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RESEARCH ARTICLE

Collaborative practices in virtual group work on dynamic geometry tasks

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Abstract

The goal of this study is to explore productive ways to engage students in groupwork using dynamic geometry tasks in online synchronous classroom environments. In particular, we aim to understand the social, mathematical, and technological aspects of student collaboration in virtual spaces. We analyzed how three online groups of students collaboratively worked on dynamic geometry tasks of exploring interactive kaleidoscope applets in Desmos and producing visual representations and written descriptions of geometric transformations used in the applets. The students shared their screens in Zoom as they shared their findings and discussed how to draw and write to represent the kaleidoscopes. We identified three emerging practices of the students collaboratively working on dynamic geometry tasks in online synchronous environments: (a) drawing in response, (b) co-construction, and (c) writing in real time. The emergent practices captured how students socially interacted with others, engaged in mathematical processes, and utilized technology tools. We also discuss inequity in students' participation in collaborative practices in an online environment and possible ways to ensure equitable learning opportunities for online students.

Keywords: dynamic geometry, task design, collaborative practice, online synchronous learning environment

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I. INTRODUCTION

Mathematics students in online synchronous classroom settings are asked to collaborate with their peers on group tasks, utilizing various mathematical and communication tools. However, there is a lack of understanding of how students can effectively collaborate in these environments and how to promote such collaborations, due to the complexity of interactions between elements involved in technology-integrated classrooms (e.g., Tomaszewski, 2023). Researchers use the concept of the didactical tetrahedron to investigate how teacher, student, content, and technology interact and to improve the integration of technology into mathematics classrooms (e.g., Ruthven, 2012). Investigating how technology can mediate students' social interaction with others and engagement in mathematical processes in online group work is crucial to advancing the knowledge of collaborative learning in online settings.

In this study, we explored productive ways to engage students in groupwork using dynamic geometry tasks in online synchronous classroom environments. In particular, we aim to understand the social, mathematical, and technological aspects of student collaboration in virtual spaces. Existing research shows that collaborative environments using Dynamic Geometry Environments (DGEs) can enhance students' collaborative learning experience in geometry by providing learners with opportunities of individual and social interactions with dynamic figures (Alqahtani & Powell, 2016; 2017; Edson et al., 2018; Medina & Stahl, 2021). We designed a dynamic geometry task sequence that can promote students' collaboration in online group work. We adopted a Desmos applet that provides interactive kaleidoscopes that demonstrate four different transformations of translation, reflection, rotation, and dilation. Students were asked to explore each kaleidoscope and collectively produce visual representations and written descriptions to explain how each kaleidoscope works. The guiding principles of the task design include providing explorative and open-ended problems at a high cognitive demand (Stein & Lane, 1996) and requiring group-generated outcomes. We implemented the task in three courses for secondary mathematics Pre-Service Teachers (PSTs) in the United States. We analyzed how students interacted with their peers, mathematics, and technology in their online group work and identified three emergent practices of drawing-in-response, co-construction, and writing in real time. The analysis allowed us to discuss group dynamics and norms in mathematical discourse and ways that technology mediates student engagement in social interactions and mathematical processes. In this study, we define virtual collaboration as students' social interaction in an online synchronous group work setting that allows the students to engage in mathematical processes meaningful for the task enactment.

II. LITERATURE REVIEW

Social Engagement in Collaboration

In the following, we present a literature review on students' social engagement in online collaborative learning environments that informed the task design and the analysis of this study. We also focus on lessons for designing and implementing online group work that can provide students with equitable opportunities to engage in collaboration in an online setting. Much has been written about student collaboration (Cohen, 1990; Johnson & Johnson, 1999; Sampson & Clark, 2009; Webb, 2013). Focusing on online environments (Akyol & Garrison, 2008; Arbaugh, 2008), it can be difficult to design online communities where learners can engage with other learners. As one avenue of research, Computer-Supported Collaborative Learning (Koschmann, 1996) examines how computer environments can support building collaborative knowledge, based on constructivist and social theories of learning. In particular, Stahl (2000) developed a conceptual framework for collaborative knowledge-building, involving personal understanding and social knowledge-building processes, theorized from empirical evidence of collaboration. Another avenue is how to foster engagement, such as developing group norms so that learners understand the expectations for communication with each other as well as their instructor, and the expectations for all learners to communicate (Richardson et al., 2010; Stephens & Roberts, 2017).

When designing collaborative group activities, it is important to aim for an equitable distribution of engagement opportunities for all learners in the group (Patterson, 2019). There may be several reasons why some students may not participate in group work effectively, or find it difficult. For example, it might be challenging for some students to articulate their thoughts effectively perhaps due to a lack of critical examination when acquiring concepts (Sampson & Clark, 2009). Some students might shy away from openly criticizing others' ideas (Sandoval, 2003). There is a need to address various student challenges when planning for effective collaboration with the goal of student learning. Students must feel safe to communicate without the pressures of social factors such as perceived academic success and social hierarchy between students (Blatchford et al., 2003). It is important to provide a collaborative space where students can interact as equals and engage with their group members using their body language and voice to communicate effectively (Barron, 2000).

When planning a collaborative activity, the instructor's role includes handling inequitable learning opportunities that arise due to limited chances for interaction, by focusing on student voice, visibility, and authority (Patterson, 2019). Voice refers to students sharing their thinking while engaging in collaborative learning (Furman & Calabrese, 2006). Students' voices and visibility in a group setting can suffer if they make arguments without providing sufficient proof to support them (Simon et al., 2006). Students themselves might choose to not criticize the work of others in the group (Sandoval, 2003), or choose to not ask for help (Webb, 2013). Sometimes during group work some students are made silent as their voices may be ignored by other students (Cohen & Lotan, 1995).

It is the instructor's responsibility to develop a space where all students have the authority to share their ideas. Students in a group must work towards having their voices heard and create a space where others' voices are heard as well so knowledge is truly constructed together as a team (Patterson, 2019). When teachers design collaborative activities the main goal is to shift teacher authority to students so they can engage in discovering ideas (Brufee, 1995; Cohen, 1990).

In sum, the literature highlighted the importance of instructors' purposeful efforts to ensure students' social engagement in online collaboration. Those efforts include designing online learning environments that promote students' active participation in collective knowledge construction, empowering students' authority, and establishing social norms for equitable distribution of opportunities to participate in group work. The literature informed us in developing instruments and instructors' material to implement the group task in this study. It also helped us address issues of inequitable student engagement that we identified in the analysis.

Technological Engagement in Collaboration: Technology Mediation

In the following, we summarize prior studies on the role of technology in mediating students' collaborative learning in online synchronous environments using DGEs. Alqahtani and Powel (2016, 2017, 2018) posit that digital technologies can influence people's interaction with their environment and with each other - and can enable learners to explore relations between mathematical objects and ideas in collaboration with others. Their work on dynamic geometry environments (Alqahtani & Powel, 2016, 2017, 2018) is grounded in the theory of instrumental genesis and mediated activities. This research includes studies that analyze learners' interaction with technological tools in mathematical contexts (for example, Barcelos et al., 2011; Hoyles & Noss, 2009; Laborde, 2007; Mariotti, 2000); how tools mediate users' interactions and meaning-making while solving shared tasks (Lonchamp, 2012; Rabardel & Beguin, 2005); learners' reactions to their environment's feedback (Hegedus & Moreno-Armella, 2010; Moreno-Armella & Hegedus, 2009), and how environments can influence learners' discourse and knowledge construction (Alqahtani & Powell, 2017; Bartolini Bussi & Mariotti, 2008; Mariotti, 2000; Sinclair & Yurita, 2008). Such studies shed light on tool usage and how it influences mathematical learning.

