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THE ZETA-DETERMINANTS OF LAPLACIANS ON THE MÖBIUS BAND AND KLEIN BOTTLE

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Abstract. We compute the zeta-determinants of the scalar Laplacians defined on the Möbius band and Klein bottle when the flat metrics are given. We consider the difference between these zeta-determinants and those of the product manifolds, and use the BFK-gluing formula to compute the difference. The zeta-determinants of product manifolds are well known and this computes the zeta-determinants on the Möbius band and Klein bottle. We finally show that the zeta-determinant on the Klein bottle satisfies the BFK-gluing formula.

1. Introduction

In this paper we are going to compute the zeta-determinants of Laplace operators defined on the Möbius band and Klein bottle. The zeta-determinants of Laplace operators are global spectral invariants, which play important roles in geometry, topology and theoretical physics including the theory of the analytic torsion ([6], [7], [11], [20], [22]). The Möbius band and Klein bottle are typical examples of mapping tori. With these reasons, we begin with the definitions of the zeta-determinant and the mapping torus.

Let (M, g^M) be an $(m-1)$-dimensional compact oriented Riemannian manifold with boundary ∂M, where ∂M may be empty. If ∂M is not empty, we impose an elliptic boundary condition on ∂M, for example, the Dirichlet or Neumann boundary condition. We denote by Δ_M a Laplace operator acting on smooth functions on M satisfying the boundary condition. The convention of defining a Laplacian is that the principal part is $-g^{ij}\frac{\partial^2}{\partial x_i \partial x_j}$ so that Δ_M is a non-negative operator. Then, it is well known that Δ_M has a discrete spectrum, which we denote by $0 = \mu_0 < \mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots \longrightarrow \infty$. For $s \in \mathbb{C}$, we define the zeta function $\zeta_{\Delta_M}(s)$ associated to Δ_M by

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$$\zeta_{\Delta_M}(s) \;=\; \sum_{k=1}^{\infty} \mu_k^{-s} \;=\; \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \left(\operatorname{Tr} e^{-t\Delta_M} - \dim \ker \Delta_M \right) \, dt.$$

Then, $\zeta_{\Delta_M}(s)$ is holomorphic for $\Re s > \frac{m-1}{2}$ and has an meromorphic continuation to $\mathbb{C}$ having a regular value at $s = 0$ ([7], [10], [22]). We define the zeta-determinant $\operatorname{Det} \Delta_M$ of Δ_M by

$$\operatorname{Det} \Delta_M \;=\; e^{-\zeta'_{\Delta_M}(0)}, \quad \text{or equivalently} \quad \log \operatorname{Det} \Delta_M \;=\; -\zeta'_{\Delta_M}(0).$$

Let $\varphi : M \to M$ be a homeomorphim. For an interval $[0, a]$ with $0 < a \in \mathbb{R}$, the mapping torus M_φ is defined by the quotient space

$$M_\varphi \;=\; M \times [0, a]/(x, 0) \sim (\varphi(x), a).$$

Equivalently, we define a $\mathbb{Z}$-action on $M \times \mathbb{R}$ by

$$\mathbb{Z} \times (M \times \mathbb{R}) \to M \times \mathbb{R}, \qquad m \cdot (x, u) \;=\; (\varphi^m(x), u + ma).$$

This action is properly discontinuous and M_φ is the quotient space. In fact, M_φ is a fiber bundle over S^1 with a fiber M and every fiber bundle over S^1 can be expressed by a mapping torus. It is well known that M_φ is a trivial bundle if and only if φ is isotopic to the identity map of M. If $M = [-b, b]$ for $0 < b \in \mathbb{R}$ and $\varphi : [-b, b] \to [-b, b]$, $\varphi(x) = -x$, then M_φ is a Möbius band. If $M = S^1(\frac{b}{\pi})$ and $\varphi : S^1(\frac{b}{\pi}) \to S^1(\frac{b}{\pi})$, $\varphi(\frac{b}{\pi} e^{i\theta}) = \frac{b}{\pi} e^{-i\theta}$, then M_φ is a Klein bottle, where $S^1(r)$ is the round circle of radius $r > 0$.

To define a Riemann metric on M_φ, we consider a more specific case. On a Riemannian manifold (M, g^M), we suppose that $\varphi : M \to M$ is an isometry. We also consider the Riemannian product $M \times \mathbb{R}$, where we give the usual flat metric on $\mathbb{R}$. The mapping torus M_φ induced from $M \times \mathbb{R}$ is called the metric mapping torus ([1]). Then, there is a natural Riemann metric on a metric mapping torus M_φ induced from the Riemannian product $M \times \mathbb{R}$. In this paper, a mapping torus means a metric mapping torus. The metric mapping tori play important roles in the study of co-symplectic and co-Kähler manifolds, which are odd dimensional analogues of symplectic and Kähler manifolds ([2], [3], [18]).

When M is a closed Riemannian manifold, the zeta-determinants of Laplace operators defined on M_φ were computed in [17] by using the BFK-gluing formula for zeta-determinants ([4]). For example, the zeta-determinant of the Laplace operator on the metric Klein bottle $\mathbb{K}$ was computed in [17]. However, this formula is not applicable if $\partial M \neq \emptyset$. With this reason we need to use a different method when we compute the zeta-determinant on the metric Möbius band $\mathbb{M}$.

In this paper, we are going to compute the zeta-determinants of the scalar Laplace operator $\Delta_{\mathbb{M},\mathrm{D}}$ defined on the metric Möbius band $\mathbb{M}$ with the Dirichlet boundary condition on $\partial\mathbb{M}$ and $\Delta_{\mathbb{K}}$ defined on the metric Klein bottle $\mathbb{K}$ by using the BFK-gluing formula and the results of [15]. Later, one of the referees made an excellent observation that $\zeta_{\Delta_{\mathbb{M},\mathrm{D}}}(s)$ and $\zeta_{\Delta_{\mathbb{K}}}(s)$ can be expressed by sums of the Epstein zeta functions, which gives a simpler way of obtaining the same results in slightly different forms by computing the Epstein zeta functions directly. However, the method presented here has its own advantage, which is that this method can be applied to a wider class of manifolds. We finally show that the zeta-determinant of $\Delta_{\mathbb{K}}$ satisfies the BFK-gluing formula. So far, we don't find a corresponding formula for $\Delta_{\mathbb{M}}$ since in case of $\mathbb{M}$ the cutting hypersurface intersects the boundary $\partial\mathbb{M}$.

2. Computation of the zeta-determinant on the metric Möbius band

For $0 < a,\ b \in \mathbb{R}$, we define a $\mathbb{Z}$-action on $\mathbb{R} \times [-b, b]$ by

$$\mathbb{Z} \times (\mathbb{R} \times [-b, b]) \to \mathbb{R} \times [-b, b], \quad k \cdot (x, y) = (x + ka,\ (-1)^k y), \tag{1}$$

where we give the trivial flat product metric on $\mathbb{R} \times [-b, b]$. Then, the orbit space is called the metric Möbius band $\mathbb{M}$. If $p : \mathbb{R} \times [-b, b] \to \mathbb{M}$ is a universal covering space, there is a natural metric on $\mathbb{M}$ whose lifting is the flat metric on $\mathbb{R} \times [-b, b]$. We define $\Omega^0_{\mathbb{M}}(\mathbb{R} \times [-b, b])$ by

$$\Omega^0_{\mathbb{M}}(\mathbb{R} \times [-b, b]) \;=\; \{f \in C^\infty(\mathbb{R} \times [-b, b]) \mid f(x, y) = f(x + a,\ -y)\}.$$

