}\left(\frac{\sigma_1+4\sigma_2}{4}\right)+\lambda\mathcal{S}(\sigma_2)}{4(1+\gamma)} - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du \right|$$
$$\leq \frac{(\sigma_2-\sigma_1)^{\frac{1}{q}}}{4(1+q)^{\frac{1}{q}}}\left(\frac{\lambda^{q+1}+(1+\gamma-\lambda)^{q+1}+(2\gamma)^{q+1}+(1-\gamma)^{q+1}}{2(1+\gamma)^{q+1}}\right)^{\frac{1}{q}}\left\|\mathcal{S}'\right\|_p.$$

Remark 2.9. *In Corollary 2.8, if we take* $\lambda = \frac{14}{39}$ *and* $\gamma = \frac{2}{13}$, *then we obtain the inequality (5.9) of Corollary 5.2 from* [1].

Corollary 2.10. *In Corollary 2.8, if we take* $\lambda = \frac{2}{5}$ *and* $\gamma = \frac{1}{5}$, *we get the following Bullen-Simpson inequality:*

$$\left| \frac{\mathcal{S}(\sigma_1)+4\mathcal{S}\left(\frac{3\sigma_1+\sigma_2}{4}\right)+2\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+4\mathcal{S}\left(\frac{\sigma_1+3\sigma_2}{4}\right)+\mathcal{S}(\sigma_2)}{12} - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du \right|$$
$$\leq \frac{(\sigma_2-\sigma_1)^{\frac{1}{q}}}{12}\left(\frac{1+2^{q+1}}{3(1+q)}\right)^{\frac{1}{q}}\left\|\mathcal{S}'\right\|_p.$$

Corollary 2.11. *In Theorem 2.3, if we take* $x = \frac{2\sigma_1+\sigma_2}{3}$, *then we obtain the following* 5*-point Newton-Cotes type inequality:*

$$\left| \frac{2\lambda\mathcal{S}(\sigma_1)+(3-2\lambda)\mathcal{S}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+6\gamma\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+(3-2\lambda)\mathcal{S}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+2\lambda\mathcal{S}(\sigma_2)}{6(1+\gamma)} - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du \right|$$
$$\leq \begin{cases} \left(\frac{2^{q+1}\lambda^{q+1}+2^{q+1}(1+\gamma-\lambda)^{q+1}+3^{q+1}\gamma^{q+1}-(2\gamma-1)^{q+1}}{6^{q+1}(1+\gamma)^{q+1}}\right)^{\frac{1}{q}}\left(\frac{2(\sigma_2-\sigma_1)}{1+q}\right)^{\frac{1}{q}} \\ \qquad \times\left\|\mathcal{S}'\right\|_p \quad \text{if } \gamma > \frac{1}{2}, \\ \left(\frac{2^{q+1}\lambda^{q+1}+2^{q+1}(1+\gamma-\lambda)^{q+1}+3^{q+1}\gamma^{q+1}+(1-2\gamma)^{q+1}}{6^{q+1}(1+\gamma)^{q+1}}\right)^{\frac{1}{q}}\left(\frac{2(\sigma_2-\sigma_1)}{1+q}\right)^{\frac{1}{q}} \\ \qquad \times\left\|\mathcal{S}'\right\|_p \quad \text{if } \gamma \leq \frac{1}{2}. \end{cases}$$

Remark 2.12. *Corollary 2.11 will be reduced to Theorem 3 from [18] and Corollary 9 from [1], if we take* $\gamma = 0$ *and* $\lambda = \frac{3}{8}$.

Corollary 2.13. *In Theorem 2.3, if we take* $x = \frac{5\sigma_1+\sigma_2}{6}$, *we obtain the following* 5*-point Newton-Cotes type inequality:*

$$\left| \frac{\lambda\mathcal{S}(\sigma_1)+(3-\lambda)\mathcal{S}\left(\frac{5\sigma_1+\sigma_2}{6}\right)+6\gamma\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+(3-\lambda)\mathcal{S}\left(\frac{\sigma_1+5\sigma_2}{6}\right)+\lambda\mathcal{S}(\sigma_2)}{6(1+\gamma)} - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du \right|$$

$$\leq \left(\frac{\lambda^{q+1}+(1+\gamma-\lambda)^{q+1}+3^{q+1}\gamma^{q+1}+(2-\gamma)^{q+1}}{6^{q+1}(1+\gamma)^{q+1}}\right)^{\frac{1}{q}} \left(\frac{2(\sigma_2-\sigma_1)}{1+q}\right)^{\frac{1}{q}} \|\mathcal{S}'\|_p .$$

Remark 2.14. *In Corollary 2.13, taking* $\lambda = 0$ *and* $\gamma = \frac{1}{3}$*, we obtain the inequality (5.7) of Corollary 5.2 from [1].*

