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ON M-ITERATIVE SCHEME FOR MAPPINGS WITH ENRICHED (C) CONDITION IN BANACH SPACES

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Abstract. In this paper, we approximate fixed points of mappings satisfying enriched (C) condition under a modified three-step M-iterative scheme in a Banach space setting. First, we establish a weak convergence theorem and then obtain several strong convergence theorems for our iterative scheme under some mild conditions. A numerical example of mappings satisfying enriched (C) condition is used that does not satisfy the ordinary (C) condition to support our main outcome. Numerical computations and graphs obtained from different iterative schemes show that the studied scheme provides a better rate of convergence as compared to some other schemes of the literature. As an application of our main result, we provide a projection type iterative scheme for solving split feasibility problems (SFP) on a Hilbert space setting.

1. Introduction

In 2008, Suzuki [21] suggested the concept of (C) condition on mappings: a mapping S on a subset $\mathcal{U}$ of a Banach space is said to satisfy the (C) condition if

$$\frac{1}{2}||u - Su|| \leq ||u - u'|| \Rightarrow ||Su - Su'|| \leq ||u - u'|| \text{ for every } u, u' \in \mathcal{U}.$$

Furthermore, S is called nonexpansive if $||Su-Su'|| \leq ||u-u'||$, for all $u, u' \in \mathcal{U}$. The class of mappings endowed with (C) condition is the natural extension of the class of nonexpansive mappings.

Suzuki [21] established the following facts for these mappings.

Proposition 1.1. *Assume that $\mathcal{U}$ is any given subset of a Banach space $\mathcal{Z}$ such that $S : \mathcal{U} \to \mathcal{U}$.*

(i) *If S satisfies the (C) condition then for any $u, u' \in \mathcal{U}$, it follows that*

$$||u - Su'|| \leq 3||u - Su|| + ||u - u'||.$$

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(ii) *If $\mathcal{Z}$ is with the Opial's condition and S satisfies the (C) condition such that for any sequence $\{u_\eta\}$ which converges weakly to a point a^*, then $Sa^* = a^*$ if the estimate $\lim_{\eta\to\infty} ||u_\eta - Su_\eta|| = 0$ holds.*

If there is a point $a^* \in \mathcal{U}$ such that $Sa^* = a^*$, then a^* is known as a fixed point for S and throughout the research, we shall write $F_S := \{a^* \in \mathcal{U} : Sa^* = a^*\}$. Browder [8] and Göhde [12] independently, however, differently proved the existence of a fixed point for nonexpansive operators on a certain subset of uniformly convex Banach space [11] (UCBS, for short). Fixed point theory of nonexpansive mappings is very important and its applications can be found in many areas of applied sciences. Thus it is always desirable to suggest some extension of these mappings in order to extend their area of applications. To this end, in the year 2019, Berinde [7] was the first who presented a novel generalization of nonexpansive mappings and he named the new class of mappings as a class of enriched nonexpansive mappings. The class of enriched nonexpansive mappings and the classes of mappings satisfying (C) condition are important classes of nonlinear mappings. Indeed, we may observe that nonexpansive mappings are essentially enriched nonexpansive mappings with $b = 0$ and also satisfies the (C) condition. Hence it is very natural to consider a generalization of these mappings. To achieve the objective, recently, Ullah et al. [24] introduced the class of mappings with enriched (C) condition as follows.

Definition 1.2. [24] *A selfmap S on a subset $\mathcal{U}$ is said to satisfy the enriched (C) condition if and only if one can estimate a real constant $b \in [0,\infty)$ such that*

$$\frac{1}{2}||u - Su|| \leq (b+1)||u - u'|| \Rightarrow ||b(u - u') + Su - Su'|| \leq (b+1)||u - u'||,$$

for every $u, u' \in \mathcal{U}$.

Ullah et al. [24] proved that every mapping that satisfies the ordinary (C) condition also satisfies the enriched (C) condition. They further showed that there are some numerical examples of mappings with enriched (C) condition that do not satisfy the (C) condition. Thus they concluded that the class of mappings satisfying enriched (C) condition is more general than many other classes of nonlinear mappings properly including the class of mappings with condition (C). They also proved existence, weak convergence and strong convergence results for these mappings in a Hilbert space setting. In this paper, we generalize their results to the more general setting of Banach spaces.

Nonexpansive and enriched nonexpansive can be characterized using continuity. For example, we know that all nonexpansive and all enriched nonexpansive mappings are continuous on their whole domains. However, the continuity is not a characterization of mappings with enriched (C) condition in general. For instance, the following example shows that a mapping satisfying enriched (C) condition may not continuous on its whole domain.