Alqahtani and Powel (2016) investigated high school mathematics teachers' engagement with a collaborative online DGE during a professional course. This work showed how these teachers explored various features of the DGE and that their engagement with DGE influenced their geometric knowledge. Later, Alqahtani and Powell (2017) employed Rabardel and Beguin's categories of tool mediation in an instrument-mediated activity (Rabardel & Beguin, 2005) to study how dynamic geometry environments can mediate activity. They found evidence of teachers' specialized content knowledge as well as Rabardel's epistemic and pragmatic mediations. Based on their analyses, Alqahtani and Powell coined "pedagogic mediation" by which teachers use the environment to help other team members understand particular geometric objects and the relations between them. Their research has implications for teachers and teacher educators as it guides them toward effective ways to implement tools in mathematics classrooms. Teachers' knowledge of tool usage and how it can shape mathematical discourse and ideas is part of their technological pedagogical content knowledge (Koehler & Mishra, 2008; Mishra & Koehler, 2006) and their mathematical knowledge for teaching (Ball et al., 2008; Hill et al., 2008), which influences students' achievement (Ball et al., 2005; Hill et al., 2005).

In sum, the literature provides theoretical ground and empirical evidence of the role of DGEs in this study to support students' collaboration by mediating mathematical activities as students interact with instruments and their peers. Technology can support collective sense-making, mathematical communication, and knowledge construction when students work on group tasks that involve their interaction with dynamic representations of geometric objects. The literature informed us in designing group tasks and shared workspace (e.g., Desmos and Google Slide on shared screen in Zoom breakout sessions) of this study that can promote students' collaboration in online synchronous environments.

Mathematical Engagement in Collaboration

In the following, we present a review of prior studies on students' mathematical engagement in technology-integrated learning environments for collaboration. In considering the classroom, modern digital curricula prioritizing collaboration are being developed, such as the Connected Mathematics Project (CMP) (Lappan et al., 2014). CMP is a middle grades problem-based curriculum from the United States, focused on student inquiry through a launch-explore-summarize lesson structure. In recent years, CMP has developed a version embedded in a digital collaborative platform, to reap the affordances of an online format beyond text. For example, Edson and Phillips (2021) reported on a teacher dashboard for monitoring students' collaboration, in contrast to student performance. Edson et al. (2018) found that the digital learning environment fostered real time collaboration and productive disciplinary engagement in mathematics. Productive disciplinary engagement refers to whether students' experiences while learning match practices from the targeted discipline (here, mathematics), such that they make intellectual progress. Bieda et al., (2020) provided an observation tool and framework for analyzing levels of students' productive disciplinary engagement, along the dimensions of problematizing, authority, accountability, and resources. Edson et al. (2018) found that students wanted an individual workspace, where they could make sense of the mathematics by themselves before sharing their work with others. This finding speaks to students' desire to think individually before collaborating with others, which impacts the design of collaborative spaces. Collectively, the CMP work on productive disciplinary engagement highlights the mathematical and social aspects of collaborative learning, while the promise of a digital curriculum provides insight into the technological aspects of working together online.

GeoGebra and Desmos are currently popular platforms for geometry but lack the ability for students to work together. One environment for promoting student collaboration is Virtual Mathematics Teams (Stahl & Çakir, 2008), a multi-user version of GeoGebra where users can take and give control of the screen with a text chat window where users can communicate. Researchers in the field of Computer-Supported Collaborative Learning (CSCL) have investigated students' social interactions in group practices of solving dynamic geometry tasks in their claims for the development of shared understanding. Stahl (2016) identified fifteen collaboration practices among students using their Virtual Mathematics Teams environment: discursive turn-taking, constituting the group as a collective unity, co-presence, group agency, etc. Medina and Stahl (2021) further analyzed groups' processes of adopting these collaborative group practices and identified the hierarchical and sequential structure of the adoption of group practices. There is a need however for best practices for students working together on mathematical tasks in an online space (Stahl, 2016). Work is needed to extend current knowledge to identify concrete practices that are actionable and accessible - for students and mathematics instructors.

In sum, the prior studies provided insights into mathematical practices that students can be engaged in while collaboratively working on group work in technological environments. The literature guided us to focus on identifying key mathematical practices that emerged as students collaboratively worked with others on their DGE tasks.

Summary of Literature Review and Research Question

Prior studies have provided insights into understanding three different aspects of collaborative mathematical practices in online synchronous environments. However, a theoretical framework that incorporates all three aspects - social, technological, and mathematical - can enhance instructional practices of designing, enacting, and evaluating mathematical tasks and collaborative learning environments that promote students' meaningful collaborative learning. Analyzing connections among these aspects of collaboration and respective environments involved in instructional decision-making such as task design, group norm settings, and digital platforms will provide teachers and educators with practical guidelines to improve their teaching practices. We therefore use a lens of social, technological, and mathematics aspects of virtual collaboration in our work. In this study, we define virtual collaboration as students' *social interaction* in *an online synchronous group work setting* that allows the students to engage in *mathematical processes* meaningful for the task enactment. There is a need for work that is more accessible for teacher educators and/or mathematics instructors (and teachers). Based on this framework, our research question is: How do students engage in *virtual collaboration* for learning geometry in an online synchronous environment? In particular, we focused on identifying students' emergent practices of virtual collaboration that demonstrate social, technological, and mathematical aspects.

III. METHODS

Settings & Participants

This study took place in two mathematical content courses and one mathematics methods course for PSTs (Table 1), at three different public universities across the United States. Participants were college students intending to be PSTs by institution, respectively. In the United States, mathematics content courses are often taught in mathematics departments and focus on mathematics and how K-12 students think about and understand these ideas. Methods courses focus on the teaching practices for specific mathematics content areas. The major course modality for all three courses was online synchronous with some asynchronous components. Each of the three authors taught one of the courses independently and implemented a Desmos task in online synchronous settings via Zoom. From each of the three courses, we selected the group that showed the most active interaction with each other, to select episodes with strong virtual collaboration. These participants are referred to using pseudonyms.

Table 1. Setting and participants across three sites

Type of course (Name)	Population	Class Size	Selected Participants
Content course (Fundamentals of Middle School Mathematics)	Math majors with middle math concentration (Grades 4 - 8)	14	Kayla, Emma, Jose, Olivia
Content course (Modern Geometry)	Math majors with secondary math concentration	9	Tom, Val, Josh
Methods course (Mathematics Teaching Methods)	Math majors with secondary math concentration	17	Bryn, Sam, Alex

Task for Exploring Properties of Transformations and Task Design

We used Desmos Classroom as our online platform to explore geometric transformations. We adapted an existing applet in the Desmos public library, titled Kaleidoscope, where there were four types of "kaleidoscopes" that demonstrated the different transformations of translation, reflection, rotation, and dilation (Figure 1) without name. In our adaptation, we created a sequence of tasks for group work in online settings. The goal of the group task was to produce a visual representation and written description to characterize each kaleidoscope. The rationale for the task selection and design was based on high cognitive demand tasks (Stein & Lane, 1996) that were explorative and open-ended and required group-generated outcomes.