Corollary 2.15. *In Corollary 2.13, taking* $\lambda = 0$ *and* $\gamma = \frac{13}{27}$*, then we obtain the following corrected Euler-Maclaurin's inequality*

$$\left|\frac{1}{80}\left(27\mathcal{S}\left(\frac{5\sigma_1+\sigma_2}{6}\right)+26\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+27\mathcal{S}\left(\frac{\sigma_1+5\sigma_2}{6}\right)\right)-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$

$$\leq \left(\frac{39^{q+1}+40^{q+1}+41^{q+1}}{240^{q+1}}\right)^{\frac{1}{q}} \left(\frac{2(\sigma_2-\sigma_1)}{1+q}\right)^{\frac{1}{q}} \|\mathcal{S}'\|_p .$$

Remark 2.16. *In Theorem 2.3, if we take* $\lambda = \gamma = 0$*, then we obtain the second inequality of (4.1) from [15].*

Theorem 2.17. *Let* $\mathcal{S} : [\sigma_1, \sigma_2] \to \mathbb{R}$ *be an absolutely continuous mapping* $[\sigma_1, \sigma_2]$*. If* $\mathcal{S}'$ *belongs to* $L_\infty [\sigma_1, \sigma_2]$*, then we have*

$$\left|\mathcal{Q}_{\lambda,\gamma}(\sigma_1, x, \sigma_2; \mathcal{S}) - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$

$$\leq \begin{cases} \frac{1}{\sigma_2-\sigma_1}\left(\Phi - \left(\frac{(1+2\gamma)(x-\sigma_1)-(\sigma_2-x)}{2(1+\gamma)}\right)^2\right)\|\mathcal{S}'\|_\infty & \text{if } \gamma > \frac{\frac{\sigma_1+\sigma_2}{2}-x}{x-\sigma_1}, \\ \frac{1}{\sigma_2-\sigma_1}\left(\Phi + \left(\frac{\sigma_2-x-(1+2\gamma)(x-\sigma_1)}{2(1+\gamma)}\right)^2\right)\|\mathcal{S}'\|_\infty & \text{if } \gamma \leq \frac{\frac{\sigma_1+\sigma_2}{2}-x}{x-\sigma_1}, \end{cases}$$

where

$$\Phi = \left(\frac{\lambda(x-\sigma_1)}{1+\gamma}\right)^2 + \left(\frac{(1+\gamma-\lambda)(x-\sigma_1)}{1+\gamma}\right)^2 + \left(\frac{\gamma(\sigma_2-\sigma_1)}{2(1+\gamma)}\right)^2,$$

$\lambda, \gamma \in [0,1]$ *and* $x \in \left[\sigma_1, \frac{\sigma_1+\sigma_2}{2}\right]$*.*

Proof. Using the absolute value on both sides of (1) and the fact that $|\mathcal{S}'|$ is bounded, we get

$$\left|\mathcal{Q}_{\lambda,\gamma}(\sigma_1, x, \sigma_2; \mathcal{S}) - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$

$$\leq \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}|\mathcal{K}(t,x)|\,|\mathcal{S}'(t)|\,dt$$

$$\leq \frac{1}{\sigma_2-\sigma_1}\|\mathcal{S}'\|_\infty\int_{\sigma_1}^{\sigma_2}|\mathcal{K}(t,x)|\,dt$$

$$
\begin{aligned}
&=\frac{1}{\sigma_2-\sigma_1}\left(\int_{\sigma_1}^{x}\left|t-\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}\right|dt+\int_{x}^{\frac{\sigma_1+\sigma_2}{2}}\left|t-\frac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)}\right|dt\right.\\
&\quad\left.+\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_1+\sigma_2-x}\left|t-\frac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)}\right|dt+\int_{\sigma_1+\sigma_2-x}^{\sigma_2}\left|t-\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}\right|dt\right)\|\mathcal{S}'\|_\infty . \qquad (11)
\end{aligned}
$$

Two cases arise. For $\gamma>\frac{\frac{\sigma_1+\sigma_2}{2}-x}{x-\sigma_1}$, (11) gives

$$
\begin{aligned}
&\left|\mathcal{Q}_{\lambda,\gamma}(\sigma_1,x,\sigma_2;\mathcal{S})-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|\\
&\le\frac{1}{\sigma_2-\sigma_1}\left(\int_{\sigma_1}^{\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}}\left(\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}-t\right)dt\right.\\
&\quad+\int_{\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}}^{x}\left(t-\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}\right)dt\\
&\quad+\int_{x}^{\frac{\sigma_1+\sigma_2}{2}}\left(t-\frac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)}\right)dt+\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_1+\sigma_2-x}\left(\frac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)}-t\right)dt\\
&\quad+\int_{\sigma_1+\sigma_2-x}^{\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}}\left(\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}-t\right)dt\\
&\quad\left.+\int_{\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}}^{\sigma_2}\left(t-\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}\right)dt\right)\|\mathcal{S}'\|_\infty\\
&=\frac{1}{\sigma_2-\sigma_1}\left(\left(\frac{\lambda(x-\sigma_1)}{1+\gamma}\right)^2+\left(\frac{(1+\gamma-\lambda)(x-\sigma_1)}{1+\gamma}\right)^2\right.\\
&\quad\left.+\left(\frac{\gamma(\sigma_2-\sigma_1)}{2(1+\gamma)}\right)^2-\left(\frac{(1+2\gamma)(x-\sigma_1)-(\sigma_2-x)}{2(1+\gamma)}\right)^2\right)\|\mathcal{S}'\|_\infty . \qquad (12)
\end{aligned}
$$