Example 1.3. [24, Example 4] *Set S on $[0,3]$ as follows:*

$$Su = \begin{cases} 0 & \text{if } u \in [0,3), \\ 1 & \text{if } u \in \{3\}. \end{cases}$$

We see that S is neither nonexpansive nor enriched nonexpansive on $\mathcal{U}$ because S is not continuous on the point $3 \in \mathcal{U}$. However, S satisfies the enriched (C) condition.

We know that once an existence of a fixed point for a mapping is established then to approximate the value of the fixed point is always requested. The simplest iterative method was presented by Picard [17] for finding fixed points of some nonlinear mappings. Soon, Banach [5] proved that the iterative scheme of Picard [17] is suitable for finding fixed points of contraction mappings. After this, many authors proved the convergence of Picard iterative method for different classes of mappings. Picard iterative scheme is simple but there are some issues which we are facing when we implement it in the class of nonexpansive mappings. For example, when $Su = -u$ then S is nonexpansive and admits a unique fixed point $u = 0$. However the Picard iterative scheme of S does not converge to 0 if the initial approximation is different form 0. On the other hand, the Picard iterative scheme normally converges very slow and requires many step of iteration to reach to the requested fixed point. To cover these issues, many author constructed general iterative schemes (see, e.g., Mann [14], Ishikawa [13], Noor [15], Agarwal [4], Abbas iteration [1] and others).

For finding fixed points of mappings satisfying (C) condition, Ullah and Arshad [25] presented a novel three-step scheme called M-iteration. Under various assumptions they proved several convergence results and provided an example of mappings with (C) condition to support these results. Further, they used this example and proved numerically that M iteration is faster convergent to a fixed point corresponding to the leading Thakur iterative method [23] and leading two step Agarwal iterative method [4]. Also, Abdeljawad et. al. [2] used a faster iterative scheme for approximating fixed points of enriched nonlinear mappings. In this paper, the aim is to prove the convergence of M-iteration for mappings satisfying enriched (C) condition. To achieve this objective, we consider a mapping S on a subset $\mathcal{U}$ of a Banach space. Suppose S_λ denotes an averaged mappings of S, that is, $S_\lambda u = (1-\lambda)u + \lambda Su$, where $\lambda = \frac{1}{b+1}$. It is well-known that the fixed point set of S_λ coincides with the fixed point set of S. Using the mapping S_λ, the Agarwal [4], Thakur [23] and the M iterative method due to Ullah and Arshad [25] can be described in the following ways, respectively:

$$(1) \qquad \begin{cases} u_1 \in \mathcal{U}, \\ v_\eta = (1-\beta_\eta)u_\eta + \beta_\eta S_\lambda u_\eta, \\ w_{\eta+1} = (1-\alpha_\eta)S_\lambda u_\eta + \alpha_\eta S_\lambda v_\eta, \eta \in \mathbb{N}, \end{cases}$$

where $\alpha_\eta, \beta_\eta \in (0,1)$ and $\lambda = \frac{1}{b+1}$,

(2)
$$\begin{cases} u_1 \in \mathcal{U}, \\ w_\eta = (1-\beta_\eta)u_\eta + \beta_\eta S_\lambda u_\eta, \\ v_\eta = S_\lambda((1-\alpha_\eta)u_\eta + \alpha_\eta w_\eta), \\ u_{\eta+1} = S_\lambda v_\eta, \eta \in \mathbb{N}, \end{cases}$$

where $\alpha_\eta, \beta_\eta \in (0,1)$ and $\lambda = \frac{1}{b+1}$,

(3)
$$\begin{cases} u_1 \in \mathcal{U}, \\ w_\eta = (1-\alpha_\eta)u_\eta + \alpha_\eta S_\lambda u_\eta, \\ v_\eta = S_\lambda w_\eta, \\ u_{\eta+1} = S_\lambda v_\eta, \eta \in \mathbb{N}, \end{cases}$$

where $\alpha_\eta \in (0,1)$ and $\lambda = \frac{1}{b+1}$.

In this paper, first we employ the iterative scheme (3) to establish existence, weak convergence and strong convergence results for mappings satisfying enriched (C) condition under various conditions. After this, we use an example in which a mapping satisfies the enriched (C) condition that does not satisfy the ordinary (C) condition. Using this example, we shall show by that our iterative method converges better to a fixed point corresponding to some other scheme. For more details on M-iterative scheme, we refer the reader to [6, 18]. Our main outcome generalizes the corresponding results of Ullah et al. [24] from Hilbert spaces to Banach spaces and Ullah and Arshad [25] results from mappings with condition (C) to mappings with enriched (C) condition.

2. Preliminaries

We now provide some known results and facts of the literature that are needed in our main outcome.

