Honam Mathematical J. **46** (2024), No. 4, pp. 522–537
https://doi.org/10.5831/HMJ.2024.46.4.522

# INVARIANT PSEUDOPARALLEL SUBMANIFOLDS OF AN ALMOST $\alpha$-COSYMPLECTIC $(\kappa, \mu, \nu)$-SPACE

Mehmet Atçeken and Gülsüm Yüca*

**Abstract.** In this article, we research the conditions for invariant submanifolds in an almost $\alpha$-cosymplectic $(\kappa, \mu, \nu)$ space to be pseudo-parallel, Ricci-generalized pseudo-parallel and 2-Ricci-generalized pseudo-parallel. We think that the results for the relations among the functions will contribute to differential geometry.

## 1. Introduction

An almost contact manifold is an odd-dimensional manifold $\widetilde{M}^{2n+1}$ which carries a field $\phi$ of endomorphism of the tangent space, a vector field $\xi$, called characteristic, and a 1-form $\eta$ satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{1}$$

where $I$ denotes the identity mapping of tangent space at each point of $M$. From (1), it follows

$$\phi\xi = 0, \quad \eta \circ \phi = 0 \quad rank(\phi) = 2n. \tag{2}$$

An almost contact manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta)$ is said to be normal if the tensor field $N = [\phi, \phi] + 2d\eta \otimes \xi = 0$, where $[\phi, \phi]$ denote the Nijenhuis tensor field of $\phi$. It is well known that any almost contact manifold $\widetilde{M}^{2n+1}(\phi, \xi, \eta)$ has a Riemannian metric such that

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{3}$$

for any vector fields $X, Y$ on $\widetilde{M}$[3]. Such metric $g$ is called compatible metric and the manifold $\widetilde{M}^{2n+1}$ together with the structure $(\phi, \eta, \xi, g)$ is called an almost contact metric manifold and is denoted by $\widetilde{M}^{2n+1}(\phi, \eta, \xi, g)$. The 2-form $\Phi$ of $\widetilde{M}^{2n+1}(\phi, \eta, \xi, g)$ is defined as $\Phi(X, Y) = g(\phi X, Y)$, and is called the

Received January 18, 2023. Accepted July 18, 2024.
2020 Mathematics Subject Classification. 53C15; 53C44, 53D10.
Key words and phrases. $\kappa, \mu, \nu)$-space, pseudoparallel, Ricci-generalized pseudoparallel and 2-pseudoparallel submanifolds.
*Corresponding author

fundamental form of $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$. If the Nijenhuis tensor vanishes, defined by

$$N_\phi(X,Y)=[\phi X,\phi Y]-\phi[\phi X,Y]-\phi[X,\phi Y]-\phi^2[X,Y]+2d\eta(X,Y)\xi,$$

then $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$ is said to be normal. It is obvious that a normal almost Kenmotsu is said to be Kenmotsu manifold. On the other hand, an almost contact metric manifold is known as Kenmotsu if and only if $(\widetilde{\nabla}_X\phi)Y=g(\phi X,Y)\xi-\eta(Y)\phi X$. An almost contact metric structure is cosymplectic if and only if $\widetilde{\nabla}\eta$ and $\widetilde{\nabla}\Phi$ are closed.

In the light of the above definitions, the generalization of almost Kenmotsu manifold, $\widetilde{M}^{2n+1}(\phi,\eta,\xi,g)$ is called almost $\alpha$-Kenmotsu manifold if $d\eta=0$ and $d\Phi=2\alpha\eta\wedge\Phi$, where $\alpha$ is a nonzero real constant[4].

As a generalization of these, a new notation has been introduced of an almost $\alpha$-cosymplectic manifold which is defined $d\eta=0$ and $d\Phi=2\alpha\eta\wedge\Phi$ for a real number $\alpha$. A normal almost $\alpha$-cosymplectic manifold is said to be $\alpha$-cosymplectic manifold, and it is either cosymplectic or $\alpha$-Kenmotsu under the condition $\alpha=0$ or $\alpha\neq 0$, respectively[6, 12].

It should be noted that almost $\alpha$-cosymplectic manifolds are generalizations of almost $\alpha$-Kenmotsu and almost cosymplectic manifolds.

It is well known that on a contact metric manifold $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$, the tensor $h$, defined by $2h=L_\xi\phi$, the following equalities are satisfied;

$$\text{(4)}\qquad \widetilde{\nabla}_X\xi=-\phi X-\phi hX,\quad h\phi+\phi h=0,\quad trh=tr\phi h=0, h\xi=0,$$

where $\widetilde{\nabla}$ is the Levi-Civita connection on $\widetilde{M}^{2n+1}$[4].

In [10], the authors studied the almost $\alpha$-cosymplectic $(\kappa,\mu,\nu)$-spaces under different conditions and gave an example in dimension 3.

Going beyond generalized $(\kappa,\mu)$-spaces, in [8], the notion of $(\kappa,\mu,\nu)$-contact metric manifold was introduced as follows[10];

$$\text{(5)}\qquad \widetilde{R}(X,Y)\xi=\eta(Y)[\kappa I+\mu h+\nu\phi h]X-\eta(X)[\kappa I+\mu h+\nu\phi h]Y,$$

for some smooth functions $\kappa,\mu$ and $\nu$ on $\widetilde{M}^{2n+1}$, where $\widetilde{R}$ denotes the Riemannian curvature tensor of $\widetilde{M}^{2n+1}$ and $X,Y$ are vector fields on $\widetilde{M}^{2n+1}$.

They proved that this type of manifold is intrinsically related to the harmonicity of the Reeb vector on contact metric 3-manifolds. Some authors have studied manifolds satisfying condition (5) but a non-contact metric structure. In this connection, P. Dacko and Z. Olszak defined an almost cosymplectic

$(\kappa,\mu,\nu)$-space as an almost cosymplectic manifold that satisfies (5), but with $\kappa,\mu$ and $\nu$ functions varying exclusively in the direction of $\xi$ in[4]. Later examples have been given for this type manifold[5].

Pseudoparallel submanifolds have been studied in different structures and working on[2, 1, 11]. In the present paper, we generalize the ambient space and research cases of existence or non-existence of pseudoparallel submanifold in $\alpha$-cosymplectic $(\kappa,\mu,\nu)$-space.

**Proposition 1.1.** *Given $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ an almost $\alpha$-cosymplectic $(\kappa,\mu,\nu)$-space, then*

$$h^2 = (\kappa+\alpha^2)\phi^2, \tag{6}$$

$$\begin{aligned}\widetilde{R}(\xi,X)Y &= \kappa[g(X,Y)\xi-\eta(Y)X]+\mu[g(hX,Y)\xi-\eta(Y)hX]\\ &+ \nu[g(\phi hX,Y)\xi-\eta(Y)\phi hX]\end{aligned} \tag{7}$$

$$(\widetilde{\nabla}_X\phi)Y = g(\alpha\phi X+hX,Y)\xi-\eta(Y)(\alpha\phi X+hX) \tag{8}$$

$$\widetilde{\nabla}_X\xi = -\alpha\phi^2X-\phi hX, \tag{9}$$

for all vector fields $X,Y$ on $\widetilde{M}^{2n+1}$[3].