Then, the Laplacian $\Delta_{\mathbb{M}}$ is described by

$$\Delta_{\mathbb{M}} \;=\; -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \qquad \mathrm{Dom}\,(\Delta_{\mathbb{M}}) = \Omega^0_{\mathbb{M}}(\mathbb{R} \times [-b, b]).$$

Here we give the Dirichlet boundary condition on the boundary of $\mathbb{M}$ and denote by $\Delta_{\mathbb{M},\mathrm{D}}$ the Laplacian $\Delta_{\mathbb{M}}$ imposed by the Dirichlet boundary condition, where D stands for the Dirichlet boundary condition. The eigenfunctions of $\Delta_{\mathbb{M},\mathrm{D}}$ are given by

$$\begin{aligned}
&\sin\left(\frac{2\pi m}{a}x\right)\ \cos\left(\frac{\pi}{b}(n - \frac{1}{2})y\right), \quad \cos\left(\frac{2\pi m}{a}x\right)\ \cos\left(\frac{\pi}{b}(n - \frac{1}{2})y\right),\\
&\sin\left(\frac{2\pi}{a}(m - \frac{1}{2})x\right)\ \sin\left(\frac{\pi n}{b}y\right), \quad \cos\left(\frac{2\pi}{a}(m - \frac{1}{2})x\right)\ \sin\left(\frac{\pi n}{b}y\right),\\
&\quad m \geq 1, \quad n \geq 1,
\end{aligned}$$

and their corresponding eigenvalues are

$$\left\{ \frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 \big(n - \frac{1}{2}\big)^2}{b^2} \mid m, n = 1, 2, 3, \cdots \right\}$$
$$\cup \left\{ \frac{4\pi^2 \big(m - \frac{1}{2}\big)^2}{a^2} + \frac{\pi^2 n^2}{b^2} \mid m, n = 1, 2, 3, \cdots \right\}$$
$$\cup \left\{ \frac{\pi^2 \big(n - \frac{1}{2}\big)^2}{b^2} \mid n = 1, 2, 3, \cdots \right\},$$

where the multiplicity of $\frac{\pi^2 \left(m - \frac{1}{2}\right)^2}{b^2}$ is 1 and those of other eigenvalues are 2. The zeta function $\zeta_{\Delta_{\mathbb{M},\mathrm{D}}}(s)$ associated to $\Delta_{\mathbb{M},\mathrm{D}}$ is given by

$$(2) \qquad \zeta_{\Delta_{\mathbb{M},\mathrm{D}}}(s) \;=\; 2 \sum_{m,n=1}^{\infty} \left(\frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 \big(n - \frac{1}{2}\big)^2}{b^2} \right)^{-s}$$
$$+ \, 2 \sum_{m,n=1}^{\infty} \left(\frac{4\pi^2 \big(m - \frac{1}{2}\big)^2}{a^2} + \frac{\pi^2 n^2}{b^2} \right)^{-s} \; + \; \sum_{n=1}^{\infty} \left(\frac{\pi^2 \big(n - \frac{1}{2}\big)^2}{b^2} \right)^{-s}.$$

On the other hand, the following $\mathbb{Z}$-action

$$\mathbb{Z} \times (\mathbb{R} \times [-b, b]) \to \mathbb{R} \times [-b, b]), \qquad k \cdot (x, y) = (x + ka, \; y)$$

gives the cylinder $\mathbb{S} := S^1(\frac{a}{2\pi}) \times [-b, b]$, whose Laplacian is denoted by $\Delta_{\mathbb{S}}$. The eigenfunctions of $\Delta_{\mathbb{S},\mathrm{D}}$ are given by

$$\sin\left(\frac{2\pi m}{a} x\right) \; \cos\left(\frac{\pi}{b}\big(n - \frac{1}{2}\big) y\right), \quad \cos\left(\frac{2\pi m}{a} x\right) \; \cos\left(\frac{\pi}{b}\big(n - \frac{1}{2}\big) y\right),$$
$$\sin\left(\frac{2\pi m}{a} x\right) \; \sin\left(\frac{\pi n}{b} y\right), \quad \cos\left(\frac{2\pi m}{a} x\right) \; \sin\left(\frac{\pi n}{b} y\right), \quad m \geq 1, \; n \geq 1,$$

and their corresponding eigenvalues are

$$\left\{ \frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 \big(n - \frac{1}{2}\big)^2}{b^2} \mid n, m = 1, 2, 3, \cdots \right\}$$
$$\cup \left\{ \frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2} \mid m, n = 1, 2, 3, \cdots \right\}$$
$$\cup \left\{ \frac{\pi^2 \big(n - \frac{1}{2}\big)^2}{b^2} \mid n = 1, 2, 3, \cdots \right\} \cup \left\{ \frac{\pi^2 n^2}{b^2} \mid n = 1, 2, 3, \cdots \right\},$$

where the multiplicities of $\frac{\pi^2 \left(m - \frac{1}{2}\right)^2}{b^2}$ and $\frac{\pi^2 m^2}{b^2}$ are 1 and those of other eigenvalues are 2. The zeta function $\zeta_{\Delta_{\mathbb{S},\mathrm{D}}}(s)$ associated to $\Delta_{\mathbb{S},\mathrm{D}}$ is given by

$$(3)\quad \zeta_{\Delta_{\mathbb{S},\mathrm{D}}}(s) = 2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2}+\frac{\pi^2\left(n-\frac{1}{2}\right)^2}{b^2}\right)^{-s} +2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2}+\frac{\pi^2 n^2}{b^2}\right)^{-s} + \sum_{n=1}^{\infty}\left(\frac{\pi^2\left(n-\frac{1}{2}\right)^2}{b^2}\right)^{-s} +\sum_{n=1}^{\infty}\left(\frac{\pi^2 n^2}{b^2}\right)^{-s}.$$

Here $\zeta_{\Delta_{\mathbb{M},\mathrm{D}}}(s)$ and $\zeta_{\Delta_{\mathbb{S},\mathrm{D}}}(s)$ are holomorphic for $\Re s > 1$ and have meromorphic continuation to the whole complex plane $\mathbb{C}$ having a regular value at $s = 0$ ([7], [10], [22]). From (2) and (3), we have the following result.

$$\begin{aligned} &\zeta_{\Delta_{\mathbb{M},\mathrm{D}}}(s) - \zeta_{\Delta_{\mathbb{S},\mathrm{D}}}(s) \\ = \;& 2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2\left(m-\frac{1}{2}\right)^2}{a^2}+\frac{\pi^2 n^2}{b^2}\right)^{-s} - 2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2}+\frac{\pi^2 n^2}{b^2}\right)^{-s} \\ & - \sum_{n=1}^{\infty}\left(\frac{\pi^2 n^2}{b^2}\right)^{-s} \\ = \;& 2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2\left(m-\frac{1}{2}\right)^2}{a^2}+\frac{\pi^2 n^2}{b^2}\right)^{-s} - 2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2}+\frac{\pi^2 n^2}{b^2}\right)^{-s} \\ & - \left(\frac{b}{\pi}\right)^{2s}\zeta_R(2s), \end{aligned}$$

where $\zeta_R(s)$ is the Riemann zeta function. Using the well known facts ([19])

$$\zeta_R(0) = -\frac{1}{2}, \qquad \zeta_R'(0) = -\frac{1}{2}\log 2\pi,$$

we obtain the following result.