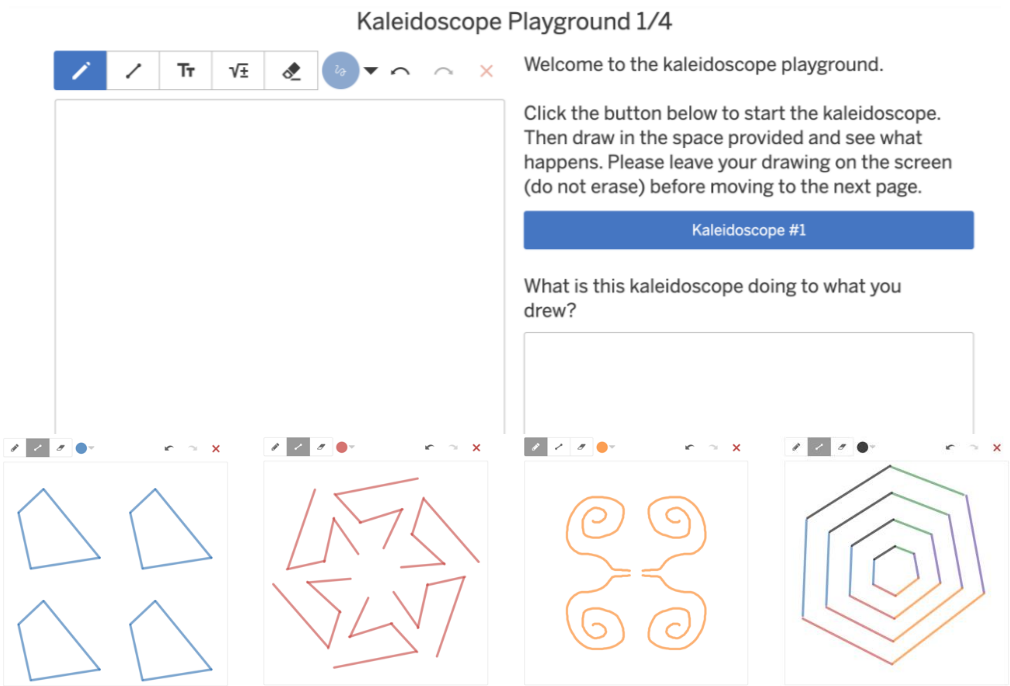
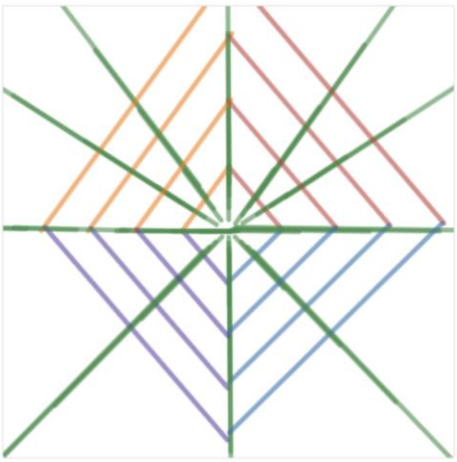


Figure 1. The task (above) and four types of kaleidoscopes (below)

The first part of the task was done individually: students freely explored the four transformations in Desmos. The applet provided space to draw figures and posed a question for students to write down their observations. In the second part of the task, students met in groups over the video-conferencing software Zoom and used their observations to discuss the behavior of each kaleidoscope, to create a group visual representation in Desmos that best showcased the kaleidoscope's behavior and a written description in Google Slides, a collaborative presentation platform, for each of the four kaleidoscopes. Figure 2 is an example from a group's illustration of the dilation kaleidoscope with the center of origin. Students' written descriptions were intended to be precursors to formally defining each transformation after the Desmos activity.



In this kaleidoscope, we observed that when drawing the original line or point it draws one line/point below and two line/points above with equal spacing between all four lines/points. This is caused by the original line distance from the origin which is then cut in half by the smaller line as well as the third line. The distance between the original line and the fourth line is the same distance from the original line and the origin. We also observed that when the lines are going towards or from the origin that the four lines overlap each other, and when the lines are facing away from the origin the lines get farther apart and space out more from the origin.

Figure 2. Group visual representation in Desmos and description for dilation kaleidoscope

Data Collection and Analysis

We collected video recordings of one 75-minute online class session and student work from each course. Video recordings showed the group's shared screens and students' videos (when turned on), with audio of the group's discussion. Videos were transcribed when students' verbal and non-verbal communications were identified. The visual representations and written descriptions that groups collectively produced in Google Slides were collected. In this analysis, we applied our definition of virtual collaboration to identify episodes with evidence of student collaboration: Virtual collaboration in this study is students' *social interaction* in *an online synchronous group work setting* that allows the students to engage in *mathematical processes* meaningful for the task enactment. The definition consists of three major components that led to the analysis of identifying episodes that show students' virtual collaboration (Table 2).

Based on this analytic frame, each researcher conducted first- and second-level coding independently and then discussed to finalize the codes. The first-level coding was to describe student interactions along one or more components of virtual collaboration (social, technological, and mathematical) using the deductive coding method (Miles et al., 2014). After identifying episodes with one or more of the above codes, we conducted second-level coding of each episode using constant comparative techniques to draw out emergent themes as potential practices. The intent was not to isolate and distinguish episodes from others, but rather to identify empirical evidence of students' virtual collaboration and practices.

Table 2. The major components of virtual collaboration of the study

Component	Meaning	Examples
Social interaction	Verbal or non-verbal communication indicating emergent group dynamics and group norms	Discuss roles of group members, ask a leading question, share observations, agree with others, ask for help, ask for reassurance, revoice each other
Online synchronous group work setting (Technology)	Technological environments given to the groups including individual workspace in Desmos, Google Slides, and any of those shared on screen over Zoom	Draw or change a figure, observe a figure, point to a figure, share screens in Zoom, write or improve a description of figures, screen capture and insert pictures
Mathematical process	Engagement in particular mathematical processes with a base on existing frameworks including CCSS Mathematical Practices and NCTM Process Standards	Make sense of problems, generate conjectures based on patterns they observe, formalize language in attention to precision, construct argument and critique others' reasoning

IV. FINDINGS: COLLABORATIVE PRACTICES IN VIRTUAL LEARNING ENVIRONMENT

In this section, we present our findings of students' virtual collaboration during dynamic geometry tasks for inquiry into geometric transformations. We found that groups working in the virtual environment of this study engaged in three collaborative practices: *drawing-in-response*, *co-construction*, and *writing in real time*. These practices emerged as groups communicated their geometric observations of the applets and made decisions regarding creating, revising, and finalizing their visuals and written descriptions. These practices involved students' social interaction with peers and mathematical inquiry into geometric transformations with the support of collaborative and mathematical technology. We provide student examples of these collaborative practices to describe how the practices emerged in groups, how the practices supported the mathematical task, and what issues and challenges emerged, in facilitating student collaboration in virtual environments.