Definition 2.1. [22, 3] *Let $\mathcal{U}$ be a convex closed nonempty subset of a UCBS $\mathcal{Z}$ and $\{u_\eta\}$ a bounded sequence contained in $\mathcal{Z}$. In this case, we denote the set $r(\mathcal{U}, \{u_\eta\}) = \inf\{\limsup_{\eta\to\infty} ||u_\eta - u|| : u \in \mathcal{U}\}$ is called the asymptotic radius and the $A(\mathcal{U}, u_\eta\}) = \{u \in \mathcal{U} : \limsup_{\eta\to\infty} ||u_\eta - u|| = r(\mathcal{U}, u_\eta)\}$ is called asymptotic center of the sequence $\{u_\eta\}$. Moreover, the set $A(\mathcal{U}, u_\eta\})$ contains one and only one element.*

Definition 2.2. [16] *A Banach space $\mathcal{Z}$ is said to be with the Opial's condition if any sequence $\{u_\eta\}$ in $\mathcal{Z}$ which admits weak limit, namely, $u \in \mathcal{Z}$ satisfies*

$$\limsup_{\eta\to\infty} ||u_\eta - u|| < \limsup_{\eta\to\infty} ||u_\eta - u'|| \quad \forall u' \in \mathcal{Z} - \{u\}.$$

Lemma 2.3. [24] *Suppose $\mathcal{U}$ is closed and convex in $\mathcal{Z}$ and $S : \mathcal{U} \to \mathcal{U}$. If S satisfies the enriched (C) condition with $b \in [0, \infty)$, then S_λ satisfies the Suzuki (C) condition, where $\lambda = \frac{1}{b+1}$.*

The following result was established by Schu [19], which is a very basic property of a given UCBS.

Lemma 2.4. *Consider a UCBS $\mathcal{Z}$ such that $0 < i \leq \alpha_\eta \leq j < 1$. In this case, if a real number $z \geq 0$ exists such that for any sequences $\{u_\eta\}$ and $\{v_\eta\}$ in $\mathcal{Z}$ satisfying $\limsup_{\eta\to\infty} ||u_\eta|| \leq z$, $\limsup_{\eta\to\infty} ||v_\eta|| \leq z$ and $\lim_{\eta\to\infty} ||\alpha_\eta u_\eta + (1-\alpha_\eta)v_\eta|| = z$. Then consequently the estimate $\lim_{\eta\to\infty} ||u_\eta - v_\eta|| = 0$ holds.*

3. Main outcome

The purpose of this section is to consider the iterative scheme (3) in the setting of mappings satisfying enriched (C) condition. Before to establish the convergence results, we need a very basic lemma as follows.

Lemma 3.1. *Suppose $\mathcal{U}$ is closed and convex in a UCBS $\mathcal{Z}$ and $S : \mathcal{U} \to \mathcal{U}$. If S satisfies the enriched (C) condition $F_S \neq \emptyset$ and $\{u_\eta\}$ is obtained from (3), then $\lim_{\eta\to\infty} ||u_\eta - a^*||$ exists for all $a^* \in F_S$.*

Proof. Consider any point $a^* \in F_S$. It follows that $a^* \in F_{S_\lambda}$. Hence using Lemma 2.3, we have S_λ satisfies the (C) condition, in particular, $\frac{1}{2}||a^* - S_\lambda a^*|| \leq ||a^* - u||$ implies $||S_\lambda a^* - S_\lambda u|| \leq ||a^* - u||$ for all $u \in \mathcal{U}$. Using this, we get

$$\begin{aligned} ||w_\eta - a^*|| &= ||(1-\alpha_\eta)u_\eta + \alpha_\eta S_\lambda u_\eta - a^*|| \\ &\leq (1-\alpha_\eta)||u_\eta - a^*|| + \alpha_\eta ||S_\lambda u_\eta - a^*|| \\ &\leq (1-\alpha_\eta)||u_\eta - a^*|| + \alpha_\eta ||u_\eta - a^*|| \\ &= ||u_\eta - a^*||, \end{aligned}$$

and

$$\begin{aligned} ||v_\eta - a^*|| &= ||S_\lambda w_\eta - a^*|| \\ &\leq ||w_\eta - a^*|| \leq ||u_\eta - a^*||. \end{aligned}$$

While using the above inequalities, we have

$$\begin{aligned} ||u_{\eta+1} - a^*|| &= ||S_\lambda v_\eta - a^*|| \\ &\leq ||v_\eta - a^*|| \leq ||u_\eta - a^*||. \end{aligned}$$

Thus, we obtained in the last that $||u_{\eta+1} - a^*|| \leq ||u_\eta - a^*||$. Thus, it follows that the sequence of real's $\{||u_\eta - a^*||\}$ is nonincreaing and also bounded. Accordingly, we get $\lim_{\eta\to\infty} ||u_\eta - a^*||$ exists, where $a^* \in F_{S_\lambda} = F_S$ is any point. □

The existence of a fixed point for the class of mappings satisfying enriched (C) condition is the following. This result is also useful for the next main result in the sequel.