For $\gamma \leq \frac{\frac{\sigma_1+\sigma_2}{2}-x}{x-\sigma_1}$, (11) gives

$$\left|\mathcal{Q}_{\lambda,\gamma}(\sigma_1,x,\sigma_2;\mathcal{S})-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$

$$\leq \frac{1}{\sigma_2-\sigma_1}\left(\int_{\sigma_1}^{\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}}\left(\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}-t\right)dt\right.$$

$$+\int_{\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}}^{x}\left(t-\frac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}\right)dt$$

$$+\int_{x}^{\frac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)}}\left(\frac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)}-t\right)dt+\int_{\frac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)}}^{\frac{\sigma_1+\sigma_2}{2}}\left(t-\frac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)}\right)dt$$

$$+\int_{\frac{\sigma_1+\sigma_2}{2}}^{\frac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)}}\left(\frac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)}-t\right)dt+\int_{\frac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)}}^{\sigma_1+\sigma_2-x}\left(t-\frac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)}\right)dt$$

$$+\int_{\sigma_1+\sigma_2-x}^{\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}}\left(\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}-t\right)dt$$

$$\left.+\int_{\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}}^{\sigma_2}\left(t-\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}\right)dt\right)\|\mathcal{S}'\|_\infty$$

$$=\frac{1}{\sigma_2-\sigma_1}\left(\left(\frac{\lambda(x-\sigma_1)}{1+\gamma}\right)^2+\left(\frac{(1+\gamma-\lambda)(x-\sigma_1)}{1+\gamma}\right)^2\right.$$

$$\left.+\left(\frac{\sigma_2-x-(1+2\gamma)(x-\sigma_1)}{2(1+\gamma)}\right)^2+\left(\frac{\gamma(\sigma_2-\sigma_1)}{2(1+\gamma)}\right)^2\right)\|\mathcal{S}'\|_\infty. \tag{13}$$

The desired result follows from (12) and (13). The proof is completed. □

Corollary 2.18. *In Theorem 2.17, if we take $x=\frac{\sigma_1+\sigma_2}{2}$, we obtain the following Simpson-like type inequality:*

$$\left|\frac{\lambda}{2(1+\gamma)}\mathcal{S}(\sigma_1)+\frac{1+\gamma-\lambda}{1+\gamma}\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+\frac{\lambda}{2(1+\gamma)}\mathcal{S}(\sigma_2)-\frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$

$$\leq\frac{\left(\lambda^2+(1+\gamma-\lambda)^2\right)(\sigma_2-\sigma_1)}{4(1+\gamma)^2}\|\mathcal{S}'\|_\infty.$$

Remark 2.19. *In Corollary 2.18, if we take*

- $\gamma = 0$, *then we obtain Corollary 4 from [1].*
- $\gamma = \lambda = 0$, *then we obtain the inequality (1) of Corollary 5 from [1].*
- $\gamma = 0$ *and* $\lambda = 1$, *then we obtain the inequality (4) of Corollary 5 from [1].*
- $\gamma = 0$ *and* $\lambda = \frac{1}{2}$, *then we obtain the inequality (6.4) of Corollary 11 from [11].*
- $\gamma = 0$ *and* $\lambda = \frac{1}{3}$, *then we obtain the inequality (2.7) of Corollary 1 from [12].*

Corollary 2.20. *In Corollary 2.18, if we take* $\gamma = 0$ *and* $\lambda = \frac{7}{15}$, *we obtain the following corrected Simpson's inequality:*

$$\left| \frac{1}{30} \left(7\mathcal{S}\left(\sigma_1\right) + 16\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right) + 7\mathcal{S}\left(\sigma_2\right)\right) - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} \mathcal{S}\left(u\right) du \right| \leq \frac{113(\sigma_2-\sigma_1)}{900} \left\|\mathcal{S}'\right\|_\infty .$$

Corollary 2.21. *In Corollary 2.18, taking* $\gamma = 0$ *and* $\lambda = \frac{3}{8}$, *we get the following spline inequality:*

$$\left| \frac{1}{16} \left(3\mathcal{S}\left(\sigma_1\right) + 10\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right) + 3\mathcal{S}\left(\sigma_2\right)\right) - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} \mathcal{S}\left(u\right) du \right| \leq \frac{17(\sigma_2-\sigma_1)}{128} \left\|\mathcal{S}'\right\|_\infty .$$