Drawing-in-response

The first example of a collaborative practice was drawing-in-response. This practice was identified when a person explains mathematical ideas involving geometric objects, and another person draws figures in a collaborative space (e.g., shared screen in

Zoom) in response to the verbal explanation. Without a direct request, the drawer often drew figures on the shared screen while making sense of the conversation, to demonstrate their interpretation, or to respond to the speaker non-verbally. In particular, when someone was verbally explaining a complicated arrangement of geometric figures, the drawer would start drawing simultaneously while the person was still explaining. We found drawing-in-response supported students communicating what diagrams they drew and what observations they made in their individual exploration. This communication featured geometric ideas about transformations on figures, such as shape, number, location, size, and orientation of figures. We illustrate how this practice emerged and gradually improved as students communicated in groups what they found interesting in the applets.

Drawing-in-response supports virtual groups' geometric communication. We present an example of how drawing-in-response aided students' communication with each other about transformations. This episode took place in the Modern Geometry content course from a group with three participants: Tom, Val, and Josh. Tom served as the group leader, sharing his Desmos screen on Zoom and moving the group along. Tom and Val were visible on camera, while Josh had his camera off. Neither the instructor nor the task set-up explicitly instructed students to draw-in-response, but this practice emerged as a way for students to be responsive to each other's thinking. Tom, Val, and Josh worked together as a group collectively to understand the translation kaleidoscope. Tom opened by asking the group for their particular approaches. Figure 3 shows Val's personal Desmos during the individual portion of the task sequence, of triangles and lines.

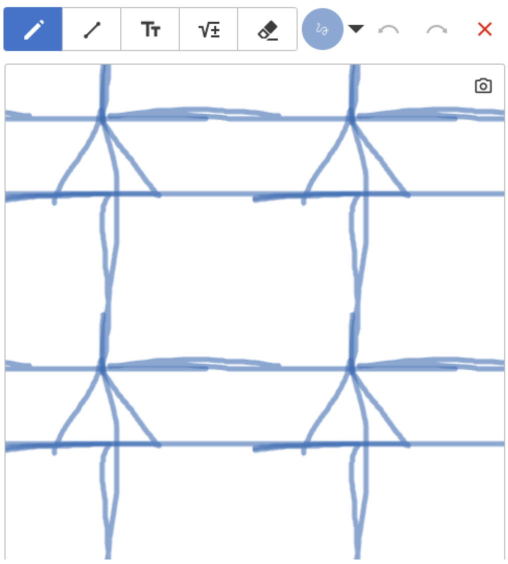


Figure 3. Val's individual Desmos drawing for the translation kaleidoscope

Val described to the group how he drew triangles, and then Tom drew them on his shared Desmos screen, of his own accord (Figure 4). Val then described more objects he had drawn - in particular lines in the middle of the screen - and his reasoning while pointing at the screen.

> Tom: Did anyone have a specific approach to this?
> Val: Well, I drew triangles first, just to see how it worked
> Tom: So like you started at a point, and then drew a triangle?
> Val: Yeah. And then I also just drew a line in the middle and then up…just to see how it would affect [the kaleidoscope] when you go all the way up and [the line] touches.

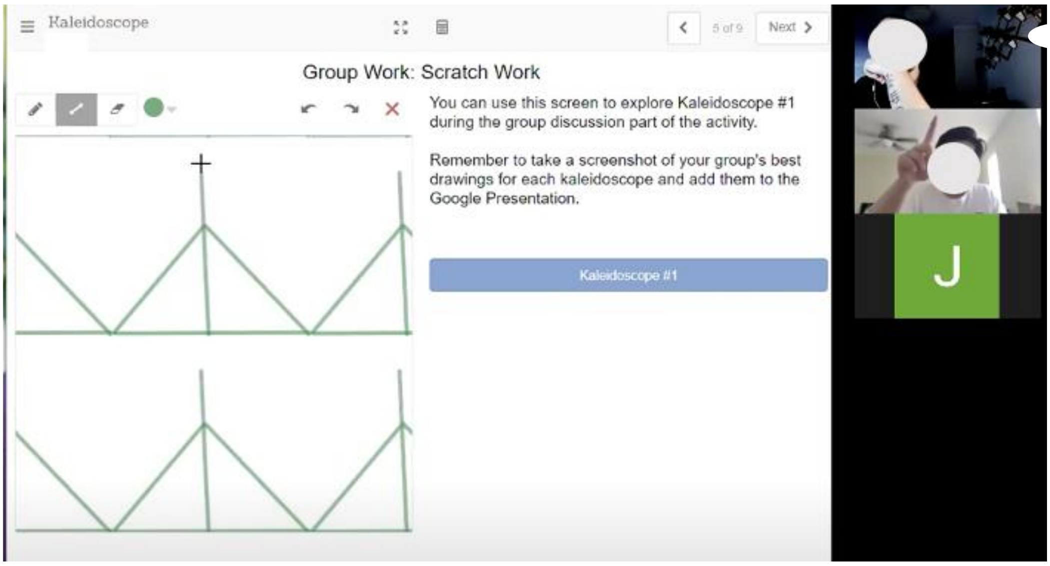


Figure 4. Val points as the leader draws what he described, triangles, and lines through the middle

The screen sharer drawing another student's contributions elicited mathematical and social aspects of virtual collaboration, for the group to understand the properties of a translation and as an acknowledgment of Val's work. In particular, this led to Val sharing more of his observations, Josh briefly acknowledging Val's observation, and Tom conjecturing about its behavior, incorporating his own individual work too (Figure 5).

> Val: For me, it sounds…it seems like you're drawing quadrants. And then, you know, it passes each quadrant if it doesn't have enough space.
> Josh: Yeah.
> Tom: Yeah, I kind of got this...like that, it was duplicating your image, because I was doing a lot with points and it duplicates your point in each quadrant.

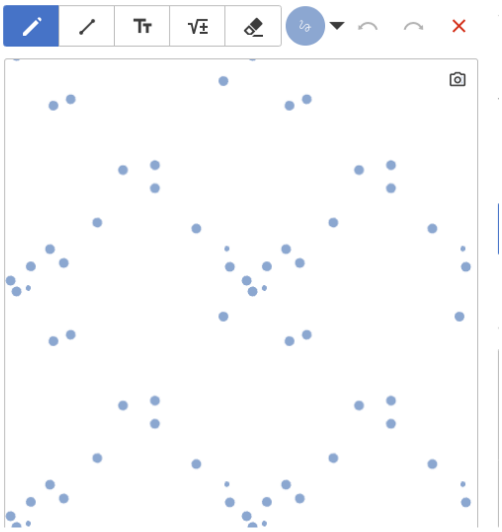


Figure 5. Tom's individual Desmos drawing, using dots

Drawing-in-response helped students mathematically notice that translations duplicated images: that orientation and distance were preserved. Technologically and socially, drawing-in-response as a practice transformed an individual's contribution into a shared visual, on which all group members could then share their mathematical observations.

