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ON A CERTAIN GENERALIZATION OF POLYGROUPS BY E-POLYGROUPS

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Abstract. In this paper, first we generalize the notion of polygroups and weak polygroups by the notion of E-polygroups and weak E-polygroups. Then, we study the isomorphism theorems on E-polygroups and weak E-polygroups. Finally, the fundamental relations on weak E-polygroups are investigated.

1. Introduction

Algebraic hyperstructures were introduced by a French mathematician, F. Marty [11] in 1934. Afterwards, this new idea was expanded rapidly and showed himself as a new view on sets. In this context, hundreds papers and several books have been written on this topic. One of the first books, dedicated especially to hypergroups is "Prolegomena of Hypergroup Theory", written by P. Corsini in 1993 [3]. Another book on "Hyperstructures and Their Representations" was published one year later [14]. Another book on these topics is "Applications of Hyperstructure Theory", written by P. Corsini and V. Leoreanu [4] and "Polygroup Theory and Related Systems" wrtten by B. Davvaz [6], also see [1, 10, 12, 13]. Algebraic hyperstructure theory has also a multiplicity of applications to other sciences, such as geometry, graphs and hypergraphs, binary relations, lattices, groups, relation algebras, artificial intelligence, probabilities and so on, for more recent details see [7, 8, 9].

In the present study, we aimed to extend algebraic hyperstructures such as polygroups and weak polygroups by the concepts of E-polygroups and weak E-polygroups, where the non-empty subset E plays a similar role of the identity. Besides mathematical applications, this approach may have of importance in physics and other sciences.

After providing some interesting examples, we investigate the characterizations of these concepts and address which properties and constructions of usual polygroups and weak polygroups still remain true for E-polygroups and weak

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E-polygroups. In particular, E-normal subpolygroups and the extensions of weak E-polygroups are discussed. We also study isomorphism of E-polygroups and conditions under which isomorphism theorems hold. Finally, the fundamental relation on a weak E-polygroup are defined as the smallest equivalence relation turning it into a group, and some results are obtained.

2. Preliminaries and notations

In this section, we summarize the general preliminary definitions of algebraic hyperstructures and we exclude special cases.

Definition 2.1. *Let H be a non-empty set and let $\mathcal{P}^*(H)$ be the set of all non-empty subsets of H, we define the concepts of hyperoperation, semi-hypergroup, hypergroup, H_v-group and regular hypergroup as following:*

(i) *A hyperoperation on H is defined as a map $\otimes : H \times H \longrightarrow \mathcal{P}^*(H)$ and the couple $(H, \otimes)$ is called a hypergroupoid. If A and B are non-empty subsets of H, then we denote $A \otimes B = \bigcup_{a \in A,\, b \in B} a \otimes b$, $x \otimes A = \{x\} \otimes A$ and $A \otimes x = A \otimes \{x\}$, where $x \in H$.*

(ii) *A hypergroupoid $(H, \otimes)$ is called a semi-hypergroup if we have $(x \otimes y) \otimes z = x \otimes (y \otimes z)$ for all x, y, z of H, which means*

$$\bigcup_{u \in x \otimes y} u \otimes z = \bigcup_{v \in y \otimes z} x \otimes v.$$

(iii) *We say that a semi-hypergroup $(H, \otimes)$ is a hypergroup if we have $x \otimes H = H \otimes x = H$ for all $x \in H$.*
A hypergroupoid $(H, \otimes)$ is an H_v-group, if for all $x, y, z \in H$, the following conditions hold:

(1) *$x \otimes (y \otimes z) \cap (x \otimes y) \otimes z \neq \emptyset$ (weak associativity),*

(2) *$x \otimes H = H \otimes x = H$ (reproduction axiom).*

(iv) *A hypergroupoid $(H, \otimes)$ is said to be commutative (or abelian) if $x \otimes y = y \otimes x$ for all $x, y \in H$.*

(v) *A hypergroup $(H, \otimes)$ is called regular if it has at least an identity, that is an element e of H, such that for all $x \in H, x \in e \otimes x \cap x \otimes e$ and moreover each element has at least one inverse, that is if $x \in H$, then there exists $x' \in H$ such that $e \in x \otimes x' \cap x' \otimes x$. The set of all identities of H is denoted by $E(H)$*

(vi) *If $x \in H, i_l(x) = \{x' : e \in x' \otimes x\}$ is the set of all left inverses of x in H (resp. $i_r(x)$) and $i(x) = i_l(x) \cap i_r(x)$.*

(vi) *A regular hypergroup $(H, \otimes)$ is called reversible if for all $(x; y; a) \in H^3$:*
(1) $y \in a \otimes x$, then there exists $a' \in i(a)$ such that $x \in a' \cap y$;
(2) $y \in x \otimes a$, then there exists $a'' \in i(a)$ such that $x \in y \otimes a''$.

Definition 2.2. *Let $(H, \otimes)$ be an H_v-group and K be a non-empty subset of H. Then K is called an H_v-subgroup of H if $(K, \otimes)$ is an H_v-group.*

Definition 2.3. *Let $(H, \otimes)$ be a hypergroup, K a nonempty subset of H. We say that K is invertible to the left if the implication $y \in K \otimes x \Longrightarrow x \in K \otimes y$ valid. We say K is invertible if K is invertible to the right and to the left.*

Proposition 2.4. [?]. *If $(H, \otimes)$ is a hypergroup such that $E(H) \neq \phi$; and K is an invertible subhypergroup of it, then $E(H) \subseteq K$.*

Proof. Suppose that $e \in E(H)$. Since $K \subseteq e \otimes K$, we have $e \in K \otimes K \subseteq K$, because K is an invertible subhypergroup. □

Definition 2.5. *Let $(H_1, \cdot)$, $(H_2, *)$ be two H_v-groups. A map $f : H_1 \to H_2$ is called an H_v-homomorphism or a weak homomorphism if*

$$f(x \cdot y) \cap f(x) * f(y) \neq \emptyset \text{ for all } x, y \in H_1.$$

f is called an inclusion homomorphism if

$$f(x \cdot y) \subseteq f(x) * f(y) \text{ for all } x, y \in H_1.$$

Finally, f is called a strong homomorphism if

$$f(x \cdot y) = f(x) * f(y) \text{ for all } x, y \in H_1.$$

If f is onto, one to one and strong homomorphism, then it is called an isomorphism. In this case, we write $H_1 \cong H_2$. Moreover, if the domain and the range of f are the same H_v-group, then the isomorphism is called an automorphism. We can easily verify that the set of all automorphisms of H, denoted by $AutH$, is a group.

Definition 2.6. [2] *A multivalued system $\langle P, \circ, e, ^{-1} \rangle$, where $e \in P$, $^{-1} : P \longrightarrow P$, $\circ : P \times P \longrightarrow \mathcal{P}^*(P)$ is called a polygroup (a weak polygroup) if the following axioms hold for all $x, y, z \in P$;*
I) $(x \circ y) \circ z = x \circ (y \circ z)$, $((x \circ y) \circ z \cap x \circ (y \circ z) \neq \emptyset)$,
II) $x \circ e = x = e \circ x$,
III) $x \in y \circ z$ *implies* $y \in x \circ z^{-1}$ and $z \in y^{-1} \circ x$.

Example 2.7. *Consider $P = \{e, 1, 2, 3\}$ and define $\circ$ on P with help of the following table;*

$\circ$	e	1	2	3
e	e	1	2	3
1	1	1	$\{e, 1, 2, 3\}$	3
2	2	$\{e, 1, 2\}$	2	$\{2, 3\}$
3	3	$\{1, 3\}$	3	$\{e, 1, 2, 3\}$

Then $\langle P, \circ, e, ^{-1} \rangle$, where $x^{-1} = x$, for every $x \in P$, is a polygroup.

A polygroup is a special case of a hypergroup.

Example 2.8. *Consider $P = \{e, 1, 2, 3\}$ and define $\circ$ on P with help of the following table;*

$\circ$	e	1	2	3
e	e	1	2	3
1	1	$\{e,1\}$	3	2
2	2	3	$\{e,2\}$	1
3	3	2	1	$\{e,3\}$

Then $\langle P, \circ, e, ^{-1} \rangle$, *where* $x^{-1} = x$, *for every* $x \in P$, *is a weak polygroup which is not a polygroup. Indeed, we have* $(1 \circ 2) \circ 3 = 3 \circ 3 = \{e, 3\}$ *and* $1 \circ (2 \circ 3) = 1 \circ 2 = \{e, 1\}$. *Therefore* $\circ$ *is not associative.*

Example 2.9. *A natural example of a polygroup is the system* $G//H$ *of all double cosets of a group* G *modulo a subgroup* H. *Namely, the polygroup* $G//H =< \{HgH : g \in G\}, \circ, H, ^{-1} >$ *where* $(HgH) \circ (Hg'H) = \{Hghg'H : h \in H\}$ *and* $(HgH)^{-1} = Hg^{-1}H$.

3. E-polygroups

Definition 3.1. *A multivalued system* $< P, \circ, E, ^{-J} >$, *where* $\emptyset \neq E \subseteq P$, $^{-J} : P \to P$ *is unitary operation on* P *and* $\circ : P \times P \to \mathcal{P}^*(P)$ *a hyperoperation on* P, *is called an* E*-polygroup if the following axioms hold for all* $x, y, z \in P$;

(IP_1) $(x \circ y) \circ z = x \circ (y \circ z)$;
(IP_2) $x \in x \circ E \cap E \circ x$;
(IP_3) $x \in y \circ z$ *implies* $y \in x \circ z^{-J}$ *and* $z \in y^{-J} \circ x$.

If instead of IP_1, *a multivalued system* $< P, \circ, E, ^{-J} >$ *satisfies the weaker condition*

(IP'_1) $(x \circ y) \circ z \cap x \circ (y \circ z) \neq \emptyset$,

it is called a weak E*-polygroup.*

Obviously, any weak E-polygroup is an E-polygroup.

Example 3.2. *Consider* $\mathbb{P} = \{e, e^{'}, 1, 2, 3\}$, $E = \{e, e^{'}\}$ *and define* $\circ_2$ *on* $\mathbb{P}$ *with help of the following table;*

$\circ_2$	e	$e^{'}$	1	2	3
e	e	$e^{'}$	1	2	3
$e^{'}$	e	$e^{'}$	1	2	3
1	1	1	1	$\{e,1,2,3\}$	3
2	2	2	$\{e,1,2\}$	2	$\{2,3\}$
3	3	3	$\{1,3\}$	3	$\{e,1,2,3\}$

Then $\langle \mathbb{P}, \circ_2, E, ^{-J} \rangle$, *where* $a^{-J} = a$, *for every* $a \in \mathbb{P}$, *is a polygroup.*

Proposition 3.3. *For every element of* x *of any weak* E*-polygroup* $< P, \circ, E, ^{-J} >$, *one has* $x \circ x^{-J} \cap E \neq \emptyset$ *and* $x^{-J} \circ x \cap E \neq \emptyset$.

Proof. By IP_2, for every $x \in P$ there exists some $e \in E$ such that $x \in e \circ x$, and by IP_3, we have $e \in x \circ x^{-J}$. Thus, $e \in (x \circ x^{-J}) \cap E \neq \emptyset$. Likewise, one can prove $(x^{-J} \circ x) \cap E \neq \emptyset$. □

Example 3.4. *Naturally, every polygroup $\langle P, \circ, e, ^{-1} \rangle$ is an E-polygroup $\langle P, \circ, E, ^{-1} \rangle$ by setting $E = \{e\}$. More generally, one can let E be any subset of P containing e.*

Example 3.5. *Assume that $(G, \cdot)$ is an abelian group with the identity e. Let E be any subset of G with more than one element such that $e \in E$ and E is closed under inversion. We define the hyperoperation $\odot$ as follows,*

$$x \odot y = \begin{cases} x \cdot y \cdot E, & if\ x \neq e,\ y \neq e \\ E, & if\ x = e \text{ or } y = e \end{cases}$$

The multivalued system $< G, \odot, E, ^{-J} >$ is a E-polygroup, where $-J$ denotes the inverse operation of $(G, \cdot)$.

Example 3.6. *A projective geometry is an incidence system (P, L, R) consisting of a set of points P, a set of lines L and incidence relation $R \subseteq P \times L$ satisfying the following axiom:*

1. *Any line contains at least three points;*
2. *Two distinct points a, b are contained in a unique line denoted by $L(a, b)$;*
3. *If a, b, c, d are distinct points and $L(a, b)$ intersects $L(c, d)$, then $L(a, c)$ must intersect $L(b, d)$ (Pasch axiom).*

Let $P' = P \cup E$, where E is a set with $P \cap E = \phi$ and define

For $a \neq b \in P$, $a \circ b = L(a, b) \setminus \{a, b\}$;
For $a \in P$, if any line contains exactly three points, put $a \circ a = E$, otherwise $a \circ a = \{a\} \cup E$;
For $a \in P'$, $E \circ a = a \circ E = \{a\}$, where $a^{-J} = a$.

Then it is easily verified that $\langle P', \circ, E, ^{-J} \rangle$ is a weak E-polygroup.

Definition 3.7. *Let $< P, \circ, E, ^{-J} >$ be an E-polygroup (a weak E-polygroup) and $E \subseteq K \subseteq P$. We say that a subset K of P is an E-subpolygroup (a weak E-subpolygroup), if the multivalued system $< K, \circ, E, ^{-J} >$ be an E-polygroup (a weak E-polygroup).*

Lemma 3.8. *Let $< P, \circ, E, ^{-J} >$ be an E-polygroup (weak E-polygroup) and $E \subseteq K \subseteq P$. Then K is an E-subpolygroup (weak E-subpolygroup) of P if and only if (i) $x \circ y \subseteq K$ for any $x, y \in K$, (ii) $x^{-J} \in K$ for any $x \in K$.*

Proof. It is straightforward. □

Definition 3.9. *An E-subpolygroup N of an E-polygroup P is E-normal in P if and only if $a^{-J} N a \subseteq N$, for every $a \in P$.*

It is easy to prove the following corollaries.

Corollary 3.10. *Let N be an E-normal in P. Then*

1. $Na = aN$, *for all $a \in P$;*
2. $(Na)(Nb) = Nab$, *for all $a, b \in P$;*
3. $Na = Nb$, *for all $b \in Na$.*

Corollary 3.11. *Let K and N be two E-subpolygroups in P with N E-normal in P. Then*
$N \cap K$ is an E-normal in K;
$NK = KN$ is an E-subpolygroup of P;
N is an E-normal in NK.

Definition 3.12. *If N is an E-normal in P, then we define the relation $x \equiv y \pmod N$ if and only if $xy^{-J} \cap N \neq \emptyset$. This relation is denoted by $xN_P y$.*

Lemma 3.13. *The relation N_P is an equivalence relation on week E-polygroups.*

Proof. Since $E \subseteq x \cdot x^{-J} \cap N$ for all $x \in P$; then $xN_P x$, i.e., N_P is reflexive. The proof of symmetric and transitivity of the relation of N_p is similar to Lemma 2.4. in [**?**]. □

Example 3.14. *Let $(G, \cdot)$ be a group and θ an equivalence on G. Let $\theta[x]$ be the equivalence class of the element $x \in G$. Suppose that $E = \theta[e]$ and $x^{-J} = x^{-1}$. It is not difficult to see that the hyperstructure $< G, \odot, E, ^{-J} >$ is a weak E-polygroup, when the hyperoperation $\odot$ is defined as follows:*

$$\odot : H \times H \to \mathcal{P}^*(P), \quad \text{by} \quad x \odot y = \{\theta[z] | z \in \theta[x] \cdot \theta[y]\}.$$

Proposition 3.15. *Let $(G, \cdot)$ be a group and θ an equivalence on G such that*

(i) *$x\theta y$ and $y \in E$ implies $x \in E$,*
(ii) *$x\theta y$ implies $x^{-J}\theta y^{-J}$.*

Let $\theta(x)$ be an equivalence class of the element $x \in G$. If $G/\theta = \{\theta(x) | x \in G\}$, then $< G/\theta, \odot, \theta(E), ^{-J} >$ is a weak E-polygroup, where the hyperoperation $\odot$ is defined as follows:

$$\odot : G/\theta \times G/\theta \to \mathcal{P}^*(G/\theta)$$
$$\theta(x) \odot \theta(y) = \{\theta(z) | z \in \theta(x) \cdot \theta(y)\},$$

and $\theta(x)^{-J} = \theta(x^{-J})$.

Proof. For all $x, y, z \in G$, we have $x \cdot (y \cdot z) \in \theta(x) \odot (\theta(y) \odot \theta(z))$ and $(x \cdot y) \cdot z \in (\theta(x) \odot \theta(y)) \odot \theta(z)$, therefore $\odot$ is week associative. It is easy to see that for any $\theta(x) \in G/\theta$, $\theta(x) \in (\theta(x) \odot \theta(E)) \cap (\theta(E) \odot \theta(x))$. Now, we show that $\theta(z) \in \theta(x) \odot \theta(y)$ implies $\theta(x) \in \theta(z) \odot \theta(y^{-J})$ and $\theta(y) \in \theta(x^{-J}) \odot \theta(z)$. Since $\theta(z) = \theta(a)$ for some $a \in \theta(x) \cdot \theta(y)$, so there exist $b \in \theta(x)$ and $c \in \theta(y)$ such that $a = b \cdot c$, so $b = a \cdot c^{-J}$ which implies that $\theta(b) = \theta(a \cdot c^{-J}) \in \theta(a) \odot \theta(c^{-J})$. Hence $\theta(x) \in \theta(z) \odot \theta(y^{-J})$. By the similar way, we obtain $\theta(y) \in \theta(x^{-J}) \odot \theta(z)$. Therefore $< G/\theta, \odot, \theta(E), ^{-J} >$ is a weak E-polygroup. □

Theorem 3.16. *Let $\mathcal{A} =< A, \cdot, E, ^{-J} >$ and $\mathcal{B} =< B, \cdot, E, ^{-J} >$ be two weak E-polygroups whose elements have been renamed so that $A \cap B = E$, where E is identity of $\mathcal{A}$ and $\mathcal{B}$. If $E \subseteq x \cdot x^{-1}$ for any $x \in P$. Then the new system $\mathcal{A}[\mathcal{B}] =< M, *, E, ^{-J} >$, which is called the extension of $\mathcal{A}$ by $\mathcal{B}$ is a weak E-polygroup.*

Proof. The second condition of Definition 3.1, is clear. It is enough to check the conditions (IP_1) and (IP_3) of Definition 3.1. Without loss of generally, we may assume $x, y, z \notin I$ and not elements belong to A. Note that

(1) If $u \in B$ and $v \in A$, then $u * v = v * u = u$.

If exactly one of x, y, z belongs to B, then (1) implies that both sides of (IP_1) equal the element in $\{x, y, z\} \cap B$. If exactly two of x, y, z belong to B, say u and v, then (1) implies that both sides of (IP_1) equal $u * v$. We assume that $x, y, z \in B - E$ and show that

(2) $u \in (x * y) * z$ implies $u \in x * (y * z)$.

If $u \notin A$, then $u \in w * z$ for some $w \in x * y$. Now, if $w \notin A$, $w \in x \cdot y$ and $u \in w \cdot z$ so $u \in (xy)z = x(yz)$ (in B) $\subseteq x * (y * z)$. Also, if $w \in A$, $u \in w * z = z$ (so $u = z$) and $E \subseteq xy$. Thus, $u = z \in (xy)z = x(yz) \subseteq x * (y * z)$. Now, suppose that $u \in A$. Then $z^{-J} \in x * y$, $z^{-J} \notin A$ so $z^{-J} \in xy$ (in B), so $E \subseteq (xy)z = x(yz)$. Thus, $x^{-J} \in yz \subseteq y * z$ and hence $u \in A \subseteq x * (y * z)$.

The proof of the opposite inclusion is similar to (2).

The condition (IP_3) is clear if $x, y, z \in A$. Since $x \in B - E$ implies y or z belongs to $B - E$ and $x \in A$ implies $z \in B - E$, we may assume at least two of x, y, z belong to $B - E$. On the other hand, if $x, y, z \in B - E$, then $x \in y * z$ implies $x \in y \cdot z$ (in B) from which (IP_3) follows. Therefore, we may assume exactly two of x, y, z belong to $B - I$. This reduced to two cases:

(3) $x \in y * z$, where $x, y \in B - E$ and $z \in A$.

By (1), $y * z = y$ so $x = y$, thus $y = x = x * z^{-J}$ using (1) again and $z \in A \subseteq x^{-J} * x = y^{-J} * x$.

(4) $x \in y * z$ where $x \in A$ and $y, z \in B - E$.

In this case $y = z^{-J}$ so the desired conclusion follows using (1). This completes the proof of (IP_3) and hence the theorem. □

The equivalence relation θ on a weak E-polygroup P is called a *full conjugation* on P if

1. $x\theta y$ implies $x^{-J}\theta y^{-J}$,
2. $z \in x \cdot y$ and $z_1 \theta z$ imply $z_1 \in x_1 \cdot y_1$ for some x_1 and y_1, where $\theta(x_1) = \theta(x)$ and $\theta(y_1) = \theta(y)$.

The collection of all θ-classes, with the induced operation from P, forms a weak E-polygroup.

Corollary 3.17. *Let $\mathcal{M}$ be a weak E-polygroup, then θ is full conjugation on P if and only if (i) $(\theta(x))^{-J} = \theta(x^{-J})$; (ii) $\theta(\theta(x)y) = \theta(x)\theta(y)$.*

Definition 3.18. *Let $\mathcal{A} =< A, \cdot, E_1, ^{-J} >$ and $\mathcal{B} =< B, *, E_2, ^{-J} >$ be two weak E_1-polygroup and weak E_2-polygroup. Let f be a mapping from*

A into B, such that $f(E_1) = E_2$. Then f is called a strong homomorphism, if $f(x \cdot y) = f(x) * f(y)$, for all $x, y \in A$.

We recall the following definition from [5].

Definition 3.19. *If A is a weak E-subpolygroup of a weak E-polygroup P, then the relation $a \equiv b \ (\mathrm{mod} A)$ if and only if there exists a set $\{c_0, c_1, \ldots, c_{k+1}\} \subseteq P$, where $c_0 = a,\ c_{k+1} = b$ such that*

$$a \cdot c_1^{-J} \cap A \neq \emptyset,\ c_1 \cdot c_2^{-J} \cap A \neq \emptyset, \ldots, c_k \cdot b^{-J} \cap A \neq \emptyset.$$

This relation is defined by aA_P^*b.

Theorem 3.20. *The relation A_P^* is an equivalence relation on weak E-polygroups.*

Proof. 1. Since for any $x \in A$, there exists $e \in E$ such that $e \in x \cdot x^{-1} \cap A$, then aA_P^*a, i.e., A_P^* is reflexive.

2. Suppose that aA_P^*b, then there exists $\{c_0, c_1, \ldots, c_{k+1}\} \subseteq P$, where $c_0 = a,\ c_{k+1} = b$ such that

$$a \cdot c_1^{-J} \cap A \neq \emptyset,\ c_1 \cdot c_2^{-J} \cap A \neq \emptyset, \ldots, c_k \cdot b^{-J} \cap A \neq \emptyset.$$

Therefore, there exists $x_i \in c_i \cdot c_{i+1}^{-1} \cap A$ $(i = 1, \ldots, k)$ which implies $x_i^{-1} \in c_{i+1} \cdot c_i^{-1}$ and $x_i^{-1} \in A$, this means that bA_P^*a.

3. Let aA_P^*b and bA_P^*c, where $a, b, c \in P$. Then, there exist $\{c_0, c_1, \ldots, c_{k+1}\} \subseteq P$ and $\{d_0, d_1, \ldots, d_{k+1}\} \subseteq P$, where $c_0 = a, c_{k+1} = b = d_0, d_{r+1} = c$ such that

$$a \cdot c_1^{-J} \cap A \neq \emptyset,\ c_1 \cdot c_2^{-J} \cap A \neq \emptyset, \ldots, c_k \cdot b^{-J} \cap A \neq \emptyset,$$

$$b \cdot d_1^{-J} \cap A \neq \emptyset,\ d_1 \cdot d_2^{-J} \cap A \neq \emptyset, \ldots, d_r \cdot c^{-J} \cap A \neq \emptyset.$$

We take $\{c_0, c_1, \ldots, c_{k+1}, d_1, d_2, \ldots, d_{r+1}\} \subseteq P$ which satisfies the condition for aA_P^*c.

□

We denote $A_P^*[x]$ the equivalence class with representative x.

Theorem 3.21. *Let P be a weak E-polygroup. If A is a weak E-subpolygroup of P, then on the set $[P : A] = \{A_P^*[a] | a \in P\}$ we define the hyperoperation $\odot$ as follows:*

$$A_P^*[a] \odot A_P^*[b] = \{A_P^*[c] | c \in A_P^*[a] \cdot A_P^*[b]\},$$

what gives the weak E-polygroup $< [P : A], \odot, A_P^*[E], ^{-J} >$, where $A_P^*[a]^{-J} = A_P^*[a^{-J}]$.

Proof. We show that $A_P^*[a] \in A_P^*[a] \odot A_P^*[e]$ for any $e \in I$. Since $x \in x \cdot e \subseteq A_P^*[x] \cdot A_P^*[e] \subseteq A_P^*[x] \cdot A_P^*[E]$, for some $e \in E$. Then $A_P^*[e] \in A_P^*[x] \odot A_P^*[e]$. So, $A_P^*[x] \in A_P^*[x] \odot A_P^*[E]$.

The proof of the conditions (i) and (iii) of Definition 3.1 of $\odot$ is similar to Theorem 2.5. in [5]. □

If A is a weak E-polygroup of P, then the weak E-polygroup $[P : A]$ is called the *quotient* weak E-subpolygroup of P by A.

Corollary 3.22. *Let ρ be strong homomorphism from a weak E_1-polygroup P_1 into a weak E_2-polygroup P_2. Then the following propositions hold:*

(i) *For all $a \in P_1$, $\rho(a^{-J}) = \rho(a)^{-J}$;*
(ii) *The* ker *of ρ is a weak E-subpolygroup of P_1;*
(iii) *Let A be a weak E-subpolygroup of P_1. The image $\rho(A) = \{\rho(x)|x \in A\}$ is a weak E-subpolygroup of P_2, the inverse image $\rho^{-1}(B) = \{x|x \in P_1,\ \rho(x) \in B\}$ is a weak E-subpolygroup of P_1.*

Theorem 3.23. *(Fundamental Homomorphism Theorem). Let P_1 and P_2 be two weak E_1-polygroup and E_2-weak polygroup and ρ be a strong homomorphism from P_1 onto P_2 with kernel K. Then $[P_1 : K] \cong P_2$.*

Proof. We prove that $\phi : [P_1 : K] \longrightarrow P_2$ by $\phi(K^*_{P_1}[x]) = \rho(x)$ for any $x \in P_1$ is well defined. If $K^*_{P_1}[x] = K^*_{P_1}[y]$, then there exists $\{z_0, z_1, \ldots, z_{k+1}\} \subseteq P_1$, where $z_0 = x$, $z_{k+1} = y$ such that

$$x \cdot z_1^{-J} \cap K \neq \emptyset,\ z_1 \cdot z_2^{-J} \cap K \neq \emptyset, \ldots, z_k \cdot y^{-J} \cap K \neq \emptyset.$$

Thus, there exist $e_1, \ldots, e_{k+1} \in E_1$ such that

$$e_1 \in \rho(x \cdot z_1^{-J}),\ e_2 \in \rho(z_1 \cdot z_2^{-J}),\ \ldots,\ e_{k+1} \in \rho(z_k \cdot y^{-J}) \text{ or } e_1 \in \rho(x) * \rho(z_1)^{-J},$$
$$e_2 \in \rho(z_1) * \rho(z_2)^{-J}, \ldots,\ e_{k+1} \in \rho(z_k) * \rho(y)^{-J}$$

and so $\rho(x) = \rho(y)$.

It is obvious that ϕ is homomorphism. Also, $\phi(K) = \phi(K^*_{P_1}[E_1]) = \rho(E_1) = E_2$.

Furthermore, if $\phi(K^*_{P_1}[x]) = \phi(K^*_{P_1}[y])$, then $\rho(x) = \rho(y)$ which implies that $x \in y^{-J} \cap K \neq \emptyset$ and so $K^*_{P_1}[x] = K^*_{P_1}[y]$. Thus, ϕ is a one to one mapping. □

4. Some results for E-polygroups

Let P_1 and P_2 be a weak E_1-polygroup and a weak E_2-polygroup, respectively. We recall that a strong homomorphism $\phi : P_1 \longrightarrow P_2$ is an isomorphism if ϕ is one to one and onto. We write $P_1 \cong P_2$ if P_1 is isomorphic to P_2.

Let P_1 be an E-polygroup, $E \subseteq a \cdot a^{-J}$ for all $a \in P_1$, then we have $\phi(E) \subseteq \phi(a) * (\phi a^{-J})$ or $E_2 \subseteq \phi(a) * \phi(a^{-J})$ which implies $\phi(a^{-J}) \in \phi(a)^{-J} * e_2$, for some $e_2 \in E_2$, therefore $\phi(a_1) = \phi(a)^{-J}$ for all $a \in P_1$. Moreover, if ϕ is a strong homomorhism from P_1 into P_2, then the kernel of ϕ is the set $\ker\phi = \{x \in P_1 | \phi(x) \in E_2\}$. It is trivial that $\ker\phi$ is a E-subpolygroup of P_1 but in general is not normal in P_1.

Lemma 4.1. *Let ϕ be a strong homomorphism from P_1 into P_2. Then $\phi(y) = \phi(z)$ implies $y = zE_1$ if and only if* $\ker\phi = E_1$.

Proof. Let $y, z \in P_1$ be such that $\phi(y) = \phi(z)$. Then $\phi(y) * \phi(y^{-J}) = \phi(z) * \phi(y^{-J})$. It follows that $\phi(e_1) \in \phi(yy^{-J}) = \phi(zy^{-J})$, for some $e_1 \in E$ and so there exists $x \in yz^{-J}$ such that $e_2 = \phi(e_1) = \phi(x)$. Thus, if $ker\phi = E_1$, then $x \in E_1$, whence $y = zE$. Now, let $x \in ker\phi$. Then $\phi(x) = e \in \phi(E_1)$. So, $x \in E_1$. □

We call the homomorphism $\phi : P_1 \longrightarrow P_2$ is a weak isomorphism if $\ker \phi = E_1$ and ϕ is onto. If ϕ is a weak isomorphism we say P_1 is weak isomorphic with P_2 and denoted by $P_1 \cong_W P_2$.

Theorem 4.2. *(First Isomorphism Theorem). Let ϕ be a strong homomorphism from P_1 into P_2 with kernel K such that K is a normal E-subpolygroup of P_1, then $P_1/K \cong_W Im\phi$.*

Proof. We define $\psi : P_1/K \cong_W Im\phi$ by setting $\psi(Kx) = \phi(x)$ for all $x \in P_1$. It is easy to see that ψ is a weak isomorphism. □

Theorem 4.3. *(Second Isomorphism Theorem). If K and N are E-subpoly groups of an E-polygroup P, with N E-normal in P, then $K/N \cap K \cong_W NK/N$.*

Proof. Since N is an E-subnormal of P, we have $NK = KN$. Consequently, NK is an E-subpolygroup of P. Further $N = NE \subseteq NK$ given that N is an E-subnormal of NK; so NK/N is defined. Define $\phi : K \longrightarrow NK/N$ by $\phi(k) = Nk$, which is a strong homomorphism. Consider any $Na \in NK/N, a \in NK$. Now $a \in NK$ given $a \in nk$ for some $n \in \mathbb{N}$, $k \in K$. Thus, by Lemma 2.10, $Na = Nnk = Nk = \phi(k)$. This shows that ϕ is also onto. If we can establish that $\ker \phi = N \cap K$, since $N \cap K$ is an E-subnormal of K, we shall get that $K/N \cap K \cong NK/N$. For any $k \in K$,

$$k \in \ker \phi \Leftrightarrow \phi(k) = N \Leftrightarrow Nk = N \Leftrightarrow k \in N \Leftrightarrow k \in N \cap K.$$

That is, $k \in \ker \phi \Leftrightarrow k \in N \cap K$. This yields $\ker \phi = N \cap K$. Hence the results follows. □

Theorem 4.4. *(Third Isomorphism Theorem). If K and N are two E-subnormals of an E-polygroup P such that $N \subseteq K$, then K/N is an E-subnormal of P/N and $(P/N)/(K/N) \cong_W P/K$.*

Proof. We leave it to reader to verify that K/N is an E-subnormal of P/N. Furthermore, $\phi : P/N \Rightarrow P/K$ defined by $\phi(Nx) = Kx$ is a strong homomorphism of P/N onto P/K such that $\ker \phi = K/N$. □

Corollary 4.5. *If N_1, N_2 are two E-subnormals of P_1, P_2 respectively, then $N_1 \times N_2$ is an E-subnormal of $P_1 \times P_2$ and $(P_1 \times P_2)/(N_1 \times N_2) \cong_W P_1/N_1 \times P_2/N_2$.*

Let $< P, \circ, E, ^{-J} >$ be a weak E-polygroup. We can define the relation β^* as the smallest equivalence relation on P such that quotient P/β^* is a group.

One can prove that the fundamental relation β^* is the transitive closure of the relation β.

The kernel of canonical map $\phi : P \to P/\beta^*$ is denoted by ω_P. It is easy to prove that the following statements

(i) $\omega_P = \beta^*(E)$;

(ii) $\beta^*(x)^{-J} = \beta^*(x^{-J})$, for all $x \in P$.

Let $M_1 =< P_1, \cdot, E_1, ^{J_1} >$ and $M_2 =< P_2, *, E_2, ^{J_2} >$ be two weak E-polygroups, then on $P_1 \times P_2$ we can define a hyperproduct similar to the hyperproduct of weak E-polygroups as follows: $(x_1, y_1) \circ (x_2, y_2) = (a, b)|a \in x_1 \cdot x_2, b \in y_1 * y_2$. We call this the direct product of P_1 and P_2. It is easy to see that $P_1 \times P_2$ equipped with the usual direct product operation becomes a weak I-polygroup.

Theorem 4.6. *Let β_1^*, β_2^* and β^* be the fundamental equivalence relations on P_1, P_2 and $P_1 \times P_2$ respectively. Then $(P1 \times P_2)/\beta^* \cong_W P_1/\beta_1^* \times P_2/\beta_2^*$.*

Proof. The proof is similar to the proof of Theorem 2.4. in [5]. □

Similar to polygroups and weak polygroups and using the fundamental equivalence relation, we can define semidirect hyperproduct of weak E-polygroups. Let $\mathcal{A} =< A, \cdot, E_1, ^{J_1} >$ and $\mathcal{B} =< B, *, E_2, ^{J_2} >$ be two weak E-polygroups. Consider the group $\mathrm{Aut}(A)$ and the fundamental group B/β_B^*, let

$$\widehat{\ }: B/\beta_B^* \to \mathrm{Aut}(A),$$

$$\beta^*(b) \mapsto \widehat{\beta^*(b)} = \widehat{b}$$

be a homomorphism of groups. Then on $\mathcal{A} \times \mathcal{B}$ we define a hyperproduct of as follows:

$$(a_1, b_1) \odot (a_2, b_2) = (x, y)|x \in a_1 \cdot \widehat{b_1}(a_2), y \in b_1 * b_2.$$

Theorem 4.7. *$\mathcal{A} \times \mathcal{B}$ equipped with the semidirect hyperproduct is a weak E-polygroup.*

Proof. The proof is similar to the proof of Theorem 2.6. in [5]. □

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