Corollary 2.22. *In Theorem 2.17, if we take* $x = \frac{3\sigma_1+\sigma_2}{4}$, *we obtain the following* 5*-point Newton-Cotes type inequality:*

$$\left| \frac{\lambda\mathcal{S}(\sigma_1)+(2-\lambda)\mathcal{S}\left(\frac{3\sigma_1+\sigma_2}{4}\right)+4\gamma\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+(2-\lambda)\mathcal{S}\left(\frac{\sigma_1+4\sigma_2}{4}\right)+\lambda\mathcal{S}(\sigma_2)}{4(1+\gamma)} - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} \mathcal{S}\left(u\right) du \right|$$
$$\leq \frac{\left(\lambda^2+(1+\gamma-\lambda)^2+4\gamma^2+(1-\gamma)^2\right)(\sigma_2-\sigma_1)}{16(1+\gamma)^2} \left\|\mathcal{S}'\right\|_\infty .$$

Corollary 2.23. *In Corollary 2.22, if we take* $\lambda = \frac{14}{39}$ *and* $\gamma = \frac{2}{13}$, *then we obtain the following Boole inequality:*

$$\left| \frac{7\mathcal{S}(\sigma_1)+32\mathcal{S}\left(\frac{\sigma_1+3\sigma_2}{4}\right)+12\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+32\mathcal{S}\left(\frac{3\sigma_1+\sigma_2}{4}\right)+7\mathcal{S}(\sigma_2)}{90} - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} \mathcal{S}\left(u\right) du \right|$$
$$\leq \frac{239(\sigma_2-\sigma_1)}{3240} \left\|\mathcal{S}'\right\|_\infty .$$

Corollary 2.24. *Within Corollary 2.22, in the event that we take* $\lambda = \frac{2}{5}$ *and* $\gamma = \frac{1}{5}$, *we get the following Bullen-Simpson inequality:*

$$\left| \frac{\mathcal{S}(\sigma_1)+4\mathcal{S}\left(\frac{3\sigma_1+\sigma_2}{4}\right)+2\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+4\mathcal{S}\left(\frac{\sigma_1+3\sigma_2}{4}\right)+\mathcal{S}(\sigma_2)}{12} - \frac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2} \mathcal{S}\left(u\right) du \right|$$
$$\leq \frac{5(\sigma_2-\sigma_1)}{72} \left\|\mathcal{S}'\right\|_\infty .$$

Corollary 2.25. *In Theorem 2.17, if we take $x=\frac{2\sigma_1+\sigma_2}{3}$, we obtain the following 5-point Newton-Cotes type inequality:*

$$\left|\frac{2\lambda\mathcal{S}(\sigma_1)+(3-2\lambda)\mathcal{S}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+6\gamma\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+(3-2\lambda)\mathcal{S}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+2\lambda\mathcal{S}(\sigma_2)}{6(1+\gamma)}\right.$$
$$\left.-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$
$$\le\begin{cases}\frac{\left(4\lambda^2+4(1+\gamma-\lambda)^2+9\gamma^2-(2\gamma-1)^2\right)(\sigma_2-\sigma_1)}{36(1+\gamma)^2}\left\|\mathcal{S}'\right\|_\infty & \text{if } \gamma>\frac{1}{2},\\ \frac{\left(4\lambda^2+4(1+\gamma-\lambda)^2+9\gamma^2+(1-2\gamma)^2\right)(\sigma_2-\sigma_1)}{36(1+\gamma)^2}\left\|\mathcal{S}'\right\|_\infty & \text{if } \gamma\le\frac{1}{2}.\end{cases}$$

Remark 2.26. *Corollary 2.25 will be reduced to Corollary 4 from [18] and Corollary 6 from [1], if we take $\gamma=0$ and $\lambda=\frac{3}{8}$.*

Corollary 2.27. *In Theorem 2.17, if we take $x=\frac{5\sigma_1+\sigma_2}{6}$, we obtain the following 5-point Newton-Cotes type inequality:*

$$\left|\frac{\lambda\mathcal{S}(\sigma_1)+(3-\lambda)\mathcal{S}\left(\frac{5\sigma_1+\sigma_2}{6}\right)+6\gamma\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+(3-\lambda)\mathcal{S}\left(\frac{\sigma_1+5\sigma_2}{6}\right)+\lambda\mathcal{S}(\sigma_2)}{6(1+\gamma)}-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$
$$\le\frac{\left(\lambda^2+(1+\gamma-\lambda)^2+9\gamma^2+(2-\gamma)^2\right)(\sigma_2-\sigma_1)}{36(1+\gamma)^2}\left\|\mathcal{S}'\right\|_\infty.$$

Corollary 2.28. *In Corollary 2.27, taking $\lambda=0$ and $\gamma=\frac{1}{3}$ we obtain the following Maclaurin's inequality*

$$\left|\frac{1}{8}\left(3\mathcal{S}\left(\frac{5\sigma_1+\sigma_2}{6}\right)+2\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+3\mathcal{S}\left(\frac{\sigma_1+5\sigma_2}{6}\right)\right)-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$
$$\le\frac{25(\sigma_2-\sigma_1)}{288}\left\|\mathcal{S}'\right\|_\infty.$$