*Proof.* From (5), we obtain

$$\widetilde{R}(X,\xi)\xi = -\kappa\phi^2X+\mu hX+\nu\phi hX. \tag{10}$$

Taking $\phi X$ instead of $X$ in (10), we have

$$\widetilde{R}(\phi X,\xi)\xi = \kappa\phi X+\mu h\phi X+\nu\phi h\phi X. \tag{11}$$

Applying $\phi$ to (11) and after the necessary revisions are made, we reach at

$$\phi\widetilde{R}(\phi X,\xi)\xi = \kappa\phi^2X+\mu hX+\nu\phi hX. \tag{12}$$

(10) and (12) give us

$$\widetilde{R}(X,\xi)\xi-\phi\widetilde{R}(\phi X,\xi)\xi = -2\kappa\phi^2X. \tag{13}$$

On the other hand, from ([13]) we know that

$$\begin{aligned}\widetilde{R}(X,\xi)\xi &= \alpha^2[\eta(X)\xi-X]+\alpha\phi hX+(\widetilde{\nabla}_\xi\phi h)X-(\widetilde{\nabla}_X\phi h)\xi\\ &= \alpha^2\phi^2X+2\alpha\phi hX+\mu hX-h^2X.\end{aligned} \tag{14}$$

This implies that

$$\widetilde{R}(\phi X,\xi)\xi = -\alpha^2\phi X+2\alpha hX+\mu\phi h\phi X+\nu hX-h^2\phi X.$$

Applying $\phi$ to the last equality, one can easily see

$$\phi\widetilde{R}(\phi X,\xi)\xi = -\alpha^2\phi^2X+2\alpha\phi hX+\mu hX+\nu\phi hX+h^2X. \tag{15}$$

From (14) and (15), we verify

$$\widetilde{R}(X,\xi)\xi-\phi\widetilde{R}(\phi X,\xi)\xi = 2[\alpha^2\phi^2-h^2]X. \tag{16}$$

(13) and (18) give us (6).

Also, from (5), we have

$$\begin{aligned} g(\widetilde{R}(X,Y)\xi,Z) &= \kappa g(\eta(Y)X-\eta(X)Y,Z)+\mu g(\eta(Y)hX-\eta(X)hY,Z) \\ &+ \nu g(\eta(Y)\phi hX-\eta(X)\phi hY,Z), \end{aligned}$$

for all $X,Y,Z \in \Gamma(T\widetilde{M})$. By using the properties of $\widetilde{R}$ and $\phi \circ h + h \circ \phi = 0$, we conclude that

$$\begin{aligned} g(\widetilde{R}(\xi,Z)X,Y) &= \kappa\eta(Y)g(X,Z)-\kappa\eta(X)g(Y,Z)+\mu\eta(Y)g(hX,Z) \\ &- \mu\eta(X)g(Y,hZ)+\nu\eta(Y)g(\phi hX,Z)-\nu\eta(X)g(Y,\phi hZ). \end{aligned} \tag{17}$$

Here, one can easily see

$$\begin{aligned} \widetilde{R}(\xi,Z)X &= \kappa[g(X,Z)\xi-\eta(X)Z]+\mu[g(hX,Z)\xi-\eta(X)hZ] \\ &+ \nu[g(\phi hX,Z)\xi-\eta(X)\phi hZ]. \end{aligned}$$

This completes the proof of (7).

Almost $\alpha$-cosymplectic manifolds are special classes of almost $\alpha$-cosymplectic $f$-manifolds, for the proof of (8), one can see [10]. So we do not need to give the proof here.

Taking $Y=\xi$ in (8), we observe

$$(\widetilde{\nabla}_X\phi)\xi = -\phi\widetilde{\nabla}_X\xi = -\alpha\phi X - hX. \tag{18}$$

Applying $\phi$ to (18) and after the necessary revisions are made, we can verify (9). Thus, the proof is completed. □

Now, let $M$ be an immersed submanifold of an almost $\alpha$-cosymplectic $(\kappa,\mu,\nu)$-space $\widetilde{M}^{2n+1}$. By $\Gamma(TM)$ and $\Gamma(T^{\perp}M)$, we denote the tangent and normal subspaces of $M$ in $\widetilde{M}$. Then, the Gauss and Weingarten formulae are, respectively, given by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \sigma(X,Y), \tag{19}$$

and

$$\widetilde{\nabla}_X V = -A_V X + \nabla^{\perp}_X V, \tag{20}$$

for all $X,Y \in \Gamma(TM)$ and $V \in \Gamma(T^{\perp}M)$, where $\nabla$ and $\nabla^{\perp}$ are the induced connections on $M$ and $\Gamma(T^{\perp}M)$ and $\sigma$ and $A$ are called the second fundamental form and shape operator of $M$, respectively, $\Gamma(TM)$ denote the set differentiable vector fields on $M$. They are related by

$$g(A_V X,Y) = g(\sigma(X,Y),V). \tag{21}$$

The covariant derivative of $\sigma$ is defined by

$$(\widetilde{\nabla}_X\sigma)(Y,Z) = \nabla^{\perp}_X\sigma(Y,Z) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z), \tag{22}$$

for all $X, Y, Z \in \Gamma(TM)$. If $\widetilde{\nabla}\sigma = 0$, then the submanifold is called its second fundamental form parallel.

By $R$, we denote the Riemannian curvature tensor of the submanifold $M$, we have the following Gauss equation

$$\begin{aligned} \widetilde{R}(X,Y)Z &= R(X,Y)Z + A_{\sigma(X,Z)}Y - A_{\sigma(Y,Z)}X + (\widetilde{\nabla}_X\sigma)(Y,Z) \\ &- (\widetilde{\nabla}_Y\sigma)(X,Z), \end{aligned} \tag{23}$$

for all $X, Y, Z \in \Gamma(TM)$.

For a $(0,k)$-type tensor field $T$, $k \geq 1$ and a $(0,2)$-type tensor field $A$ on a Riemannian manifold $(M,g)$, $Q(A,T)$-tensor field is defined by

$$\begin{aligned} & Q(A,T)(X_1, X_2, ..., X_k; X, Y) = -T((X \wedge_A Y)X_1, X_2, ..., X_k)... \\ - \; & T(X_1, X_2, ...X_{k-1}, (X \wedge_A Y)X_k), \end{aligned} \tag{24}$$

for all $X_1, X_2, ..., X_k, X, Y \in \Gamma(TM)$[9], where

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y. \tag{25}$$

**Definition 1.2.** *A submanifold of a Riemannian manifold $(M,g)$ is said to be pseudoparallel, 2-pseudoparallel, Ricci-generalized pseudoparallel and 2-Ricci-generalized pseudoparallel if*

$$\begin{aligned} \widetilde{R}\cdot\sigma \;\; &\textit{and} \;\; Q(g,\sigma) \\ \widetilde{R}\cdot\widetilde{\nabla}\sigma \;\; &\textit{and} \;\; Q(g,\widetilde{\nabla}\sigma) \\ \widetilde{R}\cdot\sigma \;\; &\textit{and} \;\; Q(S,\sigma) \\ \widetilde{R}\cdot\widetilde{\nabla}\sigma \;\; &\textit{and} \;\; Q(S,\widetilde{\nabla}\sigma) \end{aligned}$$

*are linearly dependent, respectively*[11].