Emergence of drawing-in-response. We found that drawing-in-response emerged and gradually improved in the setting of online synchronous group work without any direct instruction to establish a norm for implementing the practice. Soon after students started discussing their observations of the applets, they realized the limitation of verbal communication for describing what they constructed and how they changed diagrams in the applets. Students consistently called (or requested) to use a shared space (e.g., Desmos applet on the shared screen in Zoom) where they could draw, observe, and manipulate in response to their ongoing verbal communication about geometric ideas simultaneously. We found that students who shared their screens and took control of the applet were often expected to draw-in-response when other members verbally explained their ideas to their groups. When the drawings did not represent the discussion in the group, other members requested to revise the drawings on-screen. This indicates that less effective use of drawing-in-response can elicit students' focus on the practice itself and ways to improve it for productive disciplinary engagement in collaborative group work.

In the following paragraph, we present another group's case (Kayla, Emma, Jose, Olivia) where the initial drawing-in-response was not effective in supporting their verbal communication. Due to the limited use of technology by the students who shared the screen, drawing on the shared screen did not effectively represent what the group discussed. Other group members explicitly addressed the need for drawing-in-response and helped the drawer improve her use of technology to provide an accurate demonstration of the group's discussion. At this point, the group shifted their attention to the limitations of the current practice and looked for possible ways to improve the implementation of the practice.

At the beginning of their group discussion, Kayla explained what she drew and found interesting but noticed that it was difficult to do verbally. Kayla asked for a volunteer to share their screen and draw what she was explaining because she was not able to share her diagrams on a tablet in Zoom. This shows a call for the role of drawing figures in a shared space in supporting discussion about geometric figures in this task. Emma volunteered and shared her screen in Zoom. While Kayla explained how she drew circles and squares in the first kaleidoscope, Emma added a few more dots to her shared screen but did not draw circles or squares using continuous lines or curves (Figure 6).

> Kayla: I wish I could show you mine. I drew circles in different colors. And they were straight across. And then if I drew a square over that circle, they'd be over all the other circles. It wouldn't really change anything. You know what I mean?

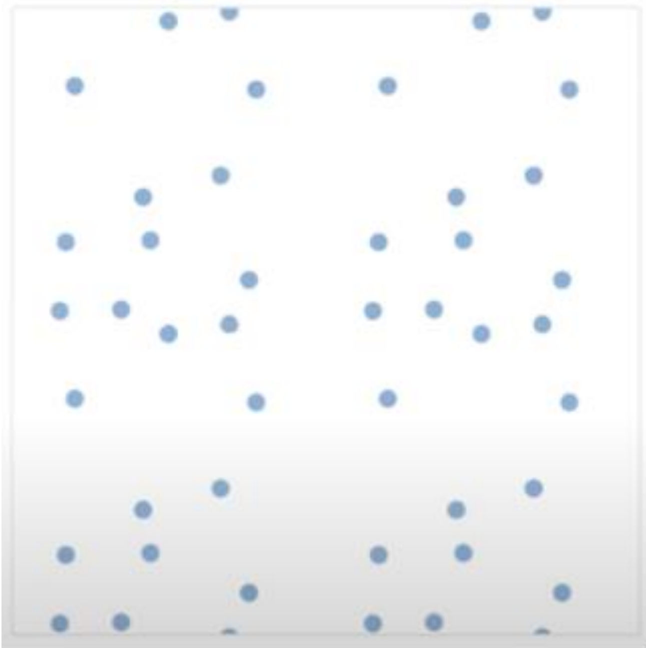


Figure 6. Emma’s drawing of dots in response to Kayla’s verbal explanation of circles and squares

Kayla found that the kaleidoscope made copies of drawings along straight lines as the drawings were translated on the plane. She shared the idea of drawing circles in different colors and squares to examine how the kaleidoscope consistently generated copies. However, Emma’s drawing on the shared screen did not represent what Kayla explained and did not show how the dots were related to each other. In the next kaleidoscope (rotation), Kayla explained how figures were copied and arranged along the circles around the center of the applet as follows.

> Kayla: So basically, that's what I wrote down for this one that it draws their drawing

repeatedly in a circle shape. Oh, and if you do it super close in the middle, it overlaps each other.

Emma drew in response when Kayla said figures would overlap each other. Emma created a dot close to the center of the screen that resulted in six dots around the center that looked slightly overlapping each other (Figure 7). It shows that Emma understood what Kayla had said about the figures overlapping along with a circle centered at the middle point of the screen. However, she drew dots close to the center so they would look to be overlapping because she was able to create only dots.

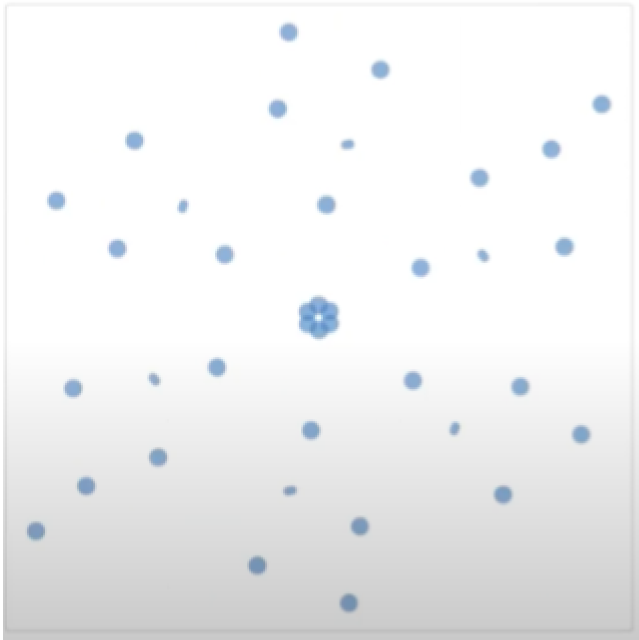


Figure 7. Emma's drawing of dots in the second kaleidoscope

Kayla: Yeah, like that.
Jose: Oh! Yeah, right in the center.

The exchange of verbal and visual communication–Kayla's verbal explanation, Emma's drawing-in-response, and other members' acclamation–shows that Emma's drawing-in-response elicited group members' attention to what Kayla explained about how figures behave near the center and signified that her comment was heard by the group. Following this, Kayla and Jose further discussed what would happen to figures created far from the center. Jose engaged more in the conversation than before he reacted to Emma's drawing.

Kayla: If you go out, it's just like a bigger circle towards the corner.
Jose: When you draw a figure towards the center, they all collapse. It looks really

nice like a flower. [...] But I drew it far apart, all sort it drew separate circles but apart from each other.
Kayla: Yeah, if you draw a circle in the bottom corner, it will be like a wider circle far apart.

This time, Emma created a dot close to the bottom corner in response to their conversation. However, this drawing of a dot did not elicit any further group reaction, maybe because the group did not notice the new dots among the numerous existing dots. If Emma had drawn circles or figures other than a single dot, there could have been more discussion on the rotational symmetry in this kaleidoscope. This shows that the limited use of drawing (only dots) was not effective in supporting the discussion between Kayla and Jose about how distance from the center affects the drawings in this kaleidoscope.