Corollary 2.29. *In Corollary 2.27, taking $\lambda=0$ and $\gamma=\frac{13}{27}$ we obtain the following corrected Euler-Maclaurin's inequality*

$$\left|\frac{1}{80}\left(27\mathcal{S}\left(\frac{5\sigma_1+\sigma_2}{6}\right)+26\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+27\mathcal{S}\left(\frac{\sigma_1+5\sigma_2}{6}\right)\right)-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$
$$\le\frac{2401(\sigma_2-\sigma_1)}{28800}\left\|\mathcal{S}'\right\|_\infty.$$

Remark 2.30. *In Theorem 2.17, if we take $\lambda=\gamma=0$, then we obtain the first inequality of (4.1) from [15].*

Theorem 2.31. *Let $\mathcal{S}:[\sigma_1,\sigma_2]\to\mathbb{R}$ be a continuous differentiable on $[\sigma_1,\sigma_2]$. If $\mathcal{S}'$belongs to $L^1[\sigma_1,\sigma_2]$, then we have*

$$\left|\mathcal{Q}_{\lambda,\gamma}(\sigma_1,x,\sigma_2;\mathcal{S})-\frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right|$$

$$\leq \max\left\{\tfrac{(1+\gamma-\lambda)}{1+\gamma}\left(\tfrac{x-\sigma_1}{\sigma_2-\sigma_1}\right), \tfrac{\lambda}{1+\gamma}\left(\tfrac{x-\sigma_1}{\sigma_2-\sigma_1}\right), \left|\tfrac{(1+2\gamma)(x-\sigma_1)-(\sigma_2-x)}{2(1+\gamma)(\sigma_2-\sigma_1)}\right|, \tfrac{\gamma}{2(1+\gamma)}\right\}\left\|\mathcal{S}'\right\|_1.$$

where $\lambda, \gamma \in [0,1]$ *and* $x \in \left[\sigma_1, \frac{\sigma_1+\sigma_2}{2}\right]$.

Proof. Using the absolute value on both sides of (1), we have

$$\begin{aligned}
&\left|\mathcal{Q}_{\lambda,\gamma}\left(\sigma_1, x, \sigma_2; \mathcal{S}\right) - \tfrac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\mathcal{S}(u)\,du\right| \\
&\leq \tfrac{1}{\sigma_2-\sigma_1}\int_{\sigma_1}^{\sigma_2}\left|\mathcal{K}(t,x)\right|\left|\mathcal{S}'(t)\right|dt \\
&\leq \tfrac{1}{\sigma_2-\sigma_1}\left(\int_{\sigma_1}^{x}\left|t-\tfrac{(1+\gamma)\sigma_1+\lambda(x-\sigma_1)}{1+\gamma}\right|\left|\mathcal{S}'(t)\right|dt\right. \\
&\quad+\int_{x}^{\frac{\sigma_1+\sigma_2}{2}}\left|t-\tfrac{(1+2\gamma)\sigma_1+\sigma_2}{2(1+\gamma)}\right|\left|\mathcal{S}'(t)\right|dt \\
&\quad+\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_1+\sigma_2-x}\left|t-\tfrac{\sigma_1+(1+2\gamma)\sigma_2}{2(1+\gamma)}\right|\left|\mathcal{S}'(t)\right|dt \\
&\quad\left.+\int_{\sigma_1+\sigma_2-x}^{\sigma_2}\left|t-\tfrac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}\right|\left|\mathcal{S}'(t)\right|dt\right) \\
&\leq \tfrac{1}{\sigma_2-\sigma_1}\left(\max\left\{\tfrac{(1+\gamma-\lambda)(x-\sigma_1)}{1+\gamma}, \tfrac{\lambda(x-\sigma_1)}{1+\gamma}\right\}\right. \\
&\quad\times\left(\int_{\sigma_1}^{x}\left|\mathcal{S}'(t)\right|dt \int_{\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}}^{\sigma_2}\left|\mathcal{S}'(t)\right|dt\right) \\
&\quad+\max\left\{\left|\tfrac{(1+2\gamma)(x-\sigma_1)-(\sigma_2-x)}{2(1+\gamma)}\right|, \tfrac{\gamma(\sigma_2-\sigma_1)}{2(1+\gamma)}\right\} \\
&\quad\left.\times\left(\int_{x}^{\frac{\sigma_1+\sigma_2}{2}}\left|\mathcal{S}'(t)\right|dt+\int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_1+\sigma_2-x}\left|\mathcal{S}'(t)\right|dt\right)\right) \\
&\leq \tfrac{1}{\sigma_2-\sigma_1}\max\left\{\tfrac{(1+\gamma-\lambda)(x-\sigma_1)}{1+\gamma}, \tfrac{\lambda(x-\sigma_1)}{1+\gamma}, \left|\tfrac{(1+2\gamma)(x-\sigma_1)-(\sigma_2-x)}{2(1+\gamma)}\right|, \tfrac{\gamma(\sigma_2-\sigma_1)}{2(1+\gamma)}\right\} \\
&\quad\times\left(\int_{\sigma_1}^{x}\left|\mathcal{S}'(t)\right|dt+\int_{x}^{\frac{\sigma_1+\sigma_2}{2}}\left|\mathcal{S}'(t)\right|dt\right.
\end{aligned}$$