Equivalently, these cases can be explained by the following way;

$$\widetilde{R}\cdot\sigma = L_1 Q(g,\sigma), \tag{26}$$

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_2 Q(g,\widetilde{\nabla}\sigma), \tag{27}$$

$$\widetilde{R}\cdot\sigma = L_3 Q(S,\sigma), \tag{28}$$

$$\widetilde{R}\cdot\widetilde{\nabla}\sigma = L_4 Q(S,\widetilde{\nabla}\sigma), \tag{29}$$

where the functions $L_1, L_2, L_3$ and $L_4$ are, respectively, defined on $M_1 = \{x \in M : \sigma(x) \neq g(x)\}$, $M_2 = \{x \in M : \widetilde{\nabla}\sigma(x) \neq g(x)\}$, $M_3 = \{x \in M : S(x) \neq \sigma(x)\}$ and $M_4 = \{x \in M : S(x) \neq \widetilde{\nabla}\sigma(x)\}$ and $S$ denotes the Ricci tensor of $M$.

Particularly, if $L_1 = 0$(resp. $L_2 = 0$), the submanifold is said to be semi-parallel(resp. 2-semiparallel)[1].

## 2. Invariant Submanifolds of an almost $\alpha$-cosymplectic $(\kappa,\mu,\nu)$ Space

Now, let $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ be an almost $\alpha$-cosymplectic $(\kappa,\mu,\nu)$-space and $M$ an immersed submanifold of $\widetilde{M}^{2n+1}$. If $\phi(T_xM)\subseteq T_xM$, for each point $x\in M$, then $M$ is said to be an invariant submanifold of $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ with respect to $\phi$. Hence, we will easily see that an invariant submanifold with respect to $\phi$ is also invariant with respect to $h$.

**Proposition 2.1.** *Let $M$ be an invariant submanifold of an almost $\alpha$-cosymplectic $(\kappa,\mu,\nu)$-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$ such that $\xi$ tangent to $M$. Then, the following equalities hold on $M$;*

$$
\begin{aligned}
R(X,Y)\xi &= \kappa[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX-\eta(X)hY] \\
&+ \nu[\eta(Y)\phi hX-\eta(X)\phi hY] \qquad (30)
\end{aligned}
$$

$$(\nabla_X\phi)Y = g(\alpha\phi X+hX,Y)\xi-\eta(Y)(\alpha\phi X+hX) \qquad (31)$$

$$\nabla_X\xi = -\alpha\phi^2X-\phi hX \qquad (32)$$

$$\phi\sigma(X,Y) = \sigma(\phi X,Y)=\sigma(X,\phi Y),\quad \sigma(X,\xi)=0, \qquad (33)$$

*where $\nabla$, $\sigma$ and $R$ denote the induced Levi-Civita connection, the shape operator and Riemannian curvature tensor of $M$, respectively.*

*Proof.* Since M is an invariant submanifold, from (9) and (19), we have

$$\sigma(X,\xi)=0,\quad\text{and}\quad A_V\xi=0,$$

for all $X\in\Gamma(TM)$ and $V\in\Gamma(T^{\perp}M)$. Also by using (23), we obtain (30).

On the other hand, tangent and normal components of expanding

$$
\begin{aligned}
(\widetilde{\nabla}_X\phi)Y &= \widetilde{\nabla}_XY-\phi\widetilde{\nabla}_XY \\
&= \nabla_X\phi Y+\sigma(X,\phi Y)-\phi\nabla_XY-\phi\sigma(X,Y) \\
&= (\nabla_X\phi)Y+\sigma(X,\phi Y)-\phi\sigma(X,Y)
\end{aligned}
$$

give us to (31) and (33), respectively.

Finally, the Gauss formulae and (9), we have

$$\widetilde{\nabla}_X\xi=\nabla_X\xi+\sigma(\xi,X)=-\alpha\phi^2X-\phi hX,$$

for all $X\in\Gamma(TM)$. Also tangent components of this give (32).
Furthermore, by using (7), (23) and the last term of (33), we get (30). □

In the rest of this paper, we will assume that $M$ is an invariant submanifold of an $\alpha$-cosymplectic $(\kappa,\mu,\nu)$-space $\widetilde{M}^{2n+1}(\phi,\xi,\eta,g)$. In this case, from (4), we have

$$\phi hX=-h\phi X, \qquad (34)$$

for all $X\in\Gamma(TM)$, that is, $M$ is also invariant with respect to the tensor field $h$.

We need the following theorem to guarantee for the second fundamental form $\sigma$ is not always identically zero.

**Theorem 2.2.** *Let $M$ be an invariant submanifold of an almost $\alpha$-cosymplectic $(\kappa, \mu, \nu)$-space $M^{2n+1}(\phi, \xi, \eta, g)$. Then, the second fundamental form $\sigma$ of $M$ is parallel $M$ is totally geodesic provided $\kappa \neq 0$.*

*Proof.* Let us suppose that $\sigma$ is parallel. From (22), we have

$$(\widetilde{\nabla}_X \sigma)(Y, Z) = \nabla^{\perp}_X \sigma(Y, Z) - \sigma(\nabla_X Y, Z) - \sigma(Y, \nabla_X Z) = 0, \tag{35}$$

for all vector fields $X, Y$ and $Z$ on $M^{2n+1}$. Setting $Z = \xi$ in (35) and taking into account (32) and (33), we have

$$\sigma(\nabla_X \xi, Y) = -\sigma(\alpha\phi^2 X + \phi h X, Y) = 0,$$

that is,

$$-\alpha\sigma(X, Y) + \phi\sigma(hX, Y) = 0. \tag{36}$$

Writing $hX$ of $X$ in (36) and by using (6) and (33), we obtain

$$\begin{aligned} -\alpha\sigma(hX, Y) &+ \phi\sigma(h^2 X, Y) = 0, \\ \alpha\sigma(hX, Y) &+ (\alpha^2 + \kappa)\phi\sigma(X, Y) = 0. \end{aligned} \tag{37}$$

From (36) and (37), we conclude that $\kappa\sigma(X, Y) = 0$, which proves our assertion. □

**Theorem 2.3.** *Let $M$ be an invariant pseudoparallel submanifold of an almost $\alpha$ cosymplectic $(\kappa, \mu, \nu)$-space $M^{2n+1}(\phi, \xi, \eta, g)$. Then, $M$ is either totally geodesic submanifold or the function $L_1$ satisfies $L_1 = \kappa \mp \sqrt{(\nu^2 - \mu^2)(\kappa + \alpha^2)}$, $\mu\nu(\kappa + \alpha^2) = 0$.*

*Proof.* We suppose that $M$ is an invariant pseudoparallel submanifold of an almost $\alpha$-cosymplectic $M^{2n+1}(\phi, \xi, \eta, g)$-space. Then, there exists a function $L_1$ on $M$ such that

$$(R(X, Y) \cdot \sigma)(U, V) = L_1 Q(g, \sigma)(U, V; X, Y),$$

for all vector fields $X, Y, U, V$ on $M$. By means of (24) and (26), we have

$$\begin{aligned} R^{\perp}(X, Y)\sigma(U, V) &- \sigma(R(X, Y)U, V) - \sigma(U, R(X, Y)V) \\ &= -L_1\{\sigma((X \wedge_g Y)U, V) + \sigma(U, (X \wedge_g Y)V)\}. \end{aligned} \tag{38}$$