When they moved to the third kaleidoscope, the group discussed how the kaleidoscope changed lines and squares. Olivia explained that she drew lines and found that four lines came out. Kayla responded to Olivia and agreed with the four lines from the four corners. This time, Kayla directly asked Emma for the first time to draw a square that she explained. However, Emma did not draw squares and created a few more dots on the screen. When Kayla and Olivia stopped their conversation and waited for other squares, it became apparent that the drawing on the screen was not effective in supporting their communication. The group's attention was shifted to Emma's way of using dots only for drawing-in-response. Kayla directly asked Emma if she could draw only dots and explained that drawing other shapes such as squares could be better for examining the kaleidoscope.

Kayla: Can you only draw dots?
Emma: Yeah
Kayla: Okay. Because I noticed if I draw a different shape, it's easier to tell that it's coming out in three other corners of a rectangular or a square. So what I put for this one was that it draws the shape four times in a square figure. The further you are from the center, the further apart the drawing is.

In the last kaleidoscope using dilation, Kayla asked Emma to draw a short line segment to show how the lengths of segments are different as they are far from the center. Other members helped Emma figure out how to draw continuous lines. The group explicitly looked into her ways of drawing figures and what device she was using.

Kayla: Can you draw a short line? Are you using your finger? What are you using?
Emma: Yeah
Kayla: Okay.
Jose: [...] trying fingertips
Kayla: There you go. You see how it starts off small in the middle, then it gets bigger or longer.

With their help, Emma finally figured out how to draw continuously and created a short line segment in yellow (Figure 8), and the applet showed three copies of the segments in different lengths. Later in the group work, Emma was able to draw figures with continuous lines and curves. Her drawing-in-response became more effective in visually representing group members' ideas than before when she used only dots.

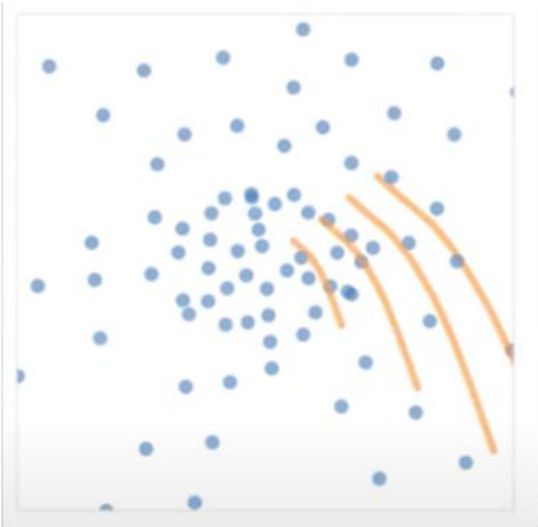


Figure 8. Emma's drawing of continuous line segments in a dilation kaleidoscope

In sum, this group's episode showed us the genesis and gradual improvement of the practice of drawing-in-response. The practice emerged as group members noticed the limitation of verbal communication to express geometric ideas. The need for drawing-in-response introduced a group norm that the drawer sharing her screen is expected to draw figures on the screen for demonstrating figures other members explain. The group noticed a problem with the less effective use of the practice that dots cannot illustrate geometric and spatial properties of figures, in their discussion on how each kaleidoscope affects the orientation, direction, and size of shapes such as lines, circles, and polygons. They paused their work on the task and discussed a better way to implement the practice–using continuous lines–to proceed in their collaborative group work. This process of improving the emergent practice of drawing-in-response shows that students were able to socially engage with others in online synchronous learning environments by establishing implicit norms for verbal and visual communication and helping each other to participate in the discussion. When only dots were used early in the group work, students had a chance to examine what shapes can be useful for investigating how kaleidoscopes affect geometric figures and found that shapes with orientation and size are effective for visualizing the effects of different transformations. This process also shows that students critically examining their use of technology could effectively support their disciplinary engagement.

Across groups, we found that implementing the practice of drawing-in-response involved all three aspects of collaboration in a virtual learning environment–social, mathematical, and technological. Drawing-in-response supports students socially interacting with each other in a collaborative environment by allowing multi-modal communication about geometric ideas. This practice can supplement verbal communication and improve the quality of their discussion for collaborative tasks by adding a visual representation that captures spatial and geometric properties of figures that are difficult to convey in verbal explanation. Drawing-in-response enables students to engage in mathematical processes as they create, observe, and interpret geometric diagrams. When drawing figures following others' verbal explanations, students are expected to make sense of mathematical ideas and to demonstrate geometric construction. When verbally explaining their ideas, students are expected to be mathematically accurate and precise for others to draw what they describe. Students are also expected to critically examine if drawings accurately represent their discussion. Drawing-in-response provides opportunities to learn how to utilize technological environments for online synchronous group work in a shared virtual space for remote students.

Co-construction

The second collaborative practice was co-construction; this happened when two or more people were involved in decision-making collaboratively while constructing a figure. This takes the form of one person sharing their screen and taking control of drawing, while others suggest how to draw, add, or modify existing figures (e.g., "Why don't we try this?", "Can you draw vertical lines from that point?"). Desmos allowed students to draw figures in different colors and use straight lines, free drawing, and dots. Co-construction occurred when a group worked together to construct a collective drawing, to understand mathematical behavior. Co-construction fundamentally consists of multiple drawing-in-response moments and/or students sharing open questions and suggestions for creating a collective image. During co-construction, students develop new ideas while drawing, whereas, with drawing-in-response, a student is checking to confirm their understanding of another student's idea or simply drawing a construction as another student narrates their idea. This practice emerged repeatedly across groups when students built off each other's contributions to investigate specific properties of the kaleidoscopes and to produce the best representation to submit for the task.

In the following episode, we present one group's implementation of co-construction where students engaged in collaborative decision-making. Students built off each other's suggestions and reached a consensus and their final construction better represented the rotational symmetry than their individual work. Later in the paper, we also present another group's implementation of co-construction where not all students' suggestions were valued equally. We address that instructors need to attend to equitable distribution of students' opportunities to contribute to group work.

Co-construction involves collaborative decision-making. Co-construction as a practice involves collaborative decision-making, evident in an episode of co-construction with the group of Tom, Val, and Josh investigating rotations. Figure 9 shows each of their individual Desmos drawings, before discussion as a group: All students drew straight lines, while some drew dots, circles, and curves respectively.

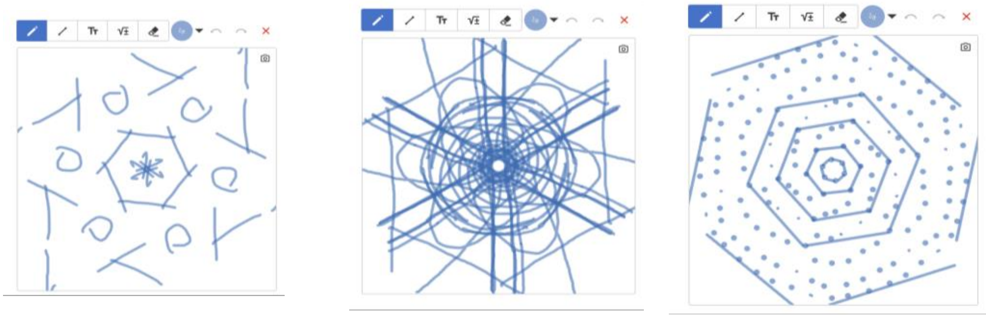


Figure 9. Josh’s (left), Val’s (middle), and Tom’s (right) individual Desmos work

Tom shared his screen and drew dots in the rotation kaleidoscope, as in his individual work. He drew blue points along a ray outward, to highlight how the kaleidoscope duplicated them like a hexagon’s vertices. Josh then shared out loud what he had drawn, which Tom then drew and led to Val’s understanding:

Tom: This one, essentially the closer you are to the origin, essentially, you're just creating a hexagon. So, as you go further and further out, you're creating this.
Josh: I just drew lines on mine, and it immediately connected them, so.
Tom: Yeah. So you can like... [draws]
Val: Ohhh, that makes sense.