$$\left. + \int_{\frac{\sigma_1+\sigma_2}{2}}^{\sigma_1+\sigma_2-x} |\mathcal{S}'(t)|\, dt + \int_{\frac{(1+\gamma)\sigma_2-\lambda(x-\sigma_1)}{1+\gamma}}^{\sigma_2} |\mathcal{S}'(t)|\, dt \right)$$

$$\leq \tfrac{1}{\sigma_2-\sigma_1} \max\left\{ \tfrac{(1+\gamma-\lambda)(x-\sigma_1)}{1+\gamma}, \tfrac{\lambda(x-\sigma_1)}{1+\gamma}, \left| \tfrac{(1+2\gamma)(x-\sigma_1)-(\sigma_2-x)}{2(1+\gamma)} \right|, \tfrac{\gamma(\sigma_2-\sigma_1)}{2(1+\gamma)} \right\} \|\mathcal{S}'\|_1 .$$

The proof is completed. □

Corollary 2.32. *In Theorem 2.31, if we take $x = \frac{\sigma_1+\sigma_2}{2}$, we obtain the following Simpson-like type inequality:*

$$\left| \tfrac{\lambda}{2(1+\gamma)} \mathcal{S}(\sigma_1) + \tfrac{1+\gamma-\lambda}{1+\gamma} \mathcal{S}\left(\tfrac{\sigma_1+\sigma_2}{2}\right) + \tfrac{\lambda}{2(1+\gamma)} \mathcal{S}(\sigma_2) - \tfrac{1}{\sigma_2-\sigma_1} \int_{\sigma_1}^{\sigma_2} \mathcal{S}(u)\, du \right|$$
$$\leq \tfrac{1+\gamma+|1+\gamma-2\lambda|}{4(1+\gamma)} \|\mathcal{S}'\|_1 ,$$

where we have used the fact that

$$\tfrac{1}{2} \max\left\{ \tfrac{1+\gamma-\lambda}{1+\gamma}, \tfrac{\lambda}{1+\gamma}, \tfrac{\gamma}{1+\gamma} \right\} = \tfrac{1}{2} \max\left\{ \tfrac{1+\gamma-\lambda}{1+\gamma}, \tfrac{\lambda}{1+\gamma} \right\} = \tfrac{1+\gamma+|1+\gamma-2\lambda|}{4(1+\gamma)}.$$

Remark 2.33. *In Corollary 2.32, if we take*

- *$\gamma = \lambda = 0$, then we obtain inequality (2.17) from [15].*
- *$\gamma = 0$ and $\lambda = 1$, then we obtain inequality (2.16) from [15].*
- *$\gamma = 0$ and $\lambda = \frac{1}{3}$, then we obtain Corollary 2.2 from [10].*

Corollary 2.34. *If we assume $\gamma = 0$, Corollary 2.32 gives the following Simpson-like type inequality:*

$$\left| \tfrac{\lambda}{2} \mathcal{S}(\sigma_1) + (1-\lambda) \mathcal{S}\left(\tfrac{\sigma_1+\sigma_2}{2}\right) + \tfrac{\lambda}{2} \mathcal{S}(\sigma_2) - \tfrac{1}{\sigma_2-\sigma_1} \int_{\sigma_1}^{\sigma_2} \mathcal{S}(u)\, du \right|$$
$$\leq \tfrac{1}{2} \max\{1-\lambda, \lambda\} \|\mathcal{S}'\|_1 = \tfrac{1+|1-2\lambda|}{4} \|\mathcal{S}'\|_1 .$$

Corollary 2.35. *In Corollary 2.32, if we take $\gamma = 0$ and $\lambda = \frac{1}{2}$, we obtain the following Bullen inequality*

$$\left| \tfrac{1}{2} \left(\tfrac{\mathcal{S}(\sigma_1)+\mathcal{S}(\sigma_2)}{2} + \mathcal{S}\left(\tfrac{\sigma_1+\sigma_2}{2}\right) \right) - \tfrac{1}{\sigma_2-\sigma_1} \int_{\sigma_1}^{\sigma_2} \mathcal{S}(u)\, du \right| \leq \tfrac{1}{4} \|\mathcal{S}'\|_1 .$$

Corollary 2.36. *In Corollary 2.32, if we take $\gamma = 0$ and $\lambda = \frac{7}{15}$, we obtain the following corrected Simpson's inequality:*

$$\left| \tfrac{1}{30} \left(7\mathcal{S}(\sigma_1) + 16\mathcal{S}\left(\tfrac{\sigma_1+\sigma_2}{2}\right) + 7\mathcal{S}(\sigma_2) \right) - \tfrac{1}{\sigma_2-\sigma_1} \int_{\sigma_1}^{\sigma_2} \mathcal{S}(u)\, du \right| \leq \tfrac{1}{60} \|\mathcal{S}'\|_1 .$$