Here taking $Y = U = \xi$ in (38) and taking into account of Proposition 2.1, we obtain

$$\begin{aligned} R^{\perp}(X, \xi)\sigma(\xi, V) &- \sigma(R(X, \xi)\xi, V) - \sigma(\xi, R(X, \xi)V) \\ &= -L_1\{\sigma((X \wedge_g \xi)\xi, V) + \sigma(\xi, (X \wedge_g \xi)V)\} \\ &= -L_1\{\sigma(X - \eta(X)\xi, V) + \sigma(\xi, \eta(V)X - g(X, V)\xi)\}, \end{aligned}$$

that is,

$$\sigma(R(X, \xi)\xi, V) = L_1\sigma(X, V). \tag{39}$$

By means of Proposition 2.1 and (5), we conclude that

$$(L_1-\kappa)\sigma(X,V)=\mu\sigma(hX,V)+\nu\sigma(\phi hX,V). \tag{40}$$

If $hX$ is substituted for $X$ at (40) and making use of (6) and (33), we obtain

$$(L-\kappa)\sigma(hX,V)=-(\kappa+\alpha^2)[\mu\sigma(X,V)+\nu\phi\sigma(X,V)]. \tag{41}$$

From (40) and (41), we reach at

$$[(L_1-\kappa)^2+(\kappa+\alpha^2)(\mu^2-\nu^2)]\sigma(X,V)=-2\mu\nu(\kappa+\alpha^2)\phi\sigma(X,V).$$

This yields to

$$(L_1-\kappa)^2+(\kappa+\alpha^2)(\mu^2-\nu^2)=0,\quad \mu\nu(\kappa+\alpha^2)=0 \ \text{ or } \ \sigma=0.$$

This completes the proof. □

From the Theorem 2.3, we have the following corollary.

**Corollary 2.4.** *Let $M$ be an invariant submanifold of an almost $\alpha$- cosymplectic $(\kappa,\mu,\nu)$-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then, $M$ is semiparallel if and only if $M$ is totally geodesic.*

**Theorem 2.5.** *Let $M$ be an invariant submanifold of an almost $\alpha$- cosymplectic $(\kappa,\mu,\nu)$-space $M^{2n+1}(\phi,\xi,\eta,g)$. If $M$ is a 2-pseudoparallel submanifold, then $M$ is either totally geodesic or the functions $\alpha,\kappa,\mu$, $\nu$ and $L_2$ satisfy $L_2=\kappa\mp\sqrt{(\kappa+\alpha^2)(\nu^2-\nu^2)}$ and $\mu\nu(\kappa+\alpha^2)=0$.*

*Proof.* Let us suppose that $M$ is a 2-pseudoparallel submanifold of $(\kappa,\mu,\nu)$-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then, by means of (27), there exists a function $L_2$ such that

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,Z)=L_2Q(g,\widetilde{\nabla}\sigma)(U,V,Z;X,Y),$$

for all vector fields $X,Y,Z,U,V$ on $M$. This implies that

$$\begin{aligned} R^{\perp}(X,Y)(\nabla_U\sigma)(V,Z) &- (\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,Z)-(\widetilde{\nabla}_U\sigma)(R(X,Y)V,Z)\\ &- (\widetilde{\nabla}_U\sigma)(V,R(X,Y)Z)=-L_2\{(\widetilde{\nabla}_{(X\wedge_g Y)U}\sigma)(V,Z)\\ &+ (\widetilde{\nabla}_U\sigma)((X\wedge_g Y)V,Z)\\ &+ (\widetilde{\nabla}_U\sigma)(V,(X\wedge_g Y)Z)\}. \end{aligned} \tag{42}$$

Taking $X=Z=\xi$ in (42), we can infer

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(V,\xi) &- (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(V,\xi)-(\nabla_U\sigma)(R(\xi,Y)V,\xi)\\ &- (\widetilde{\nabla}_U\sigma)(V,R(\xi,Y)\xi)=-L_2\{(\widetilde{\nabla}_{(\xi\wedge_g Y)U}\sigma)(V,\xi)\\ &+ (\widetilde{\nabla}_U\sigma)((\xi\wedge_g Y)V,\xi)\\ &+ (\widetilde{\nabla}_U\sigma)(V,(\xi\wedge_g Y)\xi)\}. \end{aligned} \tag{43}$$

Next, we will calculate each of these statements, respectively. Taking into account of (22), (32) and (33), we obtain

$$\begin{aligned} R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(V,\xi) &= R^{\perp}(\xi,Y)\{\nabla^{\perp}_U\sigma(V,\xi)-\sigma(\nabla_UV,\xi)-\sigma(\nabla_U\xi,V)\}\\ &= -R^{\perp}(\xi,Y)\sigma(\nabla_U\xi,V)\\ &= -R^{\perp}(\xi,Y)\sigma(-\alpha\phi^2U-\phi hU,V)\\ &= -\alpha R^{\perp}(\xi,Y)\sigma(U,V)+R^{\perp}(\xi,Y)\phi\sigma(hU,V). \end{aligned} \tag{44}$$

On the other hand, from (5), (23) and (33), by a direct calculation, we can infer

$$\begin{aligned} R(\xi,X)Y &= \kappa[g(Y,X)\xi-\eta(Y)X]+\mu[g(hY,X)\xi-\eta(Y)hX]\\ &+ \nu[g(X,\phi hY)\xi-\eta(Y)\phi hX]. \end{aligned} \tag{45}$$

Therefore,

$$\begin{aligned} (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(V,\xi) &= \nabla^{\perp}_{R(\xi,Y)U}\sigma(V,\xi)-\sigma(\nabla_{R(\xi,Y)U}V,\xi)-\sigma(\nabla_{R(\xi,Y)U}\xi,V)\\ &= -\sigma(\nabla_{R(\xi,Y)U}\xi,V)\\ &= \sigma(\alpha\phi^2R(\xi,Y)U+\phi hR(\xi,Y)U,V)\\ &= -\alpha\sigma(R(\xi,Y)U,V)+\sigma(\phi hR(\xi,Y)U,V)\\ &= -\alpha\sigma(-\kappa\eta(U)Y-\mu\eta(U)hY-\nu\eta(U)\phi hY,V)\\ &+ \sigma(-\kappa\eta(U)\phi hY-\mu\eta(U)\phi h^2Y-\nu\eta(U)\phi h\phi hY,V)\\ &= \alpha\kappa\eta(U)\sigma(V,Y)+\alpha\mu\eta(U)\sigma(hY,V)\\ &+ \alpha\nu\eta(U)\sigma(\phi hY,V)-\kappa\eta(U)\sigma(\phi hY,V)\\ &+ \mu(\kappa+\alpha^2)\eta(U)\sigma(\phi Y,V)+\nu(\kappa+\alpha^2)\sigma(V,Y). \end{aligned} \tag{46}$$