Theorem 3.2. *Suppose $\mathcal{U}$ is closed and convex in a UCBS $\mathcal{Z}$ and $S : \mathcal{U} \to \mathcal{U}$. If S satisfies the enriched (C) condition $F_S \neq \emptyset$ and $\{u_\eta\}$ is obtained from (3). Then $F_S \neq \emptyset \iff \{u_\lambda\}$ is bounded in $\mathcal{U}$ and $\lim_{\eta\to\infty} ||S_\lambda u_\eta - u_\eta|| = 0$ where $\lambda = \frac{1}{b+1}$.*

Proof. First we note from the Lemma 3.1 that for any point $a^* \in F_S$ the estimate $\lim_{\eta\to\infty} ||u_\eta - a^*||$ eventually exists and $\{u_\eta\}$ is bounded in $\mathcal{U}$. Thus we put

$$\lim_{\eta\to\infty} ||u_\eta - a^*|| = z. \tag{4}$$

But we have proved in the Lemma 3.1 that

$$||w_\eta - a^*|| \leq ||u_\eta - a^*||.$$

It follows that

$$\Rightarrow \limsup_{\eta\to\infty} ||w_\eta - a^*|| \leq \limsup_{\eta\to\infty} ||u_\eta - a^*|| = z. \tag{5}$$

Now by Lemma 2.3, S_λ satisfies the (C) condition and $\frac{1}{2}||a^* - S_\lambda a^*|| \leq ||u_\eta - a^*||$, therefore, $||S_\lambda u_\eta - a^*|| \leq ||u_\eta - a^*||$. Hence

$$\limsup_{\eta\to\infty} ||S_\lambda u_\eta - a^*|| \leq \limsup_{\eta\to\infty} ||u_\eta - a^*|| = z. \tag{6}$$

Again, from the proof of Lemma 3.1,

$$||u_{\eta+1} - a^*|| \leq ||w_\eta - a^*||.$$

It follows that

$$\Rightarrow z \leq \liminf_{\eta\to\infty} ||w_\eta - a^*||. \tag{7}$$

From (5) and (7), we get

$$z = \lim_{\eta\to\infty} ||w_\eta - a^*||. \tag{8}$$

From (8), we have

$$z = \lim_{\eta\to\infty} ||w_\eta - a^*|| = \lim_{\eta\to\infty} ||(1-\alpha_\eta)(u_\eta - a^*) + \alpha_\eta(S_\lambda u_\eta - a^*)||.$$

Applying Lemma 2.4, we obtain

$$\lim_{\eta\to\infty} ||S_\lambda u_\lambda - u_\lambda|| = 0.$$

Conversely, we suppose that $\{u_\eta\} \subseteq \mathcal{U}$ is bounded and $\lim_{\eta\to\infty} ||u_\eta - S_\lambda u_\eta|| = 0$ and need to prove that $F_S \neq \emptyset$. For this purpose, it is sufficient to prove that for every choice of $a^* \in A(\mathcal{U}, \{u_\lambda\})$, the element $S_\lambda a^*$ contained in the set $A(\mathcal{U}, \{u_\eta\})$. Using Lemma 2.3, S_λ and Proposition 1.1(i), one has

$$\begin{aligned} r(S_\lambda a^*, \{u_\eta\}) &= \limsup_{\eta\to\infty} ||u_\eta - S_\lambda a^*|| \\ &= \limsup_{\eta\to\infty} (3||u_\eta - S_\lambda u_\eta|| + ||u_\eta - a^*||) \\ &= \limsup_{\eta\to\infty} ||u_\eta - a^*|| = r(a^*, \{u_\eta\}). \end{aligned}$$

We have seen that $S_\lambda a^* \in A(\mathcal{U}, \{u_\eta\})$. But the set $A(\mathcal{U}, \{u_\eta\})$ is consist of a one point. It follows that $a^* = S_\lambda a^*$. But $F_{S_\lambda} = F_S$, hence we proved that the set $F_S \neq \emptyset$. □

The first main result related to the convergence of our method (3) is the following weak convergence theorem.