Corollary 2.37. *In Corollary 2.32, taking $\gamma = 0$ and $\lambda = \frac{3}{8}$, we get the following spline inequality:*

$$\left| \left(\frac{1}{16} \left(3\mathcal{S}(\sigma_1) + 10\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right) + 3\mathcal{S}(\sigma_2) \right) \right) - \frac{1}{\sigma_2-\sigma_1} \int_{\sigma_1}^{\sigma_2} \mathcal{S}(u)\, du \right| \le \frac{5}{16} \left\| \mathcal{S}' \right\|_1 .$$

Corollary 2.38. *In Theorem 2.31, if we take $x = \frac{3\sigma_1+\sigma_2}{4}$, we obtain the following 5-point Newton-Cotes type inequality:*

$$\left| \frac{\lambda\mathcal{S}(\sigma_1)+(2-\lambda)\mathcal{S}\left(\frac{3\sigma_1+\sigma_2}{4}\right)+4\gamma\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+(2-\lambda)\mathcal{S}\left(\frac{\sigma_1+4\sigma_2}{4}\right)+\lambda\mathcal{S}(\sigma_2)}{4(1+\gamma)} - \frac{1}{\sigma_2-\sigma_1} \int_{\sigma_1}^{\sigma_2} \mathcal{S}(u)\, du \right|$$

$$\le \max\left\{ \frac{(1+\gamma-\lambda)}{4(1+\gamma)}, \frac{\lambda}{4(1+\gamma)}, \frac{1-\gamma}{4(1+\gamma)}, \frac{\gamma}{2(1+\gamma)} \right\} \left\| \mathcal{S}' \right\|_1 .$$

Corollary 2.39. *In Corollary 2.38, if we take $\lambda = \frac{14}{39}$ and $\gamma = \frac{2}{13}$, then we obtain the following Boole inequality:*

$$\left| \frac{7\mathcal{S}(\sigma_1)+32\mathcal{S}\left(\frac{\sigma_1+3\sigma_2}{4}\right)+12\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+32\mathcal{S}\left(\frac{3\sigma_1+\sigma_2}{4}\right)+7\mathcal{S}(\sigma_2)}{90} - \frac{1}{\sigma_2-\sigma_1} \int_{\sigma_1}^{\sigma_2} \mathcal{S}(u)\, du \right|$$

$$\le \frac{11}{60} \left\| \mathcal{S}' \right\|_1 .$$

Corollary 2.40. *In Corollary 2.38, if we take $\lambda = \frac{2}{5}$ and $\gamma = \frac{1}{5}$, we get the following Bullen-Simpson inequality:*

$$\left| \frac{\mathcal{S}(\sigma_1)+4\mathcal{S}\left(\frac{3\sigma_1+\sigma_2}{4}\right)+2\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+4\mathcal{S}\left(\frac{\sigma_1+3\sigma_2}{4}\right)+\mathcal{S}(\sigma_2)}{12} - \frac{1}{\sigma_2-\sigma_1} \int_{\sigma_1}^{\sigma_2} \mathcal{S}(u)\, du \right| \le \frac{1}{6} \left\| \mathcal{S}' \right\|_1 .$$

Corollary 2.41. *In Theorem 2.31, if we take $x = \frac{2\sigma_1+\sigma_2}{3}$, we obtain the following 5-point Newton-Cotes type inequality:*

$$\left| \frac{2\lambda\mathcal{S}(\sigma_1)+(3-2\lambda)\mathcal{S}\left(\frac{2\sigma_1+\sigma_2}{3}\right)+6\gamma\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+(3-2\lambda)\mathcal{S}\left(\frac{\sigma_1+2\sigma_2}{3}\right)+2\lambda\mathcal{S}(\sigma_2)}{6(1+\gamma)} \right.$$

$$\left. - \frac{1}{\sigma_2-\sigma_1} \int_{\sigma_1}^{\sigma_2} \mathcal{S}(u)\, du \right|$$

$$\le \max\left\{ \frac{1+\gamma-\lambda}{3(1+\gamma)}, \frac{\lambda}{3(1+\gamma)}, \frac{|2\gamma-1|}{6(1+\gamma)}, \frac{\gamma}{2(1+\gamma)} \right\} \left\| \mathcal{S}' \right\|_1 .$$

Remark 2.42. *Corollary 2.41 will be reduced to Corollary 1 from* [18], *if we take $\gamma = 0$ and $\lambda = \frac{3}{8}$.*

Corollary 2.43. *In Theorem 2.31, if we take $x = \frac{5\sigma_1+\sigma_2}{6}$, we obtain the following 5-point Newton-Cotes type inequality:*

$$\left| \frac{\lambda\mathcal{S}(\sigma_1)+(3-\lambda)\mathcal{S}\left(\frac{5\sigma_1+\sigma_2}{6}\right)+6\gamma\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+(3-\lambda)\mathcal{S}\left(\frac{\sigma_1+5\sigma_2}{6}\right)+\lambda\mathcal{S}(\sigma_2)}{6(1+\gamma)} - \frac{1}{\sigma_2-\sigma_1} \int_{\sigma_1}^{\sigma_2} \mathcal{S}(u)\, du \right|$$