Furthermore, by using (32) and (45), we have

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(R(\xi,Y)V,\xi) &= \nabla^{\perp}_U\sigma(R(\xi,Y)V,\xi)-\sigma(\nabla_UR(\xi,Y)V,\xi)\\ &- \sigma(\nabla_U\xi,R(\xi,Y)V)\\ &= -\sigma(\nabla_U\xi,R(\xi,Y)V)=\sigma(\alpha\phi^2U+\phi hU,R(\xi,Y)V)\\ &= \alpha\sigma(\phi^2U,R(\xi,Y)V)+\sigma(\phi hU,R(\xi,Y)V)\\ &= -\alpha\sigma(U,-\kappa\eta(V)Y-\mu\eta(V)hY-\nu\eta(V)\phi hY)\\ &+ \sigma(\phi hU,-\kappa\eta(V)Y-\mu\eta(V)hY-\nu\eta(V)\phi hY)\\ &= \kappa\alpha\eta(V)\sigma(U,Y)+\mu\alpha\eta(V)\sigma(hY,U)\\ &+ \alpha\nu\eta(V)\sigma(U,\phi hY)-\kappa\eta(V)\sigma(\phi hU,Y)\\ &- \mu\eta(V)\sigma(\phi hU,hY)+\nu\eta(V)\sigma(hU,hY)\\ &= \kappa\alpha\eta(V)\sigma(U,Y)+\mu\alpha\eta(V)\sigma(hY,U)\\ &+ \alpha\nu\eta(V)\sigma(U,\phi hY)-\kappa\eta(V)\sigma(\phi hU,Y)\\ &+ \mu(\kappa+\alpha^2)\eta(V)\sigma(\phi U,Y)\\ &- \nu(\kappa+\alpha^2)\eta(V)\sigma(U,Y). \end{aligned} \tag{47}$$

The fourth term gives us

$$(\nabla_U\sigma)(V,R(\xi,Y)\xi) = (\nabla_U\sigma)(V,\kappa[\eta(Y)\xi-Y]-\mu hY-\nu\phi hY). \tag{48}$$

On the other hand, by the view of (25), (32) and (33), we obtain

$$\begin{aligned}(\widetilde{\nabla}_{(\xi\wedge_g Y)U}\sigma)(V,\xi) &= \nabla^{\perp}_{(\xi\wedge_g Y)U}\sigma(V,\xi)-\sigma(\nabla_{(\xi\wedge_g Y)U}V,\xi)\\ &- \sigma(V,\nabla_{(\xi\wedge_g Y)U}\xi)\\ &= \sigma(V,\alpha\phi^2(\xi\wedge_g Y)U+\phi h(\xi\wedge_g Y)U)\\ &= -\alpha\sigma(V,(\xi\wedge_g Y)U)+\sigma(V,(\xi\wedge_g Y)U)\\ &= \alpha\eta(U)\sigma(Y,V)-\eta(U)\sigma(\phi hY,V),\end{aligned} \tag{49}$$

and

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)((\xi\wedge_g Y)V,\xi) &= \nabla^{\perp}_U\sigma((\xi\wedge_g Y)V,\xi)-\sigma(\nabla_U(\xi\wedge_g Y)V,\xi)\\ &- \sigma((\xi\wedge_g Y)V,\nabla_U\xi)\\ &= \sigma(\alpha\alpha\phi^2U+\phi hU,g(Y,V)\xi-\eta(V)Y)\\ &= \alpha\eta(V)\sigma(Y,U)-\eta(V)\sigma(Y,\phi hU).\end{aligned} \tag{50}$$

Finally,

$$\begin{aligned}(\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi-Y) &= -(\widetilde{\nabla}_U\sigma)(V,Y)+(\widetilde{\nabla}_U\sigma)(V,\eta(Y)\xi)\\ &= -(\widetilde{\nabla}_U\sigma)(V,Y)+\nabla^{\perp}_U\sigma(V,\eta(Y)\xi)\\ &- \sigma(\nabla_UV,\eta(Y)\xi)-\sigma(V,\nabla_U\eta(Y)\xi)\\ &= -(\widetilde{\nabla}_U\sigma)(V,Y)-\sigma(V,U[\eta(Y)]\xi+\eta(Y)\nabla_U\xi)\\ &= -(\widetilde{\nabla}_U\sigma)(V,Y)+\eta(V)\sigma(\alpha\phi^2U+\phi hU,V)\\ &= -(\widetilde{\nabla}_U\sigma)(V,Y)-\alpha\eta(Y)\sigma(U,V)\\ &+ \eta(Y)\sigma(\phi hU,V).\end{aligned} \tag{51}$$

Substituting (44), (46), (47),(48), (49), (50) and (51) into (43), we reach at

$$\begin{aligned}&- \alpha R^{\perp}(\xi,Y)\sigma(U,V)+R^{\perp}(\xi,Y)\phi\sigma(U,V)-\kappa\alpha\eta(U)\sigma(V,Y)\\ &- \mu\alpha\eta(U)\sigma(V,hY)-\nu\alpha\eta(U)\sigma(V,\phi hY)+\kappa\eta(U)\sigma(V,\phi hY)\\ &- \mu(\kappa+\alpha^2)\eta(U)\sigma(\phi Y,V)-\nu(\kappa+\alpha^2)\eta(U)\sigma(V,Y)-\kappa\alpha\eta(V)\sigma(U,Y)\\ &- \alpha\mu\eta(V)\sigma(hY,U)-\alpha\nu\eta(V)\sigma(U,\phi hY)+\kappa\eta(V)\sigma(\phi hU,Y)\\ &- \mu(\kappa+\alpha^2)\eta(V)\sigma(\phi U,Y)+\nu(\kappa+\alpha^2)\eta(V)\sigma(U,Y)\\ &- (\nabla_U\sigma)(V,\kappa[\eta(Y)\xi-Y]-\mu hY-\nu\phi hY)=-L_2\{\alpha\eta(U)\sigma(V,Y)\\ &- \eta(U)\sigma(\phi hY,V)+\alpha\eta(V)\sigma(Y,U)-\eta(V)\sigma(Y,\phi hU)\\ &- (\nabla_U\sigma)(V,Y)-\alpha\eta(Y)\sigma(U,V)+\eta(Y)\sigma(\phi hU,V)\}.\end{aligned}$$

Here, taking $V=\xi$ in the last equality and using (33), we conclude that

$$\begin{aligned} L_2\{\alpha\sigma(U,Y) &- \sigma(Y,\phi hU)-(\widetilde{\nabla}_U\sigma)(Y,\xi)\} = \kappa\alpha\sigma(U,Y)+\alpha\mu\sigma(U,hY) \\ &+ \alpha\nu\sigma(U,\phi hY)-\kappa\alpha\sigma(\phi hY,U)+\mu(\kappa+\alpha^2)\sigma(\phi U,Y) \\ &- \nu(\kappa+\alpha^2)\sigma(U,Y) \\ &+ (\widetilde{\nabla}_U\sigma)(\xi,\kappa[\eta(Y)\xi-Y]-\mu hY-\nu\phi hY), \end{aligned} \tag{52}$$

where

$$\begin{aligned} (\widetilde{\nabla}_U\sigma)(Y,\xi) &= -\sigma(\nabla_U\xi,Y)=\sigma(\alpha\phi^2U+\phi hU,Y) \\ &= -\alpha\sigma(U,Y)+\phi\sigma(hU,Y) \end{aligned} \tag{53}$$

and

$$\begin{aligned} &(\widetilde{\nabla}_U\sigma)(\xi,\kappa[\eta(Y)\xi-Y]-\mu hY-\nu\phi hY) \\ =& -\sigma(\nabla_U\xi,\kappa[\eta(Y)\xi-Y]-\mu hY-\nu\phi hY) \\ =& \ \sigma(\alpha\phi^2U+\phi hU,\kappa[\eta(Y)\xi-Y]-\mu hY-\nu\phi hY) \\ =& -\alpha\sigma(U,\kappa[\eta(Y)\xi-Y]-\mu hY-\nu\phi hY) \\ +& \ \sigma(\phi hU,\kappa[\eta(Y)\xi-Y]-\mu hY-\nu\phi hY) \\ =& \ \kappa\alpha\sigma(U,Y)+\alpha\mu\sigma(hY,U)+\alpha\nu\sigma(\phi hY,U) \\ -& \ \kappa\sigma(\phi hU,Y)+\mu(\kappa+\alpha^2)\sigma(\phi U,Y)-\nu(\kappa+\alpha^2)\sigma(U,Y). \end{aligned} \tag{54}$$