Theorem 3.3. *Suppose $\mathcal{U}$ is convex and closed in a UCBS $\mathcal{Z}$ and $S : \mathcal{U} \to \mathcal{U}$. If S satisfies the enriched (C) condition $F_S \neq \emptyset$ and $\{u_\eta\}$ is obtained from (3). Then $\{u_\eta\}$ converges weakly to a fixed point of S provided that the Banach space $\mathcal{Z}$ satisfies the Opial's condition.*

Proof. Since $\mathcal{Z}$ is a UCBS and hence it follows that $\mathcal{Z}$ is reflexive and $\{u_\eta\}$ is bounded in $\mathcal{U}$ due to Theorem 3.2. It follows that one can find a weakly convergence subsequence which we denote here by $\{u_{\eta_i}\}$ of $\{u_\eta\}$. If u_1 denotes a weak limit of $\{u_{\eta_i}\}$ then we want to prove that u_1 is the weak limit of $\{u_\eta\}$ and a fixed point of S. For this, in the view of Theorem 3.2, it follows that $\lim_{\eta\to\infty} ||S_\lambda u_{\eta_i} - u_{\eta_i}|| = 0$. By Proposition 1.1(ii), we have $u_1 \in F_{S_\lambda}$, that is, u_1 is a fixed point of S_λ and hence a fixed point of S. Now it is remain to prove that u_1 is the weak limit of $\{u_\eta\}$. For this, we suppose that u_1 is not a weak limit of $\{u_\eta\}$ and so there exists another subsequence $\{u_{\eta_t}\}$ of $\{u_\eta\}$ which admits a weak limit u_2 such that $u_2 \neq u_1$. A similar calculation to above, we get u_2 is also a fixed point of S. Hence using the Opial's condition of $\mathcal{Z}$, we have

$$\begin{aligned} \lim_{\eta\to\infty} ||u_\eta - u_1|| &= \lim_{i\to\infty} ||u_{\eta_i} - u_1|| < \lim_{i\to\infty} ||u_{\eta_i} - u_2|| \\ &= \lim_{\eta\to\infty} ||u_\eta - u_2|| = \lim_{t\to\infty} ||u_{\eta_t} - u_2|| \\ &< \lim_{t\to\infty} ||u_{\eta_t} - u_1|| = \lim_{\eta\to\infty} ||u_\eta - u_1||. \end{aligned}$$

Subsequently, we proved that $\lim_{\eta\to\infty} ||u_\eta - u_1|| < \lim_{\eta\to\infty} ||u_\eta - u_1||$ which is a contradiction. Thus proof is finished. □

In this result, we obtain a strong convergence for our scheme (3) in the setting of mappings with enriched (C) condition.

Theorem 3.4. *Suppose $\mathcal{U}$ is convex in a UCBS $\mathcal{Z}$ and $S : \mathcal{U} \to \mathcal{U}$. If S satisfies the enriched (C) condition $F_S \neq \emptyset$ and $\{u_\eta\}$ is obtained from (3). Then $\{u_\eta\}$ converges strongly to a fixed point of S provided that $\mathcal{U}$ is compact.*

Proof. Due to the convexity of the set $\mathcal{U}$, there exists a subsequence $\{u_{\eta_i}\}$ that satisfies $\lim_{i\to\infty}||u_{\eta_i} - u_0|| = 0$, for some $u_0 \in \mathcal{U}$. Also, in the view of Theorem 3.2, $\lim_{\eta\to\infty}||u_{\eta_i} - S_\lambda u_{\eta_i}|| = 0$. According to Lemma 2.3, S_λ satisfies the (C) condition. Thus using Proposition 1.1(ii), to get

$$||u_{\eta_i} - S_\lambda u_0|| \leq 3||u_{\eta_i} - S_\lambda u_{\eta_i}|| + ||u_{\eta_i} - u_0||.$$

Now set $\lim_{i\to\infty}$ on both sides of the above estimates, we get $\lim_{i\to\infty}||u_{\eta_i} - S_\lambda u_0|| = 0$, that is, $u_{\eta_i} \to S_\lambda u_0$. It follows that $S_\lambda u_0 = u_0$. This proves that u_0 is a fixed point for S_λ and hence for S. But by Lemma 3.1, $\lim_{\eta\to\infty}||u_\eta - u_0||$ exists. Eventually, u_0 is a strong limit for $\{u_\eta\}$. This finishes the required proof. □

Theorem 3.5. *Suppose $\mathcal{U}$ is convex and closed in a UCBS $\mathcal{Z}$ and $S : \mathcal{U} \to \mathcal{U}$. If S satisfies the enriched (C) condition $F_S \neq \emptyset$ and $\{u_\eta\}$ is obtained from (3). Then $\{u_\eta\}$ converges strongly to a fixed point of S provided that $\liminf_{\eta\to\infty} dist(u_\eta, F_{S_\lambda}) = 0$.*

Proof. Since the proof of this theorem is elementary hence we neglect it. □

Senter and Dotson [20] essentially provided the following condition of selfmaps.