$\leq \max\left\{\frac{1+\gamma-\lambda}{6(1+\gamma)}, \frac{\lambda}{6(1+\gamma)}, \frac{2-\gamma}{6(1+\gamma)}, \frac{3\gamma}{6(1+\gamma)}\right\} \|\mathcal{S}'\|_1 .$

Corollary 2.44. *In Corollary 2.43, taking $\lambda = 0$ and $\gamma = \frac{1}{3}$ we obtain the following Maclaurin's inequality*

$$\left|\frac{3\mathcal{S}\left(\frac{5\sigma_1+\sigma_2}{6}\right)+2\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+3\mathcal{S}\left(\frac{\sigma_1+5\sigma_2}{6}\right)}{8} - \frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2} \mathcal{S}(u)\,du\right| \leq \frac{5}{24}\|\mathcal{S}'\|_1 .$$

Corollary 2.45. *In Corollary 2.43, taking $\lambda = 0$ and $\gamma = \frac{13}{27}$, then we obtain the following corrected Euler-Maclaurin's inequality*

$$\left|\frac{27\mathcal{S}\left(\frac{5\sigma_1+\sigma_2}{6}\right)+26\mathcal{S}\left(\frac{\sigma_1+\sigma_2}{2}\right)+27\mathcal{S}\left(\frac{\sigma_1+5\sigma_2}{6}\right)}{80} - \frac{1}{\sigma_2-\sigma_1}\int\limits_{\sigma_1}^{\sigma_2} \mathcal{S}(u)\,du\right| \leq \frac{41}{240}\|\mathcal{S}'\|_1 .$$

Remark 2.46. *In Theorem 2.31, if we take $\lambda = \gamma = 0$, then we obtain the third inequality of (4.1) from [15].*

3. Applications

For arbitrary real numbers σ_1, σ_2 we have:
The Arithmetic mean: $A(\sigma_1, \sigma_2) = \frac{\sigma_1+\sigma_2}{2}$.

The p-Logarithmic mean: $L_p(\sigma_1, \sigma_2) = \left(\frac{\sigma_2^{p+1}-\sigma_1^{p+1}}{(p+1)(\sigma_2-\sigma_1)}\right)^{\frac{1}{p}}$, $\sigma_1, \sigma_2 > 0, \sigma_1 \neq \sigma_2$ and $p \in \mathbb{R}\backslash\{-1, 0\}$.

Proposition 3.1. *Let $\sigma_1, \sigma_2 \in \mathbb{R}$ with $0 < \sigma_1 < \sigma_2$, then we have*

$$\left|3A\left(\sigma_1^2, \sigma_2^2\right) + 5A^2(\sigma_1, \sigma_2) - 8L_2^2(\sigma_1, \sigma_2)\right|$$
$$\leq \left(\frac{3^{q+1}+5^{q+1}}{8}\right)^{\frac{1}{q}} \left(\frac{\sigma_2-\sigma_1}{1+q}\right)^{\frac{1}{q}} L_p(\sigma_1, \sigma_2).$$

Proof. This follows from Corollary 2.7, applied to the function $\mathcal{S}(u) = \frac{1}{2}u^2$. □

Proposition 3.2. *Consider X a random variable with the probability density function $\mathcal{S}$ that takes values in the finite interval $[\sigma_1, \sigma_2]$ i.e. $\mathcal{S} : [\sigma_1, \sigma_2] \to [0, 1]$ with the cumulative distribution function $F(x) = \Pr(X \leq x) = \int\limits_{\sigma_1}^{x} \mathcal{S}(u)\,du$, we have*

$$\left|\frac{32\mathcal{F}\left(\frac{\sigma_1+3\sigma_2}{4}\right)+12\mathcal{F}\left(\frac{\sigma_1+\sigma_2}{2}\right)+32\mathcal{F}\left(\frac{3\sigma_1+\sigma_2}{4}\right)+7}{90} - \frac{\sigma_2-E[X]}{\sigma_2-\sigma_1}\right| \leq \frac{11}{60}.$$

Proof. Replace $\mathcal{S} = F$ in Corollary 2.39 and taking into account that $F(\sigma_1) = 0, F(\sigma_2) = 1$ and

$$E[X] = \int\limits_{\sigma_1}^{\sigma_2} u\mathcal{S}(u)\,du = \sigma_2 F(\sigma_2) - \sigma_1 F(\sigma_1) - \int\limits_{\sigma_1}^{\sigma_2} F(u)\,du = \sigma_2 - \int\limits_{\sigma_1}^{\sigma_2} F(u)\,du.$$

□

4. Conclusion

In this paper, we have introduced a novel bi-parameterized integral identity and leveraged it to derive a series of integral inequalities for functions with first derivatives in the L^p spaces, where $1 \leq p \leq \infty$. The results obtained extend and generalize several existing findings in the literature. Additionally, we have demonstrated the practical relevance of these inequalities through various applications, underscoring the significance of the proposed identity in mathematical analysis. Future work may explore further extensions and potential applications in related fields.

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