Substituting (53) and (54) into (52), we get

$$\begin{aligned} &[\alpha L_2-\kappa\alpha+\nu(\kappa+\alpha^2)]\sigma(U,Y)+[\kappa-L_2-\alpha\nu]\phi\sigma(hU,Y) \\ -& \ \mu(\kappa+\alpha^2)\phi\sigma(U,Y)-\alpha\mu\sigma(hU,Y)=0. \end{aligned} \tag{55}$$

If $hU$ is written instead of $U$ in (55) and using (6), (9) and (33), we have

$$\begin{aligned} [\alpha L_2-\kappa\alpha &+ \nu(\kappa+\alpha^2)]\sigma(hU,Y)-(\kappa+\alpha^2)[\kappa-L_2-\alpha\nu]\phi\sigma(U,Y) \\ &- \mu(\kappa+\alpha^2)\phi\sigma(hU,Y)+\alpha\mu(\kappa+\alpha^2)\sigma(U,Y)=0. \end{aligned} \tag{56}$$

From (55) and (56), for $\kappa\neq 0$, we obtain

$$[(L_2-\kappa)^2-(\kappa+\alpha^2)(\nu^2-\mu^2)]\sigma(U,Y)+2\mu\nu(\kappa+\alpha^2)\phi\sigma(U,Y)=0.$$

Since the vectors $\phi\sigma(U,Y)$ and $\sigma(U,Y)$ are orthogonal, we conclude that $M$ is a totally geodesic or

$$\mu\nu(\kappa+\alpha^2)=0,$$

and

$$L_2=\kappa\mp\sqrt{(\kappa+\alpha^2)(\nu^2-\nu^2)}.$$

Thus, the proof is completed. □

From Theorem 2.5, we have the following corollary.

**Corollary 2.6.** *Let $M$ be an invariant submanifold of an almost $\alpha$- cosymplectic $(\kappa,\mu,\nu)$-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then, $M$ is 2-semiparallel if and only if $M$ is totally geodesic.*

**Theorem 2.7.** *Let $M$ be an invariant Ricci-generalized pseudoparallel submanifold of an almost $\alpha$-cosymplectic $(\kappa,\mu,\nu)$-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then, $M$ is either totally geodesic submanifold or the functions $L_3$, $\kappa,\mu,\nu$ and $\alpha$ satisfy the condition*

$$L_3=\frac{1}{2n}\left(1\mp\frac{1}{\kappa}\sqrt{(\kappa+\alpha^2)(\nu^2-\mu^2)}\right),\quad \mu\nu(\kappa+\alpha^2)=0.$$

*Proof.* We suppose that $M$ is an invariant Ricci-generalized pseudoparallel. Then there exists a function $L_3$ on $M$ such that

$$(\widetilde{R}(X,Y)\cdot\sigma)(U,V)=L_3Q(S,\sigma)(U,V;X,Y),$$

for all vector fields $X,Y,U,V$ on $M$. This implies that

$$\begin{aligned} R^{\perp}(X,Y)\sigma(U,V) &- \sigma(R(X,Y)U,V)-\sigma(U,R(X,Y)V)\\ &= -L_3\{\sigma((X\wedge_S Y)U,V)+\sigma(U,(X\wedge_S Y)V)\}\\ &= -L_3\{\sigma(X,V)S(U,Y)-\sigma(Y,V)S(X,U)\\ &+ \sigma(U,X)S(Y,V)-\sigma(U,Y)S(X,V)\}. \end{aligned}\tag{57}$$

By a direct calculation, we obtain

$$S(X,\xi)=2n\kappa\eta(X).\tag{58}$$

Taking $U=\xi$ in (57) and by view means of (5), (33) and (58), we have

$$\sigma(R(X,Y)\xi,V)=2n\kappa L_2\{\sigma(X,V)-\sigma(Y,V)\},$$

that is,

$$\begin{aligned} 2n\kappa L_2\{\sigma(X,V)-\sigma(Y,V)\} &= \sigma(\kappa[\eta(Y)X-\eta(X)Y]+\mu[\eta(Y)hX\\ &- \eta(X)hY]+\nu[\eta(Y)\phi hX-\eta(X)\phi hY],V). \end{aligned}$$

This yields to

$$\kappa(2nL_3-1)\sigma(X,V)=\mu\sigma(hX,V)+\nu\phi\sigma(hX,V).\tag{59}$$

If $hX$ is written instead of $X$ and using (6) and (33), we get

$$\kappa(2nL_3-1)\sigma(hX,V)=-(\kappa+\alpha^2)\{\mu\sigma(X,V)-\nu\phi\sigma(X,V)\}.\tag{60}$$

From (59) and (60), we can derive

$$\begin{aligned} \{\kappa^2(2nL_3-1)^2 &+ (\kappa+\alpha^2)(\mu^2-\nu^2)\}\sigma(X,V)\\ &= -2\mu\nu(\kappa+\alpha^2)\phi\sigma(X,V). \end{aligned}$$

Since $\sigma$ and $\phi\sigma$ are orthogonal vectors, it follows that

$$\kappa^2(2nL_3-1)^2+(\kappa+\alpha^2)(\mu^2-\nu^2)=0,\quad \mu\nu(\kappa+\alpha^2)=0,$$

which proves our assertions. □

**Theorem 2.8.** *Let $M$ be an invariant 2-Ricci-generalized pseudoparallel submanifold of an almost $\alpha$-cosymplectic $(\kappa,\mu,\nu)$-space $M^{2n+1}(\phi,\xi,\eta,g)$. Then, $M$ is either totally geodesic submanifold or the function $L_4$ satisfies*

$$L_4=\frac{1}{2n}\left(1\mp\frac{1}{\kappa}\sqrt{(\kappa+\alpha^2)(\nu^2-\mu^2)}\right)\quad and\quad \mu\nu(\kappa+\alpha^2)=0.$$

*Proof.* Given $M$ is an invariant 2-Ricci-generalized pseudoparallel submanifold, we have

$$(\widetilde{R}(X,Y)\cdot\widetilde{\nabla}\sigma)(U,V,W)=L_4Q(S,\widetilde{\nabla}\sigma)(U,V,W;X,Y)$$

for all vector fields $X,Y,U,V,W$ on $M$. That means

$$\begin{aligned}R^{\perp}(X,Y)(\widetilde{\nabla}_U\sigma)(V,W) &- (\widetilde{\nabla}_{R(X,Y)U}\sigma)(V,W)-(\widetilde{\nabla}_U\sigma)(R(X,Y)V,W)\\ &- (\widetilde{\nabla}_U\sigma)(V,R(X,Y)W)=-L_4\{(\widetilde{\nabla}_{(X\wedge_S Y)U}\sigma)(V,W)\\ &+ (\widetilde{\nabla}_U\sigma)((X\wedge_S Y)V,W)\\ &+ (\widetilde{\nabla}_U\sigma)(V,(X\wedge_S Y)W)\}.\end{aligned}\tag{61}$$