Tom created and dragged orange and purple lines in the applet, as Josh said, and described what was surprising about the transformation’s behavior here: how the rays were replicated on the screen to form a hexagon. He extended the lines further to appear as rays (Figure 10): “The one thing that got me though is with the lines going beyond, it creates your hexagon…so it's technically creating just a ray out from every point.”

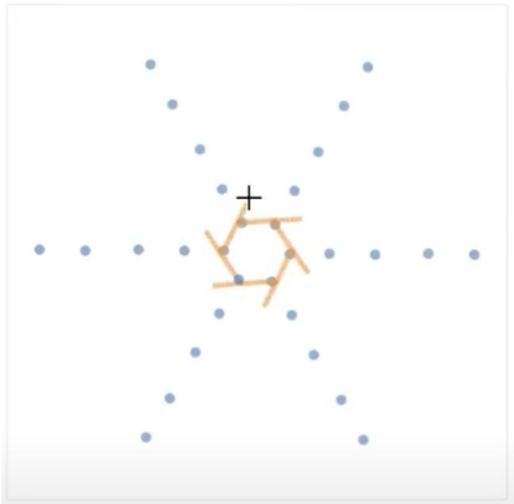


Figure 10. Tom extended the “sides” of the hexagon further to create rays

As Tom was about to reset the figure, Val suggested extending the sides of a new black hexagon on one side: "I think you should include the black hexagon though for having the rays come out of it" Val had drawn this in his individual Desmos screen prior. Val's gestures and speech supported his idea that including rays going outward from the hexagon (Figure 11) could illustrate the rotation precisely and distinguish it from other transformations.

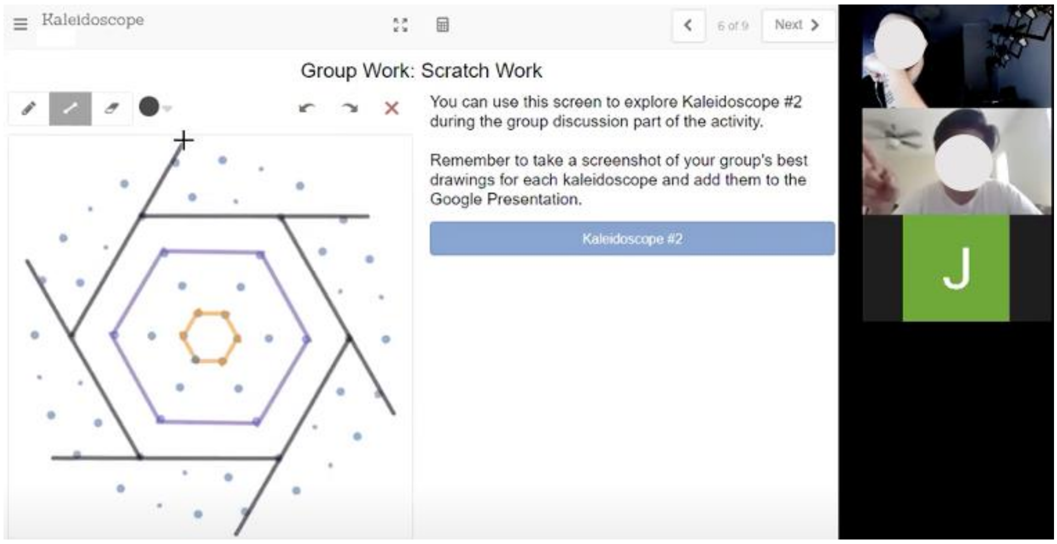


Figure 11. Val asks to include the black hexagon and rays emanating out of it, for the final group representation

We see that elements of all three students' individual work became a part of the group's drawing in Figure 11: Tim's hexagon and points, Josh's lines, and Val's lines that extended past the hexagon's sides. Val did not tell the group that he had drawn such lines, but, notably, he suggested the group include them for their final drawing after Tim happened to draw lines on the shared screen. Co-construction led to their final submission including the black hexagonal shape, to show the directional and angular properties of the rotation. This episode serves as a preliminary example of how basic co-construction emerges and how students can build off each other's contributions (verbal and pictorial) to examine mathematical behavior and make decisions together.

Writing in Real Time

The third example of the collaborative practice was writing in real time. This practice was identified when students collectively produced their written descriptions for the kaleidoscopes in a shared Google slide. This practice involved writing on the slide, suggesting changes, editing, proofreading, asking for help, and/or asking for reassurance of what was written. For example, students queried others about what to write (e.g., "Can you repeat what you just said?"). Students would add or modify existing descriptions written by another person. This practice was found when students looked for a better description or more formal language for the task of producing a written description to explain how each kaleidoscope works.

In the following section, we present an episode in which students implemented writing in real time to provide a written description of each kaleidoscope. We illustrate the students' active participation in the collective writing process where they exchanged ideas for using accurate and precise expressions and gradually improved their writing. This episode shows how the writing task in collaborative group work can support students' exercise in agency by allowing them to use written communication to make contributions and empower their voices in their group.

Writing in real time leads to iterations of refinement for formal and precise language. Writing in real time elicits the following from students: asking for reassurance from groupmates, checking one's understanding with the group, and refining and formalizing language in iterations to be more formal and precise. We return to the group of Tom, Val, and Josh to see these impacts of writing in real time. One student, Val, was writing the description of the rotation transformation's behavior and sought the help of his group in different forms at various points. Val asked for reassurance about what to write: first to generate words, then to check if what he wrote was a correct interpretation. Tom emphasized a constant distance between points and the origin, a key aspect of the definition of rotation. Val wrote this and then added in how a hexagon can form. He asked for reassurance again, asking Tom to repeat what he had said about points.

> Val: How do I word this? For every…
> Tom: I was trying to say something with, it creates a hexagon with the same radius at all six points, like, clearly it's depending on where you click from the origin is going to be the distance that you get to all six points.
> Val: OK, so…oh, can you say that again about the points?
> Tom: So like when we're creating points, as opposed to lines, every point has a given distance from the origin. And then the kaleidoscope creates equidistant points, six equidistant points, essentially from the origin. So it's going to have six points that go up from the origin. And each of those points is going to be equidistant from each other.

Val used Tom's language to write down a description but misinterpreted Tom's suggestion of "equidistant points" for "equal distance" (see Figure 12). He then asked Tom to check if what he wrote was accurate: "That sound correct?" [referring to what he wrote], kickstarting the group's cycles of refinement of language.