Lemma 2.1.

$$\log \operatorname{Det}\Delta_{\mathbb{M},\mathrm{D}} - \log \operatorname{Det}\Delta_{\mathbb{S},\mathrm{D}} = -\frac{d}{ds}\Big|_{s=0}\left\{2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2\left(m-\frac{1}{2}\right)^2}{a^2}+\frac{\pi^2 n^2}{b^2}\right)^{-s} - 2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2}+\frac{\pi^2 n^2}{b^2}\right)^{-s}\right\} - \log 2b.$$

Before going further, we briefly discuss the relation of (2) and (3) with the Epstein zeta function. The Epstein zeta function is defined as follows (p.108

in [16], [7]). For $c \in \mathbb{R}^+ \cup \{0\}$ and $\vec{r} = (r_1, \cdots, r_d) \in (\mathbb{R}^+)^d$, the Epstein zeta function $\zeta_E(s; c, \vec{r})$ is defined by

$$(4)\quad \begin{aligned} \zeta_E(s; c, \vec{r}) &:= \sum_{(m_1,\cdots,m_d)\in\mathbb{Z}^d} (c + r_1 m_1^2 + \cdots + r_d m_d^2)^{-s} \\ &= \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} \sum_{(m_1,\cdots,m_d)\in\mathbb{Z}^d} e^{-t(c+r_1 m_1^2+\cdots+r_d m_d^2)}\, dt. \end{aligned}$$

If $c = 0$, it is understood that the sum in (4) is taken over $(0, \cdots, 0) \neq (m_1, \cdots, m_d) \in \mathbb{Z}^d$. It is well known that $\zeta_E(s; c, \vec{r})$ is holomorphic for $\Re s > \frac{d}{2}$ and has a meromorphic continuation to the whole complex plane $\mathbb{C}$ having a regular value at $s = 0$. The zeta function associated to the Laplacian on flat torus is expressed by the Epstein zeta function with $c = 0$ (see (13) below). Now let $d = 2$ and $c = 0$. We refer to [16] for details of the Epstein zeta function. Then,

$$(5)\quad \zeta_E(s; 0, \vec{r}) := 4 \sum_{m,n\geq 1} (r_1 m^2 + r_2 n^2)^{-s} + 2\big(r_1^{-s} + r_2^{-s}\big)\, \zeta_R(2s).$$

It follows form (2) and (5) that

$$(6)\quad \begin{aligned} &\zeta_{\Delta_{\mathbb{M},\mathrm{D}}}(s) \\ &= 2 \sum_{m,n=1}^\infty \left(\frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{4b^2}\right)^{-s} - 2 \sum_{m,n=1}^\infty \left(\frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2}\right)^{-s} \\ &\quad + 2 \sum_{m,n=1}^\infty \left(\frac{\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2}\right)^{-s} - 2 \sum_{m,n=1}^\infty \left(\frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2}\right)^{-s} \\ &\quad + \sum_{n=1}^\infty \left(\frac{\pi^2 n^2}{4b^2}\right)^{-s} - \sum_{n=1}^\infty \left(\frac{\pi^2 n^2}{b^2}\right)^{-s} \\ &= \frac{1}{2}\zeta_E(s; 0, \vec{r}_1) + \frac{1}{2}\zeta_E(s; 0, \vec{r}_2) - \zeta_E(s; 0, \vec{r}_3) \\ &\quad + \left(\left(\frac{4\pi^2}{a^2}\right)^{-s} - \left(\frac{\pi^2}{a^2}\right)^{-s}\right) \zeta_R(2s), \end{aligned}$$

where $\vec{r}_1 = \left(\frac{4\pi^2}{a^2}, \frac{\pi^2}{4b^2}\right)$, $\vec{r}_2 = \left(\frac{\pi^2}{a^2}, \frac{\pi^2}{b^2}\right)$ and $\vec{r}_3 = \left(\frac{4\pi^2}{a^2}, \frac{\pi^2}{b^2}\right)$. Similarly, it follows by (3) and (5) that

$$\zeta_{\Delta_{\mathbb{S},\mathrm{D}}}(s) = \frac{1}{2}\zeta_E(s; 0, \vec{r}_1) - \left(\frac{4\pi^2}{a^2}\right)^{-s} \zeta_R(2s),$$

which leads to

$$\begin{aligned}\zeta_{\Delta_{\mathbb{M},\mathrm{D}}}(s) - \zeta_{\Delta_{\mathbb{S},\mathrm{D}}}(s) &= \frac{1}{2}\zeta_E(s;0,\vec{r}_2) \; - \; \zeta_E(s;0,\vec{r}_3) \\ &+ \left(2\left(\frac{4\pi^2}{a^2}\right)^{-s} - \left(\frac{\pi^2}{a^2}\right)^{-s} \right) \zeta_R(2s).\end{aligned}$$

We next compute the zeta-determinant of the right hand side of Lemma 2.1. We consider $X := S^1(\frac{b}{\pi}) \times [0, \frac{a}{2}]$ with the usual Laplacian Δ_X, which is given by

$$\Delta_X \;\; = \;\; -\frac{\partial^2}{\partial u^2} - \frac{\pi^2}{b^2}\frac{\partial^2}{\partial \theta^2},$$

where u and θ are variables for $[0, \frac{a}{2}]$ and $S^1(\frac{b}{\pi})$, respectively. We denote by $\Delta_{X,\mathrm{D},\mathrm{D}}$ ($\Delta_{X,\mathrm{D},\mathrm{N}}$) the Laplacian Δ_X with the Dirichlet boundary condition on $u = 0$ and $u = \frac{a}{2}$ (the Dirichlet boundary condition on $u = 0$ and the Neumann boundary condition on $u = \frac{a}{2}$). A simple computation shows that the eigenfunctions of $\Delta_{X,\mathrm{D},\mathrm{D}}$ are given by

$$\begin{aligned}&\left\{ \sin\left(\frac{2\pi m}{a}u\right)\cos n\theta,\; \sin\left(\frac{2\pi m}{a}u\right)\sin n\theta \mid m, n = 1,2,3,\cdots \right\} \\ &\cup \left\{ \sin\left(\frac{2\pi m}{a}u\right) \mid m = 1,2,3,\cdots \right\},\end{aligned}$$

and the eigenfunctions of $\Delta_{X,\mathrm{D},\mathrm{N}}$ are given by

$$\begin{aligned}&\left\{ \sin\left(\frac{2\pi(m-\frac{1}{2})}{a}u\right)\cos n\theta,\; \sin\left(\frac{2\pi(m-\frac{1}{2})}{a}u\right)\sin n\theta \mid m, n = 1,2,3,\cdots \right\} \\ &\qquad\qquad \cup \left\{ \sin\left(\frac{2\pi(m-\frac{1}{2})}{a}u\right) \mid m = 1,2,3,\cdots \right\}.\end{aligned}$$

Hence, the spectra of $\Delta_{X,\mathrm{D},\mathrm{D}}$ and $\Delta_{X,\mathrm{D},\mathrm{N}}$ are given as follows.

Lemma 2.2. *$\Delta_{X,\mathrm{D},\mathrm{D}}$ and $\Delta_{X,\mathrm{D},\mathrm{N}}$ are invertible operators and their spectra are given as follows.*

$$\begin{aligned}\mathrm{Spec}\,(\Delta_{X,\mathrm{D},\mathrm{D}}) &= \left\{ \frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2} \mid m, n = 1,2,3,\cdots \right\} \\ &\cup \left\{ \frac{4\pi^2 m^2}{a^2} \mid m = 1,2,3,\cdots \right\},\end{aligned}$$

$$\begin{aligned}\mathrm{Spec}\,(\Delta_{X,\mathrm{D},\mathrm{N}}) &= \left\{\frac{4\pi^2\left(m-\frac{1}{2}\right)^2}{a^2} + \frac{\pi^2 n^2}{b^2} \mid m, n = 1, 2, 3, \cdots\right\} \\ &\quad \cup \left\{\frac{4\pi^2\left(m-\frac{1}{2}\right)^2}{a^2} \mid m = 1, 2, 3, \cdots\right\},\end{aligned}$$

where the multiplicities of $\frac{4\pi^2 m^2}{a^2}$ *and* $\frac{4\pi^2\left(m-\frac{1}{2}\right)^2}{a^2}$ *are* 1 *and those of other eigenvalues are* 2.

Lemma 2.2 leads to the following result.

$$\begin{aligned}&\zeta_{\Delta_{X,\mathrm{D},\mathrm{N}}}(s) - \zeta_{\Delta_{X,\mathrm{D},\mathrm{D}}}(s) \\ &= 2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2\left(m-\frac{1}{2}\right)^2}{a^2} + \frac{\pi^2 n^2}{b^2}\right)^{-s} - 2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2}\right)^{-s} \\ &\quad + \sum_{m=1}^{\infty}\left(\frac{4\pi^2\left(m-\frac{1}{2}\right)^2}{a^2}\right)^{-s} - \sum_{m=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2}\right)^{-s} \\ &= 2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2\left(m-\frac{1}{2}\right)^2}{a^2} + \frac{\pi^2 n^2}{b^2}\right)^{-s} - 2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2}\right)^{-s} \\ &\quad + \left(\frac{2\pi}{a}\right)^{-2s}\zeta_H\left(2s, \frac{1}{2}\right) - \left(\frac{2\pi}{a}\right)^{-2s}\zeta_R(2s),\end{aligned}$$

where $\zeta_H(s,x)$ is the Hurwitz zeta function defined by $\zeta_H(s,x) = \sum_{n=0}^{\infty}(n+x)^{-s}$ for $x > 0$. It is well known (for example, p.23 in [19]) that

$$\zeta_H\left(0, \frac{1}{2}\right) = 0, \qquad \zeta_H'\left(0, \frac{1}{2}\right) = \log\Gamma\left(\frac{1}{2}\right) - \frac{1}{2}\log 2\pi \;=\; -\frac{1}{2}\log 2,$$

which leads to the following result.

$$\begin{aligned}(7)\quad &\log\mathrm{Det}\,\Delta_{X,\mathrm{D},\mathrm{N}} - \log\mathrm{Det}\,\Delta_{X,\mathrm{D},\mathrm{D}} \\ &= -\frac{d}{ds}\Big|_{s=0}\left\{2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2\left(m-\frac{1}{2}\right)^2}{a^2} + \frac{\pi^2 n^2}{b^2}\right)^{-s}\right. \\ &\quad \left. - 2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2}\right)^{-s}\right\} + 2\log\frac{2\pi}{a}\,\zeta_H\left(0, \frac{1}{2}\right) - 2\zeta_H'\left(0, \frac{1}{2}\right) \\ &\quad - 2\log\frac{2\pi}{a}\,\zeta_R(0) + 2\zeta_R'(0)\end{aligned}$$

$$= \quad -\frac{d}{ds}\Big|_{s=0}\Big\{2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2\big(m-\frac{1}{2}\big)^2}{a^2}+\frac{\pi^2 n^2}{b^2}\right)^{-s} - 2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2}+\frac{\pi^2 n^2}{b^2}\right)^{-s}\Big\} \; + \; \log\frac{2}{a}.$$

This together with Lemma 2.1 leads to the following result.

Lemma 2.3.

$$\log \operatorname{Det} \Delta_{\mathbb{M},\mathrm{D}} - \log \operatorname{Det} \Delta_{\mathbb{S},\mathrm{D}} \; = \; \log \operatorname{Det} \Delta_{X,\mathrm{D},\mathrm{N}} - \log \operatorname{Det} \Delta_{X,\mathrm{D},\mathrm{D}} - \log\frac{4b}{a}.$$

The zeta-determinant $\log \operatorname{Det} \Delta_{\mathbb{S},\mathrm{D}}$ is well known. For example, it is computed in Proposition 5.1 of [21] that

$$\log \operatorname{Det} \Delta_{\mathbb{S},\mathrm{D}} \; = \; \log\frac{4b}{a} - \frac{2b\pi}{3a} + 2\sum_{m=1}^{\infty}\log\left(1-e^{-\frac{8\pi b m}{a}}\right). \tag{8}$$

To compute $\log \operatorname{Det} \Delta_{\mathbb{M},\mathrm{D}}$, we need to compute $\log \operatorname{Det} \Delta_{X,\mathrm{D},\mathrm{N}} - \log \operatorname{Det} \Delta_{X,\mathrm{D},\mathrm{D}}$, which is discussed in [15]. For self-contained presentation, we give all details. For $0 \leq \lambda \in \mathbb{R}$, we define the Dirichlet-to-Neumann operator $Q(\lambda) : C^{\infty}\big(S^1(\frac{b}{\pi})\big) \to C^{\infty}\big(S^1(\frac{b}{\pi})\big)$ as follows. For $f \in C^{\infty}\big(S^1(\frac{b}{\pi})\big)$, we choose $\mathcal{F}(x,u) \in C^{\infty}\big(S^1(\frac{b}{\pi}) \times [0,\frac{a}{2}]\big)$ satisfying

$$(\Delta_X + \lambda)\,\mathcal{F}(x,u) \; = \; 0, \qquad \mathcal{F}(x,0) = 0, \qquad \mathcal{F}\left(x,\frac{a}{2}\right) = f.$$

We define

$$Q(\lambda) : C^{\infty}\left(S^1(\frac{b}{\pi})\right) \to C^{\infty}\left(S^1(\frac{b}{\pi})\right), \qquad Q(\lambda)(f) = \left(\frac{d}{du}\mathcal{F}(x,u)\right)\Big|_{u=\frac{a}{2}}.$$

Then, $Q(\lambda)$ is a non-negative elliptic pseudodifferential operator of order 1 by the Green formula. Here $\mathcal{F}(x,u)$ is constructed as follows. Let $\widetilde{f} \in C^{\infty}\big(S^1(\frac{b}{\pi}) \times [0,\frac{a}{2}]\big)$ be an arbitrary extension of f satisfying $\widetilde{f}(x,0) = 0$ and $\widetilde{f}(x,\frac{a}{2}) = f$. Then, $\mathcal{F}(x,u)$ is given by

$$\mathcal{F}(x,u) \; = \; \widetilde{f} - (\Delta_{X,\mathrm{D},\mathrm{D}} + \lambda)^{-1}\,(\Delta_X + \lambda)\,\widetilde{f}.$$

For the construction of $\mathcal{F}(x,u)$, we refer to p.315 of [5] or (3.1) of [21]. We define the Poisson operator $\mathcal{P}(\lambda)$ and the trace operator $\gamma_{\frac{a}{2}}$ as follows.

$$\mathcal{P}(\lambda) : C^{\infty}\left(S^1(\frac{b}{\pi})\right) \to C^{\infty}\left(S^1(\frac{b}{\pi}) \times [0, \frac{a}{2}]\right), \qquad \mathcal{P}(\lambda)(f) = \mathcal{F}(x,u),$$
$$\gamma_{\frac{a}{2}} : C^{\infty}\left(S^1(\frac{b}{\pi}) \times [0, \frac{a}{2}]\right) \to C^{\infty}\left(S^1(\frac{b}{\pi})\right),$$
$$\gamma_{\frac{a}{2}}(\phi) = \phi|_{S^1(\frac{b}{\pi}) \times \{\frac{a}{2}\}} = \phi\left(\cdot\ ,\ \frac{a}{2}\right).$$

Then, $\mathcal{P}(\lambda)$ satisfies

$$(\Delta_X + \lambda)\, \mathcal{P}(\lambda) = 0, \qquad \gamma_{\frac{a}{2}} \cdot \mathcal{P}(\lambda) = \mathrm{Id},$$

where "$\cdot$" is the operator composition. Taking derivative with respect to λ, we obtain

$$\mathcal{P}(\lambda) + (\Delta_X + \lambda)\, \frac{d}{d\lambda}\mathcal{P}(\lambda) = 0, \qquad \gamma_{\frac{a}{2}} \cdot \frac{d}{d\lambda}\mathcal{P}(\lambda) = 0,$$

which shows that

$$\frac{d}{d\lambda}\mathcal{P}(\lambda) = -\left(\Delta_{X,\mathrm{D},\mathrm{D}} + \lambda\right)^{-1} \cdot \mathcal{P}(\lambda).$$

We can rewrite $Q(\lambda)$ by

$$Q(\lambda) \;=\; \gamma_{\frac{a}{2}} \cdot \frac{d}{du}\mathcal{P}(\lambda).$$

Taking derivative with respect to λ, we obtain

$$\begin{aligned}
\frac{d}{d\lambda}Q(\lambda) &= \gamma_{\frac{a}{2}} \cdot \frac{d}{du}\frac{d}{d\lambda}\mathcal{P}(\lambda) \;=\; -\gamma_{\frac{a}{2}} \cdot \frac{d}{du}\left(\Delta_{X,\mathrm{D},\mathrm{D}} + \lambda\right)^{-1} \cdot \mathcal{P}(\lambda) \\
&= \gamma_{\frac{a}{2}} \cdot \frac{d}{du}\left(\left(\Delta_{X,\mathrm{D},\mathrm{N}} + \lambda\right)^{-1} - \left(\Delta_{X,\mathrm{D},\mathrm{D}} + \lambda\right)^{-1} \right) \cdot \mathcal{P}(\lambda) \\
&= \gamma_{\frac{a}{2}} \cdot \frac{d}{du} \cdot \mathcal{P}(\lambda) \cdot \gamma_{\frac{a}{2}} \cdot \left(\Delta_{X,\mathrm{D},\mathrm{N}} + \lambda\right)^{-1} \cdot \mathcal{P}(\lambda) \\
&= Q(\lambda) \cdot \gamma_{\frac{a}{2}} \cdot \left(\Delta_{X,\mathrm{D},\mathrm{N}} + \lambda\right)^{-1} \cdot \mathcal{P}(\lambda),
\end{aligned}$$

which shows that

$$Q(\lambda)^{-1} \cdot \frac{d}{d\lambda}Q(\lambda) \;=\; \gamma_{\frac{a}{2}} \cdot \left(\Delta_{X,\mathrm{D},\mathrm{N}} + \lambda\right)^{-1} \cdot \mathcal{P}(\lambda).$$

We note that

$$\begin{aligned}
& \frac{d}{d\lambda}\Big\{ \log \operatorname{Det}(\Delta_{X,\mathrm{D},\mathrm{N}}+\lambda) \;-\; \log \operatorname{Det}(\Delta_{X,\mathrm{D},\mathrm{D}}+\lambda) \Big\} \\
= \;& \operatorname{Tr}\Big\{ (\Delta_{X,\mathrm{D},\mathrm{N}}+\lambda)^{-1} \;-\; (\Delta_{X,\mathrm{D},\mathrm{D}}+\lambda)^{-1} \Big\} \\
= \;& \operatorname{Tr}\Big\{ \mathcal{P}(\lambda)\cdot \gamma_{\frac{a}{2}} \cdot (\Delta_{X,\mathrm{D},\mathrm{N}}+\lambda)^{-1} \Big\} \\
= \;& \operatorname{Tr}\Big\{ \gamma_{\frac{a}{2}} \cdot (\Delta_{X,\mathrm{D},\mathrm{N}}+\lambda)^{-1} \cdot \mathcal{P}(\lambda) \Big\} \;=\; \operatorname{Tr} Q(\lambda)^{-1}\cdot \frac{d}{d\lambda} Q(\lambda) \\
= \;& \frac{d}{d\lambda} \log \operatorname{Det} Q(\lambda),
\end{aligned}$$

which leads to the following result.

Lemma 2.4. *There exists a constant $c \in \mathbb{R}$ such that*

$$\log \operatorname{Det}(\Delta_{X,\mathrm{D},\mathrm{N}}+\lambda) \;-\; \log \operatorname{Det}(\Delta_{X,\mathrm{D},\mathrm{D}}+\lambda) \;=\; c \;+\; \log \operatorname{Det} Q(\lambda).$$

It is well known that each term in Lemma 2.4 has an asymptotic expansion for $\lambda \to \infty$. It is also known that the constant terms in the asymptotic expansions of $\log \operatorname{Det}(\Delta_{X,\mathrm{D},\mathrm{N}}+\lambda)$ and $\log \operatorname{Det}(\Delta_{X,\mathrm{D},\mathrm{D}}+\lambda)$ are zero (Lemma 2.1 in [13], (1.6) in [14], or (5.1) in [23]). Hence, $-c$ is the constant term in the asymptotic expansion of $\log \operatorname{Det} Q(\lambda)$. Putting $\lambda = 0$, we obtain the following result.

Lemma 2.5. *There exists a constant $c \in \mathbb{R}$ such that*

$$\log \operatorname{Det} \Delta_{X,\mathrm{D},\mathrm{N}} \;-\; \log \operatorname{Det} \Delta_{X,\mathrm{D},\mathrm{D}} \;=\; c \;+\; \log \operatorname{Det} Q(0),$$

where $-c$ is the constant term in the asymptotic expansion of $\log \operatorname{Det} Q(\lambda)$ for $\lambda \to \infty$.

A simple computation shows that the spectrum of the Dirichlet-to-Neumann operator $Q(\lambda)$ is given by

$$\operatorname{Spec}(Q(\lambda)) \;=\; \left\{ \sqrt{\frac{\pi^2 m^2}{b^2}+\lambda}\, \left(1 + \frac{2}{e^{a\sqrt{\frac{\pi^2 m^2}{b^2}+\lambda}}-1}\right) \mid m \in \mathbb{Z} \right\},$$

which shows that

$$Q(\lambda) \;=\; \sqrt{\Delta_{S(\frac{b}{\pi})}+\lambda}\, \left(\mathrm{Id} + \frac{2}{e^{a\sqrt{\Delta_{S(\frac{b}{\pi})}+\lambda}}-\mathrm{Id}}\right).$$

The following lemma is well known (for example, Lemma 3.2 in [12]).

Lemma 2.6. *For a positive definite, self-adjoint elliptic pseudodifferential operator A of positive order and a trace class operator Q with $I+Q$ invertible, the following equality holds.*

$$\log \operatorname{Det} A(I+Q) \;=\; \log \operatorname{Det} A \;+\; \log \det{}_{Fr}(I+Q).$$

Lemma 2.6 shows that

$$\log \operatorname{Det} Q(\lambda) \;=\; \frac{1}{2}\log \operatorname{Det}\left(\Delta_{S(\frac{b}{\pi})}+\lambda\right)+\sum_{m\in\mathbb{Z}}\log\left(1+\frac{2}{e^{a\sqrt{\frac{\pi^2m^2}{b^2}+\lambda}}-1}\right).$$

It is known that for $\lambda\to\infty$ the constant term in the asymptotic expansion of $\log \operatorname{Det}\left(\Delta_{S(\frac{b}{\pi})}+\lambda\right)$ is zero (Lemma 2.1 in [13], (1.6) in [14], (4.12) in [23]) and

$$\sum_{m\in\mathbb{Z}}\log\left(1+\frac{2}{e^{a\sqrt{\frac{\pi^2m^2}{b^2}+\lambda}}-1}\right) \;=\; O(e^{a\sqrt{\lambda}}).$$

This shows that the constant term in the asymptotic expansion of $\log \operatorname{Det} Q(\lambda)$ is zero and hence $c=0$. This leads to the following result, which is obtained in Theorem 2.4 of [15].

Theorem 2.7.

$$\log \operatorname{Det} \Delta_{X,\mathrm{D},\mathrm{N}} \;-\; \log \operatorname{Det} \Delta_{X,\mathrm{D},\mathrm{D}} \;=\; \log \operatorname{Det} Q(0).$$

A simple computation shows that

$$\operatorname{Spec}(Q(0)) \;=\; \left\{\frac{2}{a}\right\}\cup\left\{\frac{\pi m}{b}\left(1+\frac{2}{e^{\frac{a\pi m}{b}}-1}\right)\mid m=1,2,\cdots\right\},$$

where the multiplicity of $\frac{2}{a}$ is 1 and that of $\frac{\pi m}{b}\left(1+\frac{2}{e^{\frac{a\pi m}{b}}-1}\right)$ is 2. By Lemma 2.6, it follows that

$$\begin{aligned}(9)\quad \log \operatorname{Det} Q(0) \;&=\; \log\frac{2}{a} \;+\; \frac{1}{2}\log \operatorname{Det}\Delta_{S(\frac{b}{\pi})} \;+\; 2\sum_{m=1}^{\infty}\log\left(1+\frac{2}{e^{\frac{a\pi m}{b}}-1}\right)\\ &=\; \log\frac{2}{a} \;+\; \log 2b \;+\; 2\sum_{m=1}^{\infty}\log\left(1+\frac{2}{e^{\frac{a\pi m}{b}}-1}\right).\end{aligned}$$

Lemma 2.3 and Theorem 2.7 with (8) and (9) lead to the following result.

Theorem 2.8. *Let $\mathbb{M}$ be the metric Möbius band defined in (1) and $\Delta_{\mathbb{M}}$ be the Laplacian acting on smooth functions on $\mathbb{M}$. Then, the zeta-determinant* *of* $\Delta_{\mathbb{M},\mathrm{D}}$ *is given by*

$$\begin{aligned}\log \operatorname{Det} \Delta_{\mathbb{M},\mathrm{D}} &= \log \frac{4b}{a} - \frac{2\pi b}{3a} + 2\sum_{m=1}^{\infty} \log\left(1 - e^{-\frac{8\pi b m}{a}}\right) \\ &\quad + 2\sum_{m=1}^{\infty} \log\left(1 + \frac{2}{e^{\frac{a\pi m}{b}} - 1}\right).\end{aligned}$$

Remark : This result can be obtained in a slightly different form by taking derivatives on the Epstein zeta functions and Riemann zeta function in (6).

3. Computation of the zeta-determinant on the metric Klein bottle

In this section, we are going to compute the zeta-determinant of the Laplace operator on the metric Klein bottle $\mathbb{K}$ by using a similar method presented in the previous section. The Klein bottle $\mathbb{K}$ is obtained by the group of motions on $\mathbb{R}^2$ generated by $(x, y) \mapsto (x, y + 2b)$ and $(x, y) \mapsto (x + a, 2b - y)$ for some $0 < a, b \in \mathbb{R}$. The fundamental domain for $\mathbb{K}$ is $(0, a) \times (0, 2b)$. We denote by $\Omega^0_{\mathbb{K}}(\mathbb{R}^2)$ the set of smooth functions which are invariant under these motions, *i.e.*

$$\Omega^0_{\mathbb{K}}(\mathbb{R}^2) = \{f \in C^\infty(\mathbb{R}^2) \mid f(x, y) = f(x, y + 2b) = f(x + a, 2b - y)\}.$$

The Laplacian $\Delta_{\mathbb{K}}$ is described by

$$\Delta_{\mathbb{K}} = -\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$$

defined on $\Omega^0_{\mathbb{K}}(\mathbb{R}^2)$. The eigenfunctions are given by

$$\begin{aligned}&\cos\left(\frac{2\pi m}{a}x\right)\cos\left(\frac{\pi n}{b}y\right), \qquad \sin\left(\frac{2\pi m}{a}x\right)\cos\left(\frac{\pi n}{b}y\right), \\ &\cos\left(\frac{2\pi}{a}(m - \frac{1}{2})x\right)\sin\left(\frac{\pi n}{b}y\right), \quad \sin\left(\frac{2\pi}{a}(m - \frac{1}{2})x\right)\sin\left(\frac{\pi n}{b}y\right), \\ &\quad m \geq 1, \quad n \geq 1,\end{aligned}$$

and their corresponding eigenvalues are

$$\begin{aligned}(10) \quad &\left\{0\right\} \cup \left\{\frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2} \mid m, n = 1, 2, \cdots\right\} \cup \left\{\frac{4\pi^2 m^2}{a^2} \mid m = 1, 2, \cdots\right\} \\ &\cup \left\{\frac{\pi^2 n^2}{b^2} \mid n = 1, 2, \cdots\right\} \cup \left\{\frac{4\pi^2\left(m - \frac{1}{2}\right)^2}{a^2} + \frac{\pi^2 n^2}{b^2} \mid m, n = 1, 2, \cdots\right\},\end{aligned}$$

where the multiplicity of $\frac{m^2\pi^2}{b^2}$ is 1 and the multiplicities of other non-zero eigenvalues are 2.

The flat torus $\mathbb{T}$ is obtained by the group of motions on $\mathbb{R}^2$ generated by $(x,y) \mapsto (x, y+2b)$ and $(x,y) \mapsto (x+a, y)$. The fundamental domain is $(0,a) \times (0,2b)$. The eigenfunctions are given by

$$\begin{aligned}
&\cos\left(\frac{2\pi m}{a}x\right)\cos\left(\frac{\pi n}{b}y\right), \quad \sin\left(\frac{2\pi m}{a}x\right)\cos\left(\frac{\pi n}{b}y\right), \\
&\cos\left(\frac{2\pi m}{a}x\right)\sin\left(\frac{\pi n}{b}y\right), \quad \sin\left(\frac{2\pi m}{a}x\right)\sin\left(\frac{\pi n}{b}y\right), \quad m \geq 1, \quad n \geq 1,
\end{aligned}$$

and their corresponding eigenvalues are

$$\begin{aligned}
(11) \quad & \left\{0\right\} \cup \left\{\frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2} \mid m,n = 1,2,\cdots\right\} \cup \left\{\frac{4\pi^2 m^2}{a^2} \mid m = 1,2,\cdots\right\} \\
& \cup \left\{\frac{\pi^2 n^2}{b^2} \mid m = 1,2,\cdots\right\},
\end{aligned}$$

where the multiplicity of $\frac{4\pi^2 m^2}{a^2} + \frac{n^2\pi^2}{b^2}$ is 4 and the multiplicities of other non-zero eigenvalues are 2. From (10), (11) and (5), the zeta functions $\zeta_{\Delta_{\mathbb{K}}}(s)$ and $\zeta_{\Delta_{\mathbb{T}}}(s)$ associated to $\Delta_{\mathbb{K}}$ and $\Delta_{\mathbb{T}}$ are given as follows.

$$\begin{aligned}
(12) \quad \zeta_{\Delta_{\mathbb{K}}}(s) &= 2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2}\right)^{-s} \\
&\quad + 2\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2\left(m-\frac{1}{2}\right)^2}{a^2} + \frac{\pi^2 n^2}{b^2}\right)^{-s} \\
&\quad + 2\sum_{m=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2}\right)^{-s} + \sum_{n=1}^{\infty}\left(\frac{\pi^2 n^2}{b^2}\right)^{-s} \\
&= \frac{1}{2}\zeta_E(s;0,\vec{r}_2) + \left(2\left(\frac{4\pi^2}{a^2}\right)^{-s} - \left(\frac{\pi^2}{a^2}\right)^{-s}\right)\zeta_R(2s), \\
(13) \quad \zeta_{\Delta_{\mathbb{T}}}(s) &= 4\sum_{m,n=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2} + \frac{\pi^2 n^2}{b^2}\right)^{-s} + 2\sum_{m=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2}\right)^{-s} \\
&\quad + 2\sum_{n=1}^{\infty}\left(\frac{\pi^2 n^2}{b^2}\right)^{-s} \\
&= \zeta_E(s;0,\vec{r}_3).
\end{aligned}$$

From (12) and (13), it follows that

$$
\begin{aligned}
&\zeta_{\Delta_{\mathbb{K}}}(s) - \zeta_{\Delta_{\mathbb{T}}}(s) \\
= \;& 2\sum_{n,m=1}^{\infty}\left(\frac{4\pi^2\left(m-\frac{1}{2}\right)^2}{a^2}+\frac{\pi^2 n^2}{b^2}\right)^{-s} - 2\sum_{n,m=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2}+\frac{\pi^2 n^2}{b^2}\right)^{-s} \\
& - \sum_{n=1}^{\infty}\left(\frac{\pi^2 n^2}{b^2}\right)^{-s} \\
= \;& 2\sum_{n,m=1}^{\infty}\left(\frac{4\pi^2\left(m-\frac{1}{2}\right)^2}{a^2}+\frac{\pi^2 n^2}{b^2}\right)^{-s} - 2\sum_{n,m=1}^{\infty}\left(\frac{4\pi^2 m^2}{a^2}+\frac{\pi^2 n^2}{b^2}\right)^{-s} \\
& - \left(\frac{b}{\pi}\right)^{2s}\zeta_R(2s),
\end{aligned}
$$

which together with (7) and (9) shows that

$$
\begin{aligned}
&\log \operatorname{Det} \Delta_{\mathbb{K}} - \log \operatorname{Det} \Delta_{\mathbb{T}} \\
= \;& \log \operatorname{Det} \Delta_{X,\mathrm{D},\mathrm{N}} - \log \operatorname{Det} \Delta_{X,\mathrm{D},\mathrm{D}} - \log\frac{2}{a} - \log 2b \\
= \;& \log\frac{2}{a} + \log 2b + 2\sum_{m=1}^{\infty}\log\left(1+\frac{2}{e^{\frac{a\pi m}{b}}-1}\right) - \log\frac{2}{a} - \log 2b \\
= \;& 2\sum_{m=1}^{\infty}\log\left(1+\frac{2}{e^{\frac{a\pi m}{b}}-1}\right).
\end{aligned}
$$

It is known in Theorem 8.3 of [9] that

$$
\log \operatorname{Det} \Delta_{\mathbb{T}} = 2\log 2b - \frac{2\pi b}{3a} + 4\sum_{m=1}^{\infty}\log\left(1-e^{-\frac{4\pi b m}{a}}\right),
$$

which leads to the following result.

Theorem 3.1. *The zeta-determinant of the scalar Laplacian $\Delta_{\mathbb{K}}$ defined on $\mathbb{K}$ is given by*

$$
\begin{aligned}
\log \operatorname{Det} \Delta_{\mathbb{K}} = \;& 2\log 2b - \frac{2\pi b}{3a} + 4\sum_{m=1}^{\infty}\log\left(1-e^{-\frac{4\pi b m}{a}}\right) \\
& + 2\sum_{m=1}^{\infty}\log\left(1+\frac{2}{e^{\frac{a\pi m}{b}}-1}\right).
\end{aligned}
$$

Remark: (1) Theorem 3.1 is obtained in Theorem 2.8 of [17] by using a different method. In [17] the zeta-determinant on a metric mapping torus is computed by using the BFK-gluing formula for zeta-determinants proved in [4] (see also [5] and [8]).

(2) This result can be obtained in a slightly different form by taking derivatives on the Epstein zeta function and Riemann zeta function in (12).

We finally discuss the BFK-gluing formula for the zeta-determinant on the metric Klein bottle $\mathbb{K}$, which is also obtained as follows. For $r > 0$, let $S^1(r)$ be the round circle of radius r. We define

$$\varphi : S^1\left(\frac{b}{\pi}\right) \to S^1\left(\frac{b}{\pi}\right), \qquad \varphi\left(\frac{b}{\pi}e^{i\theta}\right) = \frac{b}{\pi}e^{-i\theta}.$$

Then the metric mapping torus associated to φ is $\mathbb{K} = S^1\left(\frac{b}{\pi}\right)_{\varphi}$. We now cut $\mathbb{K}$ along $S^1\left(\frac{b}{\pi}\right) \times \{0\}$ to obtain $S^1\left(\frac{b}{\pi}\right) \times [0, a]$. Let $\operatorname{Det} \Delta_{S^1\left(\frac{b}{\pi}\right)\times[0,a],\mathrm{D}}$ be the zeta-determinant of the scalar Laplacian defined on $S^1\left(\frac{b}{\pi}\right) \times [0, a]$ with the Dirichlet boundary condition. The BFK-gluing formula express $\log \operatorname{Det} \Delta_{\mathbb{K}} - \log \operatorname{Det} \Delta_{S^1\left(\frac{b}{\pi}\right)\times[0,a],\mathrm{D}}$ by the zeta-determinant of some Dirichlet-to-Neumann operator defined on $S^1\left(\frac{b}{\pi}\right) \times \{0\}$ with some extra term. We are going to investigate this formula. It is known (for example, Proposition 5.1 in [21], (8)) that

$$\begin{aligned}
\log \operatorname{Det} \Delta_{S^1\left(\frac{b}{\pi}\right)\times[0,a],\mathrm{D}} &= \log \frac{a}{b} - \frac{a\pi}{6b} + 2\sum_{m=1}^{\infty} \log\left(1 - e^{-\frac{2a\pi m}{b}}\right), \\
\log \operatorname{Det} \Delta_{S^1\left(\frac{b}{\pi}\right)\times S^1\left(\frac{a}{2\pi}\right)} &= 2\log 2b - \frac{2b\pi}{3a} + 4\sum_{m=1}^{\infty} \log\left(1 - e^{-\frac{4b\pi m}{a}}\right) \\
= \log \operatorname{Det} \Delta_{S^1\left(\frac{a}{2\pi}\right)\times S^1\left(\frac{b}{\pi}\right)} &= 2\log a - \frac{a\pi}{6b} + 4\sum_{m=1}^{\infty} \log\left(1 - e^{-\frac{a\pi m}{b}}\right).
\end{aligned}$$

In fact, the last equality is obtained from the second equality by replacing a with $2b$ and b with $\frac{a}{2}$, and vice versa. The above equalities with Theorem 3.1 leads to the following result.

$$\begin{aligned}
(14) \qquad \log \operatorname{Det} \Delta_{S^1\left(\frac{b}{\pi}\right)\times[0,a],\mathrm{D}} &= \left\{2\log a - \frac{a\pi}{6b} + 4\sum_{m=1}^{\infty} \log\left(1 - e^{-\frac{a\pi m}{b}}\right)\right\} \\
&+ \left\{2\sum_{m=1}^{\infty} \log \frac{1 - e^{-\frac{2a\pi m}{b}}}{\left(1 - e^{-\frac{a\pi m}{b}}\right)^2} - \log ab\right\} \\
= &\left\{2\log 2b - \frac{2b\pi}{3a} + 4\sum_{m=1}^{\infty} \log\left(1 - e^{-\frac{4b\pi m}{a}}\right)\right\} \\
&+ \left\{2\sum_{m=1}^{\infty} \log \frac{1 + e^{-\frac{a\pi m}{b}}}{1 - e^{-\frac{a\pi m}{b}}} - \log ab\right\}
\end{aligned}$$

$$= \quad 2\log 2b - \frac{2b\pi}{3a} + 4\sum_{m=1}^{\infty}\log\left(1 - e^{-\frac{4b\pi m}{a}}\right) + 2\sum_{m=1}^{\infty}\log\frac{1+e^{-\frac{a\pi m}{b}}}{1-e^{-\frac{a\pi m}{b}}} - \log ab$$
$$= \quad \log \operatorname{Det} \Delta_{\mathbb{K}} - \log a - \log b.$$

We next define the Dirichlet-to-Neumann operator $\mathfrak{R} : C^{\infty}\left(S^1\left(\frac{b}{\pi}\right)\right) \to C^{\infty}\left(S^1\left(\frac{b}{\pi}\right)\right)$ as follows. For $f \in C^{\infty}\left(S^1\left(\frac{b}{\pi}\right)\right)$, we choose $\psi(\theta, u) \in C^{\infty}\left(S^1\left(\frac{b}{\pi}\right)\times [0,a]\right)$ satisfying

$$\left(-\frac{\partial^2}{\partial u^2} - \frac{\pi^2}{b^2}\frac{\partial^2}{\partial \theta^2}\right)\psi = 0, \qquad \psi(\theta,0) = f(\theta), \qquad \psi(\theta,a) = f(\varphi^{-1}(\theta)).$$

Then, $\mathfrak{R}(f)$ is defined by

$$\mathfrak{R}(f) \quad = \quad \left(\frac{\partial}{\partial u}\psi\right)(\varphi(\theta), a) \; - \; \left(\frac{\partial}{\partial u}\psi\right)(\theta, 0).$$

Then, it is well known that $\mathfrak{R}$ is a non-negative elliptic pseudodifferential operator of order 1. If $f =$ constant, then $\psi(\theta, u) = f$ and hence

$$\mathfrak{R}f \quad = \quad 0.$$

For $f = \cos m\theta$ or $\sin m\theta$ with $m \neq 0$, $\psi(\theta, u)$ is given by

$$\psi(\theta, u) \quad = \quad \frac{e^{-\frac{m\pi}{b}(u-a)} - e^{\frac{m\pi}{b}(u-a)}}{e^{\frac{m\pi}{b}a} - e^{-\frac{m\pi}{b}a}} f(\theta) \; + \; \frac{e^{\frac{m\pi}{b}u} - e^{-\frac{m\pi}{b}u}}{e^{\frac{m\pi}{b}a} - e^{-\frac{m\pi}{b}a}} f(\varphi^{-1}(\theta)),$$

which shows that

$$\frac{\partial}{\partial u}\psi(\theta, u)$$
$$= \quad \frac{m\pi}{b}\left(-\frac{e^{-\frac{m\pi}{b}(u-a)} + e^{\frac{m\pi}{b}(u-a)}}{e^{\frac{m\pi}{b}a} - e^{-\frac{m\pi}{b}a}} f(\theta) \; + \; \frac{e^{\frac{m\pi}{b}u} + e^{-\frac{m\pi}{b}u}}{e^{\frac{m\pi}{b}a} - e^{-\frac{m\pi}{b}a}} f(\varphi^{-1}(\theta))\right).$$

Hence,

$$\mathfrak{R}(f) \quad = \quad \frac{2m\pi}{b}\left\{\frac{e^{\frac{m\pi}{b}a} + e^{-\frac{m\pi}{b}a}}{e^{\frac{m\pi}{b}a} - e^{-\frac{m\pi}{b}a}} f(\theta) - \frac{f(\varphi(\theta)) + f(\varphi^{-1}(\theta))}{e^{\frac{m\pi}{b}a} - e^{-\frac{m\pi}{b}a}}\right\}.$$

If $f = \cos m\theta$, then $f(\varphi(\theta)) = f(\varphi^{-1}(\theta)) = f(\theta)$ and

$$\mathfrak{R}(f) \quad = \quad \frac{2m\pi}{b}\left(\frac{e^{\frac{m\pi}{b}a} + e^{-\frac{m\pi}{b}a} - 2}{e^{\frac{m\pi}{b}a} - e^{-\frac{m\pi}{b}a}}\right) f(\theta)$$
$$= \quad \frac{2m\pi}{b}\left(\frac{e^{\frac{m\pi}{2b}a} - e^{-\frac{m\pi}{2b}a}}{e^{\frac{m\pi}{2b}a} + e^{-\frac{m\pi}{2b}a}}\right) f(\theta) \quad = \quad \frac{2m\pi}{b}\left(\frac{1 - e^{-\frac{m\pi}{b}a}}{1 + e^{-\frac{m\pi}{b}a}}\right) f(\theta).$$

Similarly, if $f = \sin m\theta$, then $f(\varphi(\theta)) = f(\varphi^{-1}(\theta)) = -f(\theta)$ and

$$\begin{aligned}\mathfrak{R}(f) &= \frac{2m\pi}{b}\left(\frac{e^{\frac{m\pi}{b}a}+e^{-\frac{m\pi}{b}a}+2}{e^{\frac{m\pi}{b}a}-e^{-\frac{m\pi}{b}a}}\right)f(\theta)\\ &= \frac{2m\pi}{b}\left(\frac{e^{\frac{m\pi}{2b}a}+e^{-\frac{m\pi}{2b}a}}{e^{\frac{m\pi}{2b}a}-e^{-\frac{m\pi}{2b}a}}\right)f(\theta) = \frac{2m\pi}{b}\left(\frac{1+e^{-\frac{m\pi}{b}a}}{1-e^{-\frac{m\pi}{b}a}}\right)f(\theta).\end{aligned}$$

Hence, the spectrum of $\mathfrak{R}$ is given by

(15) $\operatorname{Spec}(\mathfrak{R})$

$$= \left\{0\right\} \cup \left\{\frac{2m\pi}{b}\left(\frac{1-e^{-\frac{m\pi}{b}a}}{1+e^{-\frac{m\pi}{b}a}}\right), \frac{2m\pi}{b}\left(\frac{1+e^{-\frac{m\pi}{b}a}}{1-e^{-\frac{m\pi}{b}a}}\right) \mid m=1,2,3,\cdots\right\}.$$

Lemma 2.6 shows that

$$\log \operatorname{Det} \mathfrak{R} = \log b. \tag{16}$$

Since the area of $S^1\left(\frac{b}{\pi}\right) \times [0,a]$ is $2ab$ and the length of $S^1\left(\frac{b}{\pi}\right) \times \{0\}$ is $2b$, which shows that

$$\log \frac{\operatorname{Area}\left(S^1\left(\frac{b}{\pi}\right)\times[0,a]\right)}{\operatorname{Length}\left(S^1\left(\frac{b}{\pi}\right)\times\{0\}\right)} = \log a.$$

From (14), (15) and (16), we obtain the following result.

Theorem 3.2. *The difference of the zeta-determinants of* $\log \operatorname{Det} \Delta_{\mathbb{K}}$ *and* $\log \operatorname{Det} \Delta_{S^1\left(\frac{b}{\pi}\right)\times[0,a],\mathrm{D}}$ *satisfies the following relation.*

$$\begin{aligned}&\log \operatorname{Det} \Delta_{\mathbb{K}} - \log \operatorname{Det} \Delta_{S^1\left(\frac{b}{\pi}\right)\times[0,a],\mathrm{D}}\\ &= \log \operatorname{Det} \mathfrak{R} + \log \frac{\operatorname{Area}\left(S^1\left(\frac{b}{\pi}\right)\times[0,a]\right)}{\operatorname{Length}\left(S^1\left(\frac{b}{\pi}\right)\times\{0\}\right)}.\end{aligned}$$

Remark : This formula is called the BFK-gluing formula for zeta-determinants. Theorem 3.2 shows that the BFK-gluing formula also holds on a non-orientable manifold. However, so far, we don't find a similar formula for a Möbius band because in this case the cutting hypersurface intersects the boundary.

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