Taking $X=V=\xi$ in (61), we obtain

$$\begin{aligned}R^{\perp}(\xi,Y)(\widetilde{\nabla}_U\sigma)(\xi,W) &- (\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(\xi,W)-(\widetilde{\nabla}_U\sigma)(R(\xi,Y)\xi,W)\\ &- (\widetilde{\nabla}_U\sigma)(\xi,R(\xi,Y)W)=-L_4\{(\widetilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi,W)\\ &+ (\widetilde{\nabla}_U\sigma)(\xi\wedge_S Y)\xi,W)\\ &+ (\widetilde{\nabla}_U\sigma)(\xi,(\xi\wedge_S Y)W)\}.\end{aligned}\tag{62}$$

Now, let us calculate each of these terms separately. First,

$$\begin{aligned}R^{\perp}(\xi,Y)\{-\sigma(\nabla_U\xi,W)\} &= R^{\perp}(\xi,Y)\sigma(\alpha\phi^2U+\phi hU,W)\\ &= -\alpha R^{\perp}(\xi,Y)\sigma(U,W)\\ &+ R^{\perp}(\xi,Y)\sigma(\phi hU,W).\end{aligned}\tag{63}$$

Making use of (6), (32) and (45), we can calculate the second term as

$$\begin{aligned}(\widetilde{\nabla}_{R(\xi,Y)U}\sigma)(W,\xi) &= -\sigma(\nabla_{R(\xi,Y)U}\xi,W)=\alpha\sigma(\phi^2R(\xi,Y)U,W)\\ &+ \sigma(\phi h\nabla_{R(\xi,Y)U},W)\\ &= \alpha\kappa\eta(U)\sigma(Y,W)+\alpha\mu\eta(U)\sigma(hY,W)\\ &+ \alpha\nu\eta(U)\sigma(\phi hY,W)\\ &- \kappa\eta(U)\sigma(\phi hY,W)+\mu(\kappa+\alpha^2)\eta(U)\sigma(\phi Y,W)\\ &- \nu\eta(U)\sigma(\phi h\phi hY,W)\\ &= \alpha\kappa\eta(U)\sigma(Y,W)+\alpha\mu\eta(U)\sigma(hY,W)\\ &+ \alpha\nu\eta(U)\sigma(\phi hY,W)\\ &- \kappa\eta(U)\sigma(\phi hY,W)+\mu\eta(U)(\kappa+\alpha^2)\sigma(\phi Y,W)\\ &- \nu(\kappa+\alpha^2)\eta(U)\sigma(Y,W),\end{aligned}\tag{64}$$

In view of (9) and (22), we observe

$$\begin{aligned}
&(\widetilde{\nabla}_U\sigma)(R(\xi,Y)\xi,W)\\
&= (\widetilde{\nabla}_U\sigma)(\kappa[\eta(Y)\xi-Y]-\mu hY-\nu\phi hY,W). \qquad (65)
\end{aligned}$$

In the same way,

$$\begin{aligned}
(\widetilde{\nabla}_U\sigma)(R(\xi,Y)W,\xi) &= -\sigma(\nabla_U\xi,R(\xi,Y)W)=\sigma(\alpha\phi^2U+\phi hU,R(\xi,Y)W)\\
&= \alpha\kappa\eta(W)\sigma(U,Y)+\alpha\mu\eta(W)\sigma(hY,W)\\
&+ \alpha\nu\eta(W)\sigma(U,\phi hY)-\kappa\eta(W)\sigma(\phi hU,Y)\\
&- \mu\eta(W)\sigma(\phi h^2U,Y)+\nu\eta(W)\sigma(h^2U,Y)\\
&= \alpha\kappa\eta(W)\sigma(U,Y)+\alpha\mu\eta(W)\sigma(hY,W)\\
&+ \alpha\nu\eta(W)\sigma(U,\phi hY)-\kappa\eta(W)\sigma(\phi hU,Y)\\
&+ \mu(\kappa+\alpha^2)\eta(W)\sigma(\phi U,Y)\\
&+ \nu(\kappa+\alpha^2)\eta(W)\sigma(U,Y), \qquad (66)
\end{aligned}$$

$$\begin{aligned}
(\widetilde{\nabla}_{(\xi\wedge_S Y)U}\sigma)(\xi,W) &= -\sigma(\nabla_{(\xi\wedge_S Y)U}\xi,W)\\
&= \sigma(\alpha\phi^2(\xi\wedge_S Y)U+\phi h(\xi\wedge_S Y)U,W)\\
&= -\alpha\sigma(S(Y,U)\xi-S(\xi,U)Y,W)\\
&+ \sigma(\phi h[S(Y,U)\xi-S(\xi,U)Y],W)\\
&= 2n\kappa\eta(U)\{\alpha\sigma(Y,W)-\sigma(\phi hY,W)\}, \qquad (67)
\end{aligned}$$

$$\begin{aligned}
(\widetilde{\nabla}_U\sigma)((\xi\wedge_S Y)\xi,W) &= -\sigma(\nabla_U(\xi\wedge_S Y)\xi,W)\\
&= (\widetilde{\nabla}_U\sigma)(S(\xi,Y)\xi-S(\xi,\xi)Y,W)\\
&= 2n\{(\nabla_U\sigma)(\kappa\eta(Y)\xi,W)-(\nabla_U\sigma)(\kappa Y,W)\}\\
&= 2n\{-\sigma(U[\kappa\eta(Y)]\xi+\kappa\eta(Y)\nabla_U\xi,W)\\
&- (\nabla_U\sigma)(\kappa Y,W)\}\\
&= 2n\{-\kappa\alpha\eta(Y)\sigma(U,W)+\kappa\eta(Y)\sigma(\phi hU,W)\\
&- (\nabla_U\sigma)(\kappa Y,W)\}. \qquad (68)
\end{aligned}$$

Finally,

$$\begin{aligned}
(\widetilde{\nabla}_U\sigma)(\xi,(\xi\wedge_S Y)W) &= -\sigma(\nabla_U\xi,(\xi\wedge_S Y)W)\\
&= \sigma(\alpha\phi^2U+\phi hU,S(Y,W)\xi-S(\xi,W)Y)\\
&= 2n\kappa\alpha\eta(W)\sigma(U,Y)\\
&- 2n\kappa\eta(W)\sigma(\phi hU,Y). \qquad (69)
\end{aligned}$$