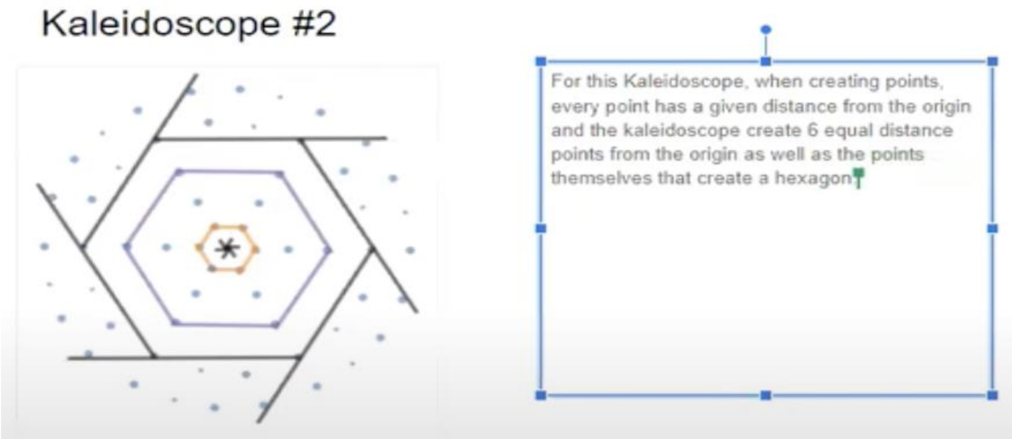


Figure 12. Val’s first pass at describing a rotation

In the next phase, the group engaged in a back-and-forth conversation about how to state that OP’ = OP (given point P, center of rotation O, and image P’). The students rewrote “equal distance points” to “points that are a given distance.” We then see another moment of asking for reassurance, as Val asked Tom to read over if what he had written sounded correct. Tom read the description out loud and tried to improve the last written sentence, “equal distance points from the origin as well as the points themselves.”

> Tom: Let's see. So for every point...I'd say, rather than "has a given distance," say "When creating points…the point you create has a given distance or a certain distance from the origin. The kaleidoscope then creates five more points with that same distance?
> Val: Okay, I get what you mean, “in creating…” I'll say “when creating a point.”
> Tom: “That point has a given distance?”
> Val: “That point has a given distance from the origin and the kaleidoscope creates five equal distant points. Five points that are also equal distance from the origin as well as…”
> Tom: “From"
> Val: “As well as equal distance”
> Tom: From each other? Or from the points themselves?
> Val: Yeah. Let's go with "each other…which creates a hexagon." It's nice.

Through this back-and-forth, we see that Tom shared language and Val contributed by trying to refine it further, particularly what distance is relative to a point and from where. Tom emphasized that they needed to specify the location of the five points they referenced in addition to the original, for how these six points together appear like a hexagon.

The group then entered a second round of refinement, to specify the kaleidoscope’s behavior when drawing lines (as opposed to points, previously). This led to the group articulating a second key property of the definition of a rotational transformation: that given point P, center of rotation O, and image P’, angle POP’ is the degree of rotation and is constant. We see a formalization of this idea, starting from “lines go in whichever direction” to the notion of “angle.” Val asked for reassurance about what to write regarding lines. Tom attempted to describe the behavior out loud but struggled to articulate what to call the lines, requesting the third member, Josh help. Josh described the behavior in terms of “same angles,” formalizing Tom’s ideas.

> Val: And then - ah thank you - "when drawing lines, we observed"... the lines are just raised from the points?
> Tom: I mean essentially, yeah, you're creating some sort of segment…And they go in whichever direction from the point and it does that at each of those other five points. So if you created a one-unit line, heading with a certain vector, it's going to create a one-unit line. Yeah, I don’t know, Josh, you have any input on what's going on with these lines here?
> Josh: How I look at it is, it's kind of like an angle from the point. So if your center angle is to create the hexagon outwards, then it's the same angle that's formed from there.
> Tom: Okay.
> Val: Hmm, that makes so much sense.
> Tom: I like that.
> Josh: Yeah. That is your center point, and then the hexagon's a certain angle and then the outward one is a different angle.
> Tom: There you go.

The pattern evident here is the writer asking for reassurance on how to phrase the transformation’s behavior, another student thinking out loud as they describe behavior and exchanging a back-forth with the writer, and a third student providing precise and succinct wording. We see a last cycle of refinement over how lines are affected by the transformation and the final description in Figure 13:

> Val: Ahh, okay. So when drawing lines, we observed that the lines create an angle.
> Tom: Let's see, when drawing lines, I guess depending on the direction drawn, all of the angles will be the same on each of the given points.
> Val: Okay, so when drawing lines we observed that depending on how the lines are drawn, we see that the angles that - wait, um.
> Tom: It's like a tough one to word 'cause...
> Val: Yeah.
> Tom: Like we're drawing a line and you're gonna draw the line, you can draw it in any direction and depending on how you draw that, the angle from, you could say the x-axis, is going to be the same on each of those lines.
> Josh: You can say like “The angle formed is the same for all six points in relation to your starter?”
> Tom: Okay, yeah, there we go.

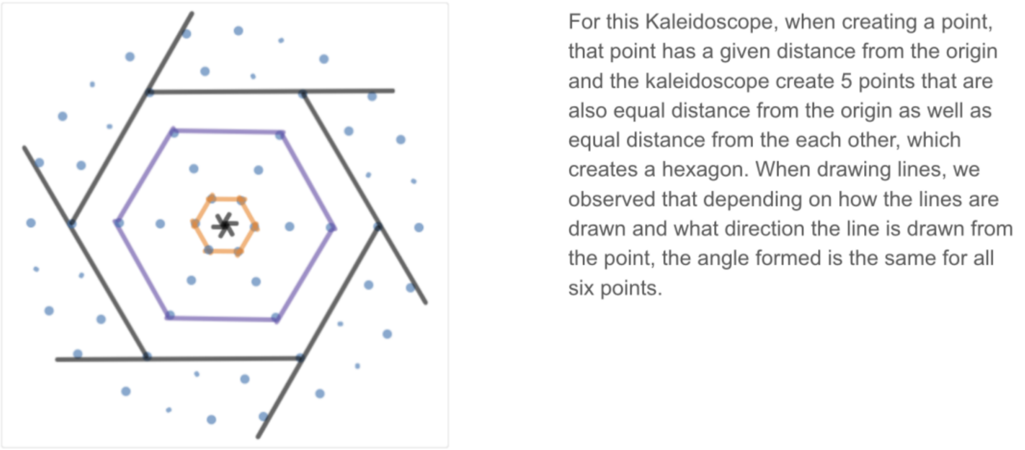


Figure 13. Group's final description for rotation transformation

Needing to write down a description of the rotation spurred students' language about how to describe its behavior on points and lines. This led to the formalizing of ideas and language around constant distance and angles, key components in a definition of rotation.

In summary, writing in real time involved students' social engagement with each other including assigning different roles (e.g., writing, editing, proofreading, etc.) and discussing what content they want to include and how to express them. Technology provided students with a shared workspace where they could read, type, and edit text synchronously, so everyone in their groups could access the space and make changes in the writing in real time. Students attended to precision in their mathematical language by looking for accurate vocabulary and formal expressions to deