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STATISTICALLY ω-FRÉCHET AND STATISTICALLY ω-SEQUENTIAL SPACES: STATISTICAL CONVERGENCES OF ω-NETS

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Abstract. In this paper, we introduce some modifications of statistically Fréchet and statistically sequential spaces which were defined in [5], termed statistically ω-Fréchet and statistically ω-sequential spaces concerned with ω-nets. Our work includes the construction of an illustrative example that distinguishes these two properties. Furthermore, we establish general relationships between statistically ω-Fréchet and statistically ω-sequential properties, demonstrating their equivalence in the class of statistically ω-transitive spaces.

1. Introduction

Fast ([1]) introduced the notion of statistical convergences of sequences of real numbers in 1951, and Kostyrko et al ([4]) extended the statistical convergence to sequences in metric spaces. In 2008, Maio and Kočinac ([5]) investigated the statistical convergence in topological and uniform spaces.

Throughout this paper, the cardinality of a given set A is denoted by $|A|$. For any $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, denote $A(n) = \{k \in A : k \leq n\}$. Then we say that

$$\underline{\delta}(A) = \liminf_{n\to\infty} \frac{|A(n)|}{n}$$

and

$$\overline{\delta}(A) = \limsup_{n\to\infty} \frac{|A(n)|}{n}$$

are the *lower* and *upper asymptotic density* of A respectively. When $\underline{\delta}(A) = \overline{\delta}(A)$,

$$\delta(A) = \lim_{n\to\infty} \frac{|A(n)|}{n}$$

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is called the *asymptotic density* of A. One can prove easily that all the three densities, if they exist, are in $[0, 1]$, and $\delta(\mathbb{N} \setminus A) = 1 - \delta(A)$ for any $A \subseteq \mathbb{N}$.

Definition 1.1 ([1])**.** A sequence $\langle x_n : n \in \mathbb{N}\rangle$ in a topological space X is said to *converge statistically* (or *s-converge*) to $p \in X$, if for every neighborhood U of p, $\delta(A) = 0$, where $A = \{n \in \mathbb{N} : x_n \notin U\}$.

Definition 1.2 ([5])**.** A sequence $\langle x_n : n \in \mathbb{N}\rangle$ in a topological space X is said to *s^*-converge* to $p \in X$ if there exists a subset $A \subseteq \mathbb{N}$ with $\delta(A) = 1$ such that the sequence $\langle x_n : n \in A\rangle$ converges to p, that is, for every neighborhood U of p there is $n_0 \in \mathbb{N}$ such that $n \geq n_0$ and $n \in A$ imply $x_n \in U$.

It was shown in [2] that s-convergence and s^*-convergence are equivalent when $X = \mathbb{R}$. Furthermore, Maio and Kočinac proved in [5] that they are equivalent in the class of first countable spaces.

In Section 2, we introduce two concepts pertaining to the statistical convergence of ω-nets: s-convergence and s^*-convergence. We investigate their general relationships, establishing their equivalence in the class of first countable spaces.

Section 3 is dedicated to the study of statistically ω-Fréchet spaces and statistically ω-sequential spaces. Specifically, we observe that every statistically ω-Fréchet space is statistically ω-sequential; however, we establish that the converse is not true. Additionally, we demonstrate that every statistically ω-Fréchet space is statistically Fréchet. We explore some characterization of the statistically ω-sequential property. Finally, we prove an equivalence of the statistically ω-Fréchet property and statistically ω-sequentiality in the class of statistically ω-transitive spaces.

2. Statistical Convergence of ω-Nets

In this section, we introduce the notion of statistical convergence of ω-nets in topological spaces. All spaces are assumed to be Hausdorff, ω is the first countably infinite ordinal, and $[\omega]^{<\omega}$ is the collection of all finite subsets of ω.

A subfamily $\mathcal{F} = \{F_k : k \in \mathbb{N}\}$ of $[\omega]^{<\omega}$ is said to be *strictly increasing* if $F_1 = \emptyset$; $F_k \subsetneq F_{k+1}$ for each $k \in \mathbb{N}$; and $|F_{k+1}| = |F_k| + 1$ for each $k \in \mathbb{N}$.

For a strictly increasing subfamily $\mathcal{F} = \{F_k : k \in \mathbb{N}\}$ of $[\omega]^{<\omega}$, let $\mathcal{A} \subseteq \mathcal{F}$. For each $k \in \mathbb{N}$ define $\mathcal{A}_{\mathcal{F}}(k) = \{F_n \in \mathcal{A} : F_n \subseteq F_k\}$. Note that $n \leq k$ if and only if $F_n \subseteq F_k$.

Remark 2.1. We may take $\mathcal{A}$ as an arbitrary subfamily of $[\omega]^{<\omega}$. In this case, we define $\mathcal{A}_{\mathcal{F}}(k) = \{F_n \in \mathcal{A} \cap \mathcal{F} : F_n \subseteq F_k\}$.

Now we say that

$$\underline{\delta}(\mathcal{A}_{\mathcal{F}}) = \liminf_{k\to\infty} \frac{|\mathcal{A}_{\mathcal{F}}(k)|}{k}$$

and

$$\overline{\delta}(\mathcal{A}_{\mathcal{F}}) = \limsup_{k\to\infty} \frac{|\mathcal{A}_{\mathcal{F}}(k)|}{k}$$

are the *lower* and *upper asymptotic density* of $\mathcal{A}$ over $\mathcal{F}$ respectively. When $\underline{\delta}(\mathcal{A}_{\mathcal{F}}) = \overline{\delta}(\mathcal{A}_{\mathcal{F}})$,

$$\delta(\mathcal{A}_{\mathcal{F}}) = \lim_{k\to\infty} \frac{|\mathcal{A}_{\mathcal{F}}(k)|}{k}$$

is called the *asymptotic density* of $\mathcal{A}$ over $\mathcal{F}$.

All the three densities, if they exist, are in $[0,1]$. Moreover, we have the following:

Lemma 2.2. *Let $\mathcal{F} = \{F_k : k \in \mathbb{N}\}$ be a strictly increasing subfamily of $[\omega]^{<\omega}$ and let $\mathcal{A}$ be a subfamily of $\mathcal{F}$. Assume that $\delta(\mathcal{A}_{\mathcal{F}})$ exists. Then $\delta(\mathcal{B}_{\mathcal{F}}) = 1 - \delta(\mathcal{A}_{\mathcal{F}})$ where $\mathcal{B} = \mathcal{F} \setminus \mathcal{A}$.*

Proof. Suppose $\delta(\mathcal{A}_{\mathcal{F}}) = c \in [0,1]$. Then for any $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that $k \geq k_0$ implies

$$c - \epsilon < \frac{|\mathcal{A}_{\mathcal{F}}(k)|}{k} < c + \epsilon.$$

Since $k = |\mathcal{A}_{\mathcal{F}}(k)| + |\mathcal{B}_{\mathcal{F}}(k)|$, we have

$$1 - \frac{|\mathcal{A}_{\mathcal{F}}(k)|}{k} = \frac{|\mathcal{B}_{\mathcal{F}}(k)|}{k}.$$

Hence

$$1 - c - \epsilon < \frac{|\mathcal{B}_{\mathcal{F}}(k)|}{k} < 1 - c + \epsilon.$$

Therefore $\delta(\mathcal{B}_{\mathcal{F}}) = 1 - \delta(\mathcal{A}_{\mathcal{F}})$. □

Let X be a topological space. An *ω-net* in X is a function $\xi : [\omega]^{<\omega} \to X$ such that $[\omega]^{<\omega}$ is directed by $\subseteq$. The ω-net ξ is usually denoted by $\langle x_F : F \in [\omega]^{<\omega}\rangle$, or $\langle x_F\rangle$ where $x_F = \xi(F)$ for all $F \in [\omega]^{<\omega}$.

We say that an ω-net $\langle x_F\rangle$ in a space X *converges* to $p \in X$ ([3]) if for any open neighborhood V of p in X, there exists $F \in [\omega]^{<\omega}$ such that

$$x_G \in V \text{ for all } G \in [\omega]^{<\omega} \text{ with } G \supseteq F.$$

Now we introduce some concepts of statistical convergences of ω-nets as follows:

Definition 2.3. An ω-net $\langle x_F\rangle$ in a space X is said to *converge statistically* (or, *s-converge*) to $p \in X$ if for every open neighborhood U of p and for every strictly increasing subfamily $\mathcal{F} = \{F_k : k \in \mathbb{N}\}$ of $[\omega]^{<\omega}$,

$$\delta(\mathcal{A}_{\mathcal{F}}) = 0$$

where $\mathcal{A} = \{F_k \in \mathcal{F} : x_{F_k} \notin U\}$.

Note that the limit of an s-convergent ω-net is uniquely determined in Hausdorff spaces.

Definition 2.4. An ω-net $\langle x_F \rangle$ in a space X is said to *s^*-converge* to $p \in X$ if for any strictly increasing subfamily $\mathcal{F} = \{F_k : k \in \mathbb{N}\}$ of $[\omega]^{<\omega}$, there exists a subfamily $\mathcal{A}$ of $\mathcal{F}$ satisfying the following:

- $\delta(\mathcal{A}_{\mathcal{F}}) = 1$;
- for every open neighborhood U of p there exists $F_{k_0} \in \mathcal{F}$ such that $F_k \supseteq F_{k_0}$ and $F_k \in \mathcal{A}$ imply $x_{F_k} \in U$.

Theorem 2.5. *If an ω-net $\langle x_F \rangle$ in a space X s^*-converges to p, then $\langle x_F \rangle$ s-converges to p.*

Proof. Let U be an open neighborhood of p and let $\mathcal{F} = \{F_k : k \in \mathbb{N}\}$ be a strictly increasing subfamily of $[\omega]^{<\omega}$. Since $\langle x_F \rangle$ s^*-converges to p, there exists a subfamily $\mathcal{A}$ of $\mathcal{F}$ satisfying the following:

- $\delta(\mathcal{A}_{\mathcal{F}}) = 1$;
- for every open neighborhood U of p there exists $F_{k_0} \in \mathcal{F}$ such that $F_k \supseteq F_{k_0}$ and $F_k \in \mathcal{A}$ imply $x_{F_k} \in U$.

Denote $\mathcal{B} = \{F_k \in \mathcal{F} : x_{F_k} \notin U\}$ and $\mathcal{C} = \{F_1, F_2, \cdots, F_{k_0}\} \cup (\mathcal{F} \setminus \mathcal{A})$. Then $\mathcal{B}_{\mathcal{F}}(k) \subseteq \mathcal{C}_{\mathcal{F}}(k)$ for each $k \in \mathbb{N}$. It is easy to prove that $\delta(\mathcal{C}_{\mathcal{F}}) = 0$ by using Lemma 2.2, and hence $\delta(\mathcal{B}_{\mathcal{F}}) = 0$. Therefore the ω-net $\langle x_F \rangle$ s-converges to p. □

Theorem 2.6. *Let X be a first countable space. Then the converse of Theorem 2.5 holds.*

Proof. We assume that an ω-net $\langle x_F \rangle$ in X s-converges to $p \in X$. Let $\mathcal{F} = \{F_k : k \in \mathbb{N}\}$ be a strictly increasing subfamily of $[\omega]^{<\omega}$ and take a countable decreasing local base $\{U_i : i \in \mathbb{N}\}$ of p in X. For each $i \in \mathbb{N}$, we denote

$$\mathcal{A}^i = \{F_k \in \mathcal{F} : x_{F_k} \in U_i\}.$$

Then it is clear that $\mathcal{A}^1 \supseteq \mathcal{A}^2 \supseteq \mathcal{A}^3 \supseteq \cdots$ and $\delta({\mathcal{A}^i}_{\mathcal{F}}) = 1$ for each $i \in \mathbb{N}$. Choose $F_{k_1} \in \mathcal{A}^1$. Then there exists $F_{k_2} \in \mathcal{A}^2$ such that

- $F_{k_2} \supsetneq F_{k_1}$;
- for every $F_n \in \mathcal{F}$ such that $F_n \supsetneq F_{k_2}$, the following holds:

$$\frac{|{\mathcal{A}^2}_{\mathcal{F}}(n)|}{n} = \frac{|\{F_m \in \mathcal{A}^2 : m \leq n\}|}{n} > \frac{1}{2}.$$

By induction, we obtain the family $\{F_{k_i} : i \in \mathbb{N}\}$ such that

- $F_{k_{i+1}} \supsetneq F_{k_i}$ for each $i \in \mathbb{N}$;
- $F_{k_i} \in \mathcal{A}^i$ for each $i \in \mathbb{N}$;

- for every $F_n \in [\omega]^{<\omega}$ such that $F_n \supsetneq F_{k_i}$, the following holds:
$$\frac{|\mathcal{A}^i{}_{\mathcal{F}}(n)|}{n} = \frac{|\{F_m \in \mathcal{A}^i : m \le n\}|}{n} > 1 - \frac{1}{i}.$$

Now we define a subfamily $\mathcal{A}$ of $\mathcal{F}$ as following:

- for each $F_n \subseteq F_{k_1}$, $F_n \in \mathcal{A}$;
- if $i \ge 1$ and $F_{k_i} \subsetneq F_n \subseteq F_{k_{i+1}}$, then
$$F_n \in \mathcal{A} \text{ if and only if } F_n \in \mathcal{A}^i.$$

Then if $n \in \mathbb{N}$ is such that $F_{k_i} \subseteq F_n \subseteq F_{k_{i+1}}$, then we have
$$\frac{|\mathcal{A}_{\mathcal{F}}(n)|}{n} \ge \frac{|\mathcal{A}^i{}_{\mathcal{F}}(n)|}{n} > 1 - \frac{1}{i},$$
and hence $\delta(\mathcal{A}_{\mathcal{F}}) = 1$.

Let U be an open neighborhood of p and let $U_i \subseteq U$. If $F_n \in \mathcal{A}$ with $F_n \supseteq F_{k_i}$, then there exists $k_j \ge k_i$ with $F_{k_j} \subseteq F_n \subseteq F_{k_{j+1}}$. Hence we have $x_{F_n} \in U_j \subseteq U_i \subseteq U$. Therefore the ω-net $\langle x_F \rangle$ s^*-converges to p. □

3. Statistically ω-Fréchet and Statistically ω-Sequential Spaces

In this section, we extend the definitions of κ-Fréchet and κ-sequential spaces using the framework of statistical convergence, particularly when $\kappa = \omega$. In [3], Hodel investigated the notions and properties of κ-Fréchet and κ-net spaces when κ is any infinite cardinal. The definitions are as follows: A space X is said to be *κ-Fréchet* if for every $p \in \overline{A}$, there exists a κ-net $\langle x_F : F \in [\kappa]^{<\omega}\rangle$ in A which converges to p. A space X is said to be a *κ-net space* if for any non-closed subset A of X, there exist a point $p \in \overline{A} \setminus A$ and a κ-net $\langle x_F : F \in [\kappa]^{<\omega}\rangle$ in A such that $\langle x_F \rangle$ converges to p. For convenience, we shall use the terminology "κ-sequential" instead of "κ-net space".

Maio and Kočinac in [5] introduced the following definitions and investigated some relevant properties. In fact, they used the term "Fréchet-Urysohn" or "FU". But we shall take the term " Fréchet" for simplicity.

Definition 3.1 ([5])**.** A space X is said to be *statistically Fréchet* (or for brevity, *s-Fréchet*) if for each $A \subseteq X$ and each $p \in \overline{A}$ there is a sequence in A s-converging to p.

Definition 3.2 ([5])**.** A space X is said to be *statistically sequential* (or for brevity, *s-sequential*) if for each non-closed $A \subseteq X$, there are a point $p \in \overline{A}$ and a sequence in A s-converging to p.

We introduce the following new concepts for statistical convergences of ω-nets from our observations.

Definition 3.3. A space X is said to be *statistically ω-Fréchet* (or, *sω-Fréchet*) if for every $p \in \overline{A}$, there exists an ω-net $\langle x_F \rangle$ in A which s-converges to p.

Definition 3.4. A space X is said to be *statistically ω-sequential* (or, *sω-sequential*) if for every non-closed subset A of X, there exist a point $p \in \overline{A} \setminus A$ and an ω-net $\langle x_F \rangle$ in A which s-converges to p.

By definitions, every $s\omega$-Fréchet space is $s\omega$-sequential. But the converse does not hold.

Example 3.5. There exists an $s\omega$-sequential space which is not $s\omega$-Fréchet.

Let $S = \{n^2 : n \text{ is a non-negative integer}\}$. For any $F \in [\omega]^{<\omega}$, let $\langle x_{F,G} : G \in [\omega]^{<\omega} \rangle$ be an ω-net in the Euclidean space $\mathbb{R}$ defined by

- if $|F| \notin S$, then $\langle x_{F,G} : G \in [\omega]^{<\omega} \rangle$ s-converges to $x_F = \frac{1}{|F|+1}$;
- if $|F| \in S$, then $\langle x_{F,G} : G \in [\omega]^{<\omega} \rangle$ s-converges to $x_F = |F| + 1$.

First, we show that the ω-net $\langle x_F \rangle$ s-converges to 0: Let $U = (-\epsilon, \epsilon)$ be an open interval for any positive real number $\epsilon < 1$ and let $\mathcal{F} = \{F_k : k \in \mathbb{N}\}$ be a strictly increasing subfamily of $[\omega]^{<\omega}$. Put $\mathcal{A} = \{F_k \in \mathcal{F} : x_{F_k} \notin U\}$. Then

$$\frac{|\mathcal{A}_{\mathcal{F}}(n)|}{n} = \frac{|\{F_k \in \mathcal{A} : k \le n\}|}{n} = \frac{|\{F_k \in \mathcal{F} : |F_k| \in S, k \le n\}|}{n} + \frac{|\{F_k \in \mathcal{F} : |F_k| \notin S, \frac{1}{|F_k|+1} \ge \epsilon, k \le n\}|}{n}.$$

So we obtain

$$\frac{|\{F_k \in \mathcal{F} : |F_k| \notin S, \frac{1}{|F_k|+1} \ge \epsilon, k \le n\}|}{n} \to 0$$

because the numerator can be regarded as a constant for sufficiently large n. We also have the following from the facts that $|F_n| = n - 1$ and $\delta(S) = 0$:

$$\frac{|\{F_k \in \mathcal{F} : |F_k| \in S, k \le n\}|}{n} = \frac{|\{l^2 \in S : l^2 \le |F_n|\}|}{n} < \frac{|\{l^2 \in S : l^2 \le |F_n|\}|}{|F_n|} \to 0,$$

and hence $\delta(\mathcal{A}_{\mathcal{F}}) = 0$. Thus the ω-net $\langle x_F \rangle$ s-converges to 0.

Now we define a topology on the set

$$X = \{x_{F,G} : F, G \in [\omega]^{<\omega}\} \cup \{x_F : F \in [\omega]^{<\omega}\} \cup \{0\}$$

as follows:

(1) each point $x_{F,G}$ is isolated;

(2) a basic open neighborhood of x_F is of the form

$$U_F = \{x_F\} \cup O_F$$

where $O_F = \{x_{F,G} : \delta(\{G \in \mathcal{G} : x_{F,G} \notin U_F\}_{\mathcal{G}}) = 0$ for every strictly increasing $\mathcal{G} \subseteq [\omega]^{<\omega}\}$;

(3) a basic open neighborhood of 0 is of the form

$$V = \{0\} \cup \{x_F : \delta(\{F \in \mathcal{F} : x_F \notin V\}_{\mathcal{F}}) = 0 \text{ for every strictly increasing } \mathcal{F} \subseteq [\omega]^{<\omega}\} \cup \bigcup\{O_F : x_F \in V\}$$

where $O_F = \{x_{F,G} : \delta(\{G \in \mathcal{G} : x_{F,G} \notin V\}_{\mathcal{G}}) = 0$ for every strictly increasing $\mathcal{G} \subseteq [\omega]^{<\omega}\}$.

It is clear that the space X is Hausdorff.

Claim 1: X is not $s\omega$-Fréchet.

Let $A = \{x_{F,G} : F, G \in [\omega]^{<\omega}\}$. Then $0 \in \overline{A}$. We show that no ω-net in A s-converges to 0. Suppose, by the contrary, that an ω-net $\langle y_H = x_{F_H,G_H} : H \in [\omega]^{<\omega}\rangle$ in A s-converges to 0. Fix an $F \in [\omega]^{<\omega}$. Let $A_F = \{x_{F,G} \in A : G \in [\omega]^{<\omega}\}$. Since $x_F \neq 0$, $\langle y_H\rangle$ does not s-converge to x_F. Without loss of generality, we can assume that there exists an ω-net ξ_F in $\{y_H : H \in [\omega]^{<\omega}\} \cap A_F$ such that ξ_F does not s-converge to x_F. Take a basic open neighborhood $U_F = \{x_F\} \cup O_F$ of x_F and a strictly increasing subfamily $\mathcal{G}_F$ of $[\omega]^{<\omega}$ such that

$$\delta(\mathcal{B}(F)_{\mathcal{G}_F}) \neq 0$$

where $\mathcal{B}(F) = \{G_H \in \mathcal{G}_F : \xi_F(H) = x_{F,G_H} \notin U_F\}$.

Since F is arbitrary, we can denote

$$O = \cup\{O_F : F \in [\omega]^{<\omega}\}$$

and

$$\mathcal{B}' = \cup\{\mathcal{B}(F) : F \in [\omega]^{<\omega}\}.$$

Then $\mathcal{B}' = \{G_H \in \cup\{\mathcal{G}_F : F \in [\omega]^{<\omega}\} : \xi_F(H) = x_{F,G_H} \notin O\}$ and $\delta(\mathcal{B}'_{\mathcal{G}_0}) \neq 0$ for some strictly increasing subfamily $\mathcal{G}_0$ of $[\omega]^{<\omega}$ (precisely, $\delta(\mathcal{B}'_{\mathcal{G}_F}) \neq 0$ for $\mathcal{G}_F$ chosen in the above).

Let $V = \{0\} \cup \{x_F : F \in [\omega]^{<\omega}\} \cup O$. Firstly, we show that V is a basic open neighborhood of 0. Let $\mathcal{F}$ be any strictly increasing subfamily of $[\omega]^{<\omega}$ and let $\mathcal{A} = \{F \in \mathcal{F} : x_F \notin V\}$. Since $x_F \in V$ for all $F \in [\omega]^{<\omega}$, $\mathcal{A} = \emptyset$. Hence $\delta(\mathcal{A}_{\mathcal{F}}) = 0$, that is, $\{x_F : F \in [\omega]^{<\omega}\} = \{x_F : \delta(\mathcal{A}_{\mathcal{F}}) = 0\}$. It is clear that $O = \cup\{O_F : x_F \in V\}$, where O_F is the set defined in (3). Thus V is a basic open neighborhood of 0.

But $\mathcal{B} = \{G_H \in \mathcal{G}_0 : y_H = x_{F_H,G_H} \notin V\} \supseteq \mathcal{B}'$ implies that $\delta(\mathcal{B}_{\mathcal{G}_0}) \geq \delta(\mathcal{B}'_{\mathcal{G}_0}) \neq 0$, which is a contraction. Therefore X is not $s\omega$-Fréchet.

<u>Claim 2: X is $s\omega$-sequential.</u>

Let B be a non-closed subset of X and let $C_F = \{x_{F,G} : G \in [\omega]^{<\omega}\} \cup \{x_F\}$ for each $F \in [\omega]^{<\omega}$.

Case 1: When $B \cap C_F$ is not closed in X for some $F \in [\omega]^{<\omega}$, $x_F \in \overline{B \cap C_F} \subseteq \overline{B} \setminus B$. Note that $B \cap C_F$ can be regarded as an ω-net $\langle y_H = x_{F,G_H} : H \in [\omega]^{<\omega}\rangle$ satisfying the following:

- $\langle y_H\rangle$ consists of all members of $B \cap C_F$; and
- $H \subseteq I$ if and only if $G_H \subseteq G_I$ for each $H, I \in [\omega]^{<\omega}$.

Then the ω-net $\langle y_H\rangle$ in B s-converges to x_F by the definition of basic open neighborhoods of x_F.

Case 2: When $B \cap C_F$ is closed in X for each $F \in [\omega]^{<\omega}$, we shall show that $B \cap \{x_F : F \in [\omega]^{<\omega}\} \neq \emptyset$. Suppose that $B \cap \{x_F : F \in [\omega]^{<\omega}\} = \emptyset$. Notice that $0 \in \overline{B} \setminus B$ since B is not closed in X. Also we obtain that, for each $F \in [\omega]^{<\omega}$, $C_F \setminus B$ is a non-empty open neighborhood of x_F. Then the set $U = \{0\} \cup \bigcup\{C_F \setminus B : F \in [\omega]^{<\omega}\}$ is an open neighborhood of 0. Since $B \cap U = \emptyset$, $0 \notin \overline{B}$. This is a contradiction. Hence $B \cap \{x_F : F \in [\omega]^{<\omega}\} \neq \emptyset$.

We now consider an ω-net $\langle y_H = x_{F_H} : H \in [\omega]^{<\omega}\rangle$ satisfying following:

- $\langle y_H\rangle$ consists of all members of $B \cap \{x_F : F \in [\omega]^{<\omega}\}$; and
- $H \subseteq I$ if and only if $F_H \subseteq F_I$ for each $H, I \in [\omega]^{<\omega}$.

Then the ω-net $\langle y_H\rangle$ in B s-converges to 0 by the definition of basic open neighborhoods of 0.

Therefore the space X is $s\omega$-sequential. □

The following is a special case of a theorem given in [3].

Theorem 3.6 ([3])**.** *Let X be a space and let $p \in X$. Given any sequence $\langle x_n : n \in \omega\rangle$ in X, there is an ω-net $\langle y_F\rangle$ in X such that the following hold:*

(1) *$\{y_F\} \subseteq \{x_n\}$;*
(2) *$\langle x_n\rangle$ converges to p if and only if $\langle y_F\rangle$ converges to p.*

We have a slightly different theorem.

Theorem 3.7. *Let X be a space and let $p \in X$. For any ω-net $\langle y_F\rangle$ in X, there exists a sequence $\langle x_n : n \in \mathbb{N}\rangle$ such that the following hold:*

(1) *$\{x_n\} \subseteq \{y_F\}$;*
(2) *if the ω-net $\langle y_F\rangle$ s-converges to p, then the sequence $\langle x_n\rangle$ s-converges to p;*
(3) *if the ω-net $\langle y_F\rangle$ s^*-converges to p, then the sequence $\langle x_n\rangle$ s^*-converges to p.*

Proof. For each $n \in \mathbb{N}$, take $x_n = y_{F_n}$ where

$$F_n = \{k \in \mathbb{Z} : 0 \leq k < n-1\}.$$

For example, we have $F_1 = \emptyset$ and $F_2 = \{0\}$. Then (1) is obvious.

To prove (2), let U be an open neighborhood of p and let $A = \{n \in \mathbb{N} : x_n \notin U\}$. We must show that $\delta(A) = 0$. Since the family $\mathcal{F} = \{F_n : n \in \mathbb{N}\}$ is a strictly increasing subfamily of $[\omega]^{<\omega}$ and since the ω-net $\langle y_F \rangle$ s-converges to p, we have

$$\delta(\mathcal{A}_{\mathcal{F}}) = 0$$

where $\mathcal{A} = \{F_n \in \mathcal{F} : y_{F_n} \notin U\}$. In other words, for any $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that

$$k \geq k_0 \quad \Rightarrow \quad \frac{|\mathcal{A}_{\mathcal{F}}(k)|}{k} = \frac{|\{F_n \in \mathcal{A} : n \leq k\}|}{k} < \epsilon.$$

From the fact that $y_{F_n} \notin U$ if and only if $x_n \notin U$, it follows that

$$F_n \in \mathcal{A} \text{ if and only if } n \in A.$$

Hence

$$k \geq k_0 \quad \Rightarrow \quad \frac{|\{n \in A : n \leq k\}|}{k} = \frac{|\{F_n \in \mathcal{A} : n \leq k\}|}{k} < \epsilon.$$

Therefore the sequence $\langle x_n \rangle$ s-converges to p.

To prove (3), we assume that the ω-net $\langle y_F \rangle$ s^*-converges to p. Then for the strictly increasing subfamily $\mathcal{F}$ which was constructed in (2), there exists a subfamily $\mathcal{A}$ of $\mathcal{F}$ satisfying the following:

- $\delta(\mathcal{A}_{\mathcal{F}}) = 1$;
- for every open neighborhood U of p there exists $F_{k_0} \in \mathcal{F}$ such that $F_k \supseteq F_{k_0}$ and $F_k \in \mathcal{A}$ imply $y_{F_k} \in U$.

Take $A = \{n \in \mathbb{N} : F_n \in \mathcal{A}\}$. Then one can prove the following by the similar argument with (2):

- $\delta(A) = 1$;
- for every open neighborhood U of p there exists $k_0 \in \mathbb{N}$ such that $k \geq k_0$ and $k \in A$ imply $x_k \in U$.

Therefore the sequence $\langle x_n \rangle$ s^*-converges to p. □

We can prove the following two corollaries by using Theorem 3.7.

Corollary 3.8. *If a space X is $s\omega$-Fréchet, then X is s-Fréchet.*

Proof. Let $p \in \overline{A}$. Since X is statistically ω-Fréchet, there exists an ω-net $\langle y_F \rangle$ in A which s-converges to p. By Theorem 3.7 (2), we can find a sequence $\langle x_n \rangle$ in A which s-converges to p. Therefore X is statistically Fréchet. □

Corollary 3.9. *Let X be a space. If for every $p \in \overline{A}$, there exists an ω-net in A which s^*-converges to p, then for every $p \in \overline{A}$, there exists a sequence in A which s^*-converges to p.*

Proof. It is immediate from Theorem 3.7 (3). □

Recall that a subset A of a space X is said to be *sequentially closed* if a sequence $\langle x_n \rangle$ in A converges to p, then p belongs to A. It is well known that a space X is sequential if and only if every sequentially closed subset of X is a closed subset.

Definition 3.10. A subset A of a space X is said to be *statistically ω-sequentially closed* provided that if an ω-net $\langle x_F \rangle$ in A s-converges to p, then p belongs to A.

The following is a characterization of the statistically ω-sequential property.

Theorem 3.11. *A space X is statistically ω-sequential if and only if every statistically ω-sequentially closed subset of X is a closed subset.*

Proof. ($\Rightarrow$) Suppose that A is a non-closed subset of a statistically ω-sequential space X. Then there exist a point $p \in \overline{A} \setminus A$ and an ω-net $\langle x_F \rangle$ in A which s-converges to p. Since $p \notin A$, A is not statistically ω-sequentially closed.

($\Leftarrow$) We prove it by the way of contraposition. Suppose that X is not statistically ω-sequential. Then there exists a non-closed subset A of X such that for every point $p \in \overline{A} \setminus A$, there is no ω-net in A which s-converges to p. Let $\langle x_F \rangle$ be an ω-net in A which s-converges to a point q, then it is clear that $q \in \overline{A}$, but $q \notin \overline{A} \setminus A$. Hence $q \in A$. Therefore A is a non-closed subset which is statistically ω-sequentially closed. □

Definition 3.12. A space X is said to be *statistically ω-transitive* if the following holds for every ω-net $\langle x_F \rangle$ in X: if $\langle x_F \rangle$ s-converges to p, and for each $F \in [\omega]^{<\omega}$ there is an ω-net $\langle x_{F,G} : G \in [\omega]^{<\omega} \rangle$ in X such that $\langle x_{F,G} \rangle$ s-converges to x_F, then there is an ω-net in $\{x_{F,G} : F, G \in [\omega]^{<\omega}\}$ that s-converges to p.

Finally, an equivalence between statistically ω-Fréchet and statistically ω-sequential properties is established in the class of statistically ω-transitive spaces.

Theorem 3.13. *For any space X, the following are equivalent:*

(1) *X is statistically ω-Fréchet;*
(2) *X is statistically ω-sequential and statistically ω-transitive.*

Proof. (1) $\Rightarrow$ (2) By definitions, every statistically ω-Fréchet space is statistically ω-sequential. We shall show that X is statistically ω-transitive. Let $\langle x_F \rangle$ be an ω-net in X which s-converges to p. For each $F \in [\omega]^{<\omega}$, let $\langle x_{F,G} : G \in [\omega]^{<\omega} \rangle$ be an ω-net in X such that $\langle x_{F,G} \rangle$ s-converges to x_F.

Denote $A = \{x_{F,G} : F, G \in [\omega]^{<\omega}\}$. Then $p \in \overline{A}$. Since X is statistically ω-Fréchet, there exists an ω-net in A that s-converges to p.

(2) $\Rightarrow$ (1) Let $p \in \overline{A}$ and let L be the set of all points q in X that $\langle x_F \rangle$ s-converges to q for some ω-net $\langle x_F \rangle$ in A. Then $A \subseteq L$.

Claim : L is statistically ω-sequentially closed.

Let $\langle x_F \rangle$ be an ω-net in L that $\langle x_F \rangle$ s-converges to r. By the definition of L, we can find an ω-net $\langle x_{F,G} : G \in [\omega]^{<\omega} \rangle$ in A that $\langle x_{F,G} \rangle$ s-converges to x_F. Since X is statistically ω-transitive, there exists an ω-net in $\{x_{F,G} : F, G \in [\omega]^{<\omega}\}$ (particularly, in A) that s-converges to r. So r belongs to L. Hence L is statistically ω-sequentially closed.

Since X is statistically ω-sequential, L is a closed subset of X. Hence $p \in \overline{A} \subseteq \overline{L} = L$. Thus there exists an ω-net in A that s-converges to p. Therefore X is statistically ω-Fréchet. □

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