Definition 3.6. [20] *Take any convex subset $\mathcal{U}$ of a UCBS $\mathcal{Z}$ a selfmap $S : \mathcal{U} \to \mathcal{U}$ is said to satisfy the condition (I) provided that one can find a function ξ such that $\xi r = 0$ whenever $r = 0$, $\xi r > 0$ for every choice of $r > 0$ and $||u - Su|| \geq \xi(dist(u, F_S))$ for all elements u in the set $\mathcal{U}$.*

We close the section with the result that is proved under the condition (I).

Theorem 3.7. *Suppose $\mathcal{U}$ is convex and closed in a UCBS $\mathcal{Z}$ and $S : \mathcal{U} \to \mathcal{U}$. If S satisfies the enriched (C) condition $F_S \neq \emptyset$ and $\{u_\eta\}$ is obtained from (3). Then $\{u_\eta\}$ converges strongly to a fixed point of S provided that S_λ satisfies condition (I).*

Proof. In the view of Theorem 3.2, we have $\liminf_{\eta\to\infty}||u_\eta - S_\lambda u_\eta|| = 0$. Thus, by condition (I) of S_λ, it follows that $\liminf_{\eta\to\infty} \mathrm{dist}(u_\eta, F_{S_\lambda}) = 0$. Applying Theorem 3.5 that, $\{u_\eta\}$ strongly converges to a fixed point of S. □

4. Rate of convergence

The purpose of this section is to use a numerical example of a mapping satisfying enriched (C) condition that does not satisfy the ordinary (C) condition. Using this example, we shall show by that our iterative scheme converges better to a fixed point corresponding to some other methods.

Example 4.1. *Define a mapping* $S : [0.1, 2] \to [0.1, 2]$ *by* $Su = \frac{1}{5u}$*. We shall prove that* S *is enriched nonexpansive but not nonexpansive. First note that, when* $u = 1$ *and* $u' = 0.1$*, we have* $|Su - Su'| = |\frac{1}{5} - \frac{1}{0.5}| = |0.2 - 2| = 1.8 > 0.9 = |u - u'|$*. Hence* S *is not nonexpansive. Next, we prove that* S *is enriched nonexpansive. For this, if* $u = u'$*, then we have nothing to prove. Hence, we assume that* $u \neq u'$*. Consider*

$$|b(u - u') + \frac{1}{5u} - \frac{1}{5u'}| \leq (b + 1)|u - u'|.$$

if and only if

$$|b(u - u') + \frac{u' - u}{5uu'}| \leq (b + 1)|u - u'|,$$

if and only if

$$|u - u'| \times |b - \frac{1}{5uu'}|| \leq (b + 1)|u - u'|,$$

if and only if

$$|b - \frac{1}{5uu'}|| \leq (b + 1).$$

This hold for $b = 20$*. The unique fixed point of* S *is* $\sqrt{\frac{1}{5}} = 0.4472135955$*. Now using Example 4.1, we construct a table. Clearly, our method converges to the fixed point of* S *for any starting value. Moreover, our method provides better accuracy as compared some other schemes of the literature. This fact is displayed in the Table 1 and Figure 1.*

Table 1. Numerical convergence using Example 4.1.

η	**M**	**Thakur**	**Agarwal**
1	0.7	0.7	0.7
2	0.6633824473932	0.6697228677604	0.6688025787270
3	0.6314392941424	0.6426220547695	0.6409914365938
4	0.6037005007725	0.6184379026916	0.6162780317783
5	0.5797227795521	0.5969226327351	0.5943877892181
6	0.5590897417974	0.5778402643516	0.5750600534082
7	0.5414126329067	0.5609667669523	0.5580482960237
8	0.5263313023766	0.5460903622527	0.5431204716532
9	0.5135150761551	0.5330118902029	0.5300594135505
10	0.5026632670040	0.5215451561327	0.5186631719480
11	0.4935051560201	0.5115171895820	0.5087452176588
12	0.4857993820072	0.5027683637935	0.5001344594774
13	0.4793327651699	0.4951523458354	0.4926750509523
14	0.4739186556180	0.4885358678055	0.4862259865906
15	0.4693949320803	0.4827983269980	0.4806605069641
16	0.4656217853869	0.4778312360231	0.4758653452678
17	0.4624794116320	0.4735375520390	0.4717398547327
18	0.4598657192032	0.4698309179408	0.4681950578139

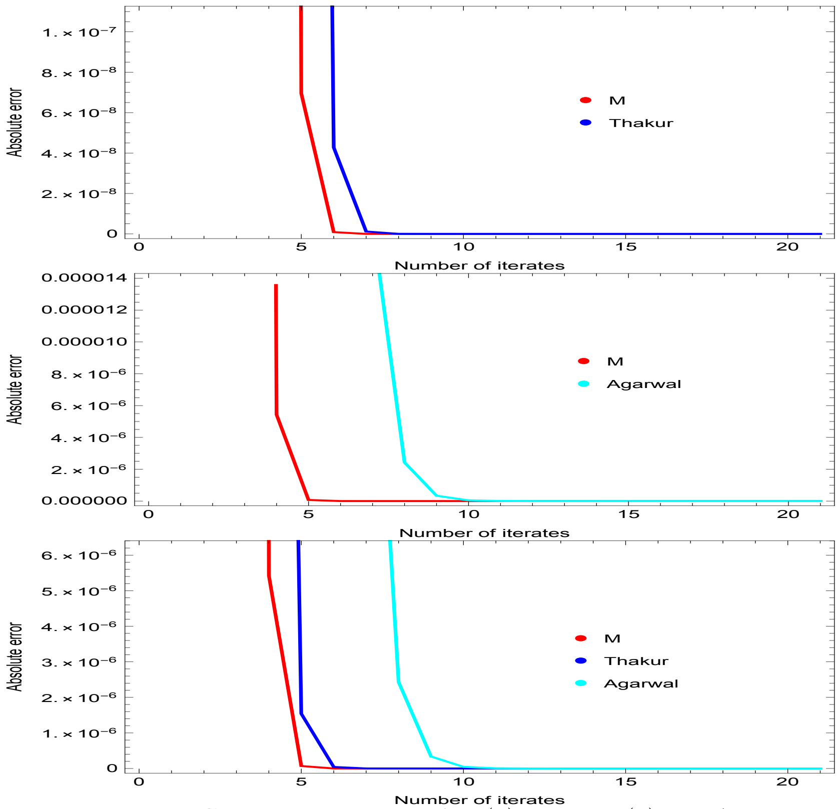

Figure 1. Graphical behavior of M (3), Thakur (2) and Agarwal (1) iterative schemes.

5. Application

In this section, we suggest an iterative scheme based on (3) for solving a split feasibility problem. It is known that if $\mathcal{Z}_1$ and $\mathcal{Z}_1$ denotes essentially two Hilbert spaces and $G : \mathcal{Z}_1 \to \mathcal{Z}_2$ is linear and bounded. Subsequently, a split feasibility problem (SFP) reads as follows (see, e.g., [10] and others):

$$\text{Search } a^* \in \mathcal{C} \text{ such that } Ga^* \in \mathcal{Q}, \tag{9}$$

provided that $\mathcal{C}$ and $\mathcal{Q}$ are compact, nonempty and convex.

As many knows, in [26], there are many problems that can be reduced to the form of SFPs. Thus here, we may assume that the SFP (9) admits atleast one solution and essentially we denote its solution set by $\mathcal{S}_s$. In [26], the author proved that if $a^* \in \mathcal{C}$ is a solution for (9) then a^* solved the equation $u = P^{\mathcal{C}}(I_d - \mu G^*(I_d - P^{\mathcal{Q}})G)u$ and vice versa provided that $P^{\mathcal{C}}$ and $P^{\mathcal{Q}}$ stand respectively for the nearest point projection (NPP) onto $\mathcal{C}$ and $\mathcal{Q}$, $\mu > 0$ and the operator G^* stands for adjoint operator of G. Byrne [9] proved that when ξ denotes a spectral radius of $G{*}G$ such that $0 < \mu < \frac{2}{\xi}$ the operator $S = P_{\mathcal{C}}(I_{id} - \mu G^*(I - P_{\mathcal{Q}})G)$ is nonexpansive. In this case, the $\mathcal{CQ}$ iterative scheme that generates a sequence as follows converges to a solution of the problem (9):

$$u_{\eta+1} = P^{\mathcal{C}}(I_d - \mu G^*(I_d - P^{\mathcal{Q}})G)u_\eta, \eta \geq 1.$$

In this research, our approach is to consider mappings with enriched (C) condition that generally not on their domains (as we observed in Example 1.3), instead of nonexpansive mappings which are continuous on their domains. Next we suggest a new type of iterative method which is based on (3).