Consequently, substituting (63), (64), (65), (66), (67), (68) and (69) into (62), we reach at

$$\begin{aligned} &- \alpha R^{\perp}(\xi, Y)\sigma(U, W) + R^{\perp}(\xi, Y)\sigma(\phi hU, W) - \alpha\kappa\eta(U)\sigma(Y, W) \\ &- \alpha\mu\eta(U)\sigma(hY, W) - \alpha\nu\eta(U)\sigma(\phi hY, W) + \kappa\eta(U)\sigma(\phi hY, W) \\ &- \mu\eta(U)(\kappa + \alpha^2)\sigma(\phi Y, W) + \nu(\kappa + \alpha^2)\eta(U)\sigma(Y, W) \\ &- (\nabla_U\sigma)(\kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY, W) - \alpha\kappa\eta(W)\sigma(U, Y) \\ &- \alpha\mu\eta(W)\sigma(hY, W) - \alpha\nu\eta(W)\sigma(U, \phi hY) + \kappa\eta(W)\sigma(\phi hU, Y) \\ &- \mu(\kappa + \alpha^2)\eta(W)\sigma(\phi U, Y) + \nu(\kappa + \alpha^2)\eta(W)\sigma(U, Y) \\ &= -L_4\{2n\kappa\alpha\eta(U)\sigma(Y, W) - 2n\kappa\eta(U)\sigma(\phi hY, W) - 2n\kappa\alpha\eta(Y)\sigma(U, W) \\ &+ 2n\kappa\eta(Y)\sigma(\phi hU, W) - 2n(\nabla_U\sigma)(\kappa Y, W) + 2n\alpha\kappa\eta(W)\sigma(U, Y) \\ &- 2n\kappa\eta(W)\sigma(\phi hU, Y)\}. \end{aligned}$$

In the last equality, putting $W = \xi$, we have

$$\begin{aligned} 2nL_4\{(\nabla_U\sigma)(\kappa Y, \xi) &- \kappa\alpha\sigma(U, Y) + \kappa\sigma(\phi hU, Y)\} = \nu(\kappa + \alpha^2)\sigma(U, Y) \\ &- \alpha\kappa\sigma(U, Y) - \alpha\mu\sigma(hY, U) - \alpha\nu\sigma(\phi hU, Y) \\ &- \mu(\kappa + \alpha^2)\sigma(\phi U, Y) + \kappa\sigma(\phi hU, Y) \\ &- (\nabla_U\sigma)(\kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY, \xi), \end{aligned} \tag{70}$$

where

$$\begin{aligned} (\nabla_U\sigma)(\kappa Y, \xi) &= -\sigma(\nabla_U\xi, \kappa Y) = \sigma(\alpha\phi^2 U + \phi hU, \kappa Y) \\ &= -\alpha\kappa\sigma(U, Y) + \kappa\sigma(\phi hU, Y), \end{aligned} \tag{71}$$

and

$$\begin{aligned} &(\nabla_U\sigma)(\kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY, \xi) \\ &= -\sigma(\nabla_U\xi, \kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY) \\ &= \sigma(\alpha\phi^2 U + \phi hU, \kappa[\eta(Y)\xi - Y] - \mu hY - \nu\phi hY) \\ &= \alpha\kappa\sigma(U, Y) + \alpha\mu\sigma(U, hY) + \alpha\nu\sigma(U, \phi hY) \\ &- \kappa\sigma(\phi hU, Y) + \mu(\kappa + \alpha^2)\sigma(\phi U, Y) - \nu(\kappa + \alpha^2)\sigma(U, Y). \end{aligned} \tag{72}$$

(71) and (72) are put in (70), we conclude that

$$\begin{aligned} [\kappa\alpha(2nL_4 - 1) &+ (\kappa + \alpha^2)(\nu - \mu\phi)]\sigma(U, Y) \\ &- [\kappa(2nL_4 - 1)\phi + \alpha(\nu\phi + \mu)]\sigma(hU, Y) = 0. \end{aligned} \tag{73}$$

Here $hU$ is written instead of $U$ and taking into account of (6) and (33), we have

$$\begin{aligned} [\kappa\alpha(2nL_4 - 1) &+ (\kappa + \alpha^2)(\nu - \mu\phi)]\sigma(hU, Y) \\ &+ [\kappa(2nL_4 - 1)\phi + \alpha(\nu\phi + \mu)](\kappa + \alpha^2)\sigma(U, Y) = 0. \end{aligned} \tag{74}$$

From (73) and (74), it follows for $\kappa \neq 0$,

$$[\kappa^2(2nL_4 - 1)^2 + (\mu^2 - \nu^2)(\kappa + \alpha^2)]\sigma(U, V) + 2\mu\nu(\kappa + \alpha^2)\phi\sigma(U, V) = 0.$$

This proves our assertion. □

## References

[1] M. Atçeken and P. Uygun, *Characterizations for totally geodesic submanifolds of* $(\kappa, \mu)$*-paracontact metric manifolds*, Korean J. Math. **28** (2020), no. 3, 555–571.

[2] M. Atçeken, Ü. Yildirim, and S. Dirik, *Pseudoparallel invariant submanifolds of* $(LCS)_n$*-manifolds*, Korean J. Math. **28** (2020), no. 2, 275–284.

[3] A. Carriazo and V. Martin-Molina, *Almost cosymplectic and almost Kenmotsu* $(\kappa, \mu, \nu)$*-spaces*, Mediterr. J. Math.**10** (2013) 1551–1571.

[4] P. Dacko and Z. Olszak, *On almost cosympletic* $(\kappa, \mu, \nu)$*-spaces*, Banach Center Publ., 211–220, 69 Polish Academy of Sciences, Institute of Mathematics, Warsaw, 2005.

[5] P. Dacko and Z. Olszak, *On almost cosymplectic* $(-1, \mu, o)$*-spaces*, Cent. Eur. J. Math. **3** (2005), no. 2, 318–330.

[6] D. Janssens and L. Vanhecke, *Almost contact structures and curvature tensors*, Kodai Math. J.**4** (1981), no.1, 1–27.

[7] T. W. Kim and H. K. Pak, *Canonical foliations of certain classes of almost contact metric structures*, Acta Math. Sin. (Engl. Ser.) **21** (2005), no. 4, 841–846.

[8] T. Koufogiorgos, M. Markellos, and V. J. Papantoniou, *The harmonicity of the Reeb vector fields on contact 3-manifolds*, Pasific J. Math. **234** (2008), no.2, 325–344.

[9] T. Mert and M. Atçeken, *Some important properties of almost Kenmotsu* $(\kappa, \mu, \nu)$ *space on the concircular curvature tensor*, Fundam. J. Math. Appl. **6** (2023), no.1, 51–60.

[10] H. Ozturk, C. Murathan, N. Aktan, and A. T. Vanlı, *Almost* $\alpha$*-cosymplectic* $f$*-manifolds.* Analele S. Tint. Ifice Ale Universitat II. Al.I. Cuza Din Iasi (S.N.) Matematica, Tomul LX, 2014.

[11] S. Sular, C. Özgür, and C. Murathan, *Pseudoparallel anti-invariant submanifolds of Kenmotsu manifolds*, Hacet. J. Math. Stat. **39** (2010), no.4, 535–543.

[12] I. Vaisman, *Conformal changes of almost contact metric structures*, Lecture Notes in Math., **792** Springer, Berlin, 1980, ,435–443.

[13] S. Ozturk and H. Ozturk, *Certain class of almost* $\alpha$*-cosymplectic manifolds.* J. Math. (2021), Art. ID 9277175.

Mehmet ATÇEKEN
Aksaray University, Faculty of Arts and Sciences,
Department of Mathematics,
68100, Aksaray, TURKEY
E-mail: mehmet.atceken382@gmail.com

Gülsüm YÜCA
Aksaray University, Faculty of Arts and Sciences,
Department of Mathematics,
68100, Aksaray, TURKEY
E-mail: gulsumbicer@gmail.com