Theorem 5.1. *Suppose the SFP (9) is such that $\mathcal{S}_s \neq \emptyset$, $0 < \mu < \frac{2}{\xi}$ and $P^{\mathcal{C}}(I_{id} - \mu G^*(I_{id} - P^{\mathcal{Q}})G)$ is endowed with enriched (C) condition. Assume that there exists $\alpha_\eta \in (0,1)$ so that $\{u_\eta\}$ a sequence generated as:*

$$\begin{cases} u_1 \in \mathcal{C}, \\ w_\eta = (1-\alpha_\eta)u_\eta + \alpha_\eta P_{\mathcal{C}}(I_{id} - \mu G^*(I_{id} - P^{\mathcal{Q}})G)u_\eta, \\ v_\eta = P^{\mathcal{C}}(I_{id} - \mu G^*(I_{id} - P_{\mathcal{Q}})G)w_\eta, \\ u_{\eta+1} = P^{\mathcal{C}}(I_{id} - \mu G^*(I_{id} - P^{\mathcal{Q}})G)v_\eta, \eta \geq 1. \end{cases}$$

Subsequently, $\{u_\eta\}$ converges weakly to some $a^* \in \mathcal{S}_s$, that is, to some solution of SFP (9).

Proof. We know that every Hilbert space satisfies the Opial's property. Now we put $S = P^{\mathcal{C}}(I_d - \mu G^*(I_d - P^{\mathcal{Q}})G)$. Then by assumption S_λ satisfies the enriched (C) condition. Therefore all the requirements for Theorem 3.3 are available and hence $\{u_\eta\}$ converges weakly to a point of F_S. But $F_S = \mathcal{S}_s$, it follows that $\{u_\eta\}$ converges weakly to a solution a^* of the SFP (9). □

Once a weak convergence for an iterative scheme is established then one looks for the strong convergence. In the literature, many authors studied only weak convergence for the problem (9) and some of these authors took interest in the strong convergence [26]. Our next result establishes a strong convergence result for (9) as follows.

Theorem 5.2. *Suppose the SFP (9) is such that* $\mathcal{S}_s \neq \emptyset$, $0 < \mu < \frac{2}{\xi}$ *and* $P^{\mathcal{C}}(I_{id} - \mu G^*(I_{id} - P^{\mathcal{Q}})G)$ *is endowed with enriched (C) condition. Assume that there exists* $\alpha_\eta \in (0, 1)$ *so that* $\{u_\eta\}$ *a sequence generated as:*

$$\begin{cases} u_1 \in \mathcal{C}, \\ w_\eta = (1 - \alpha_\eta)u_\eta + \alpha_\eta P_{\mathcal{C}}(I_{id} - \mu G^*(I_{id} - P^{\mathcal{Q}})G)u_\eta, \\ v_\eta = P^{\mathcal{C}}(I_{id} - \mu G^*(I_{id} - P^{\mathcal{Q}})G)w_\eta, \\ u_{\eta+1} = P^{\mathcal{C}}(I_d - \mu G^*(I_{id} - P^{\mathcal{Q}})G)v_\eta, \eta \geq 1. \end{cases}$$

Subsequently, $\{u_\eta\}$ *converges strongly to some* $a^* \in \mathcal{S}_s$, *that is, to some solution of SFP (9).*

Proof. Suppose that $S = P^{\mathcal{C}}(I_{id} - \mu G^*(I_{id} - P^{\mathcal{Q}})G)$. Then by assumption S_λ satisfies the enriched (C) condition. Therefore all the requirements for Theorem 3.4 are available and hence $\{u_\eta\}$ converges strongly to a point of F_S. But $F_S = \mathcal{S}_s$, it follows that $\{u_\eta\}$ converges weakly to a solution a^* of the SFP (9). □

6. Conclusions

In this article, we have started the approximation of fixed points for the recently suggested class of nonlinear mappings so-called mappings satisfying the enriched (C) condition. We utilized the iteration scheme M using the techniques of averaged mappings, and proved its weak and strong convergence. Some numerical computations are suggested for validation of the main outcome. The observations show that the effectiveness of M iterative scheme is still very high as compared the other iterative schemes. The presented outcome improves and extends the results of Ullah et al. [24] form Hilbert spaces to Banach spaces and also improve the rate of convergence. Moreover, the results of this paper extends the corresponding results of Thakur et al. [23] and Ullah and Arshad [25] form mappings satisfying (C) condition to the more general setting of mappings satisfying the enriched (C) condition. Eventually, we solve a SFP in a general setting of enriched mappings and general M-iterative scheme.

7. Open problems

Now, we pose an open problem which is as follows.
Problem 1. Can we improve all results of this paper to nonlinear spaces?

Data Availability.

The data in this paper will be made available by the author upon reasonable request.

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Conflicts of Interest.

The authors declare that they have no conflicts of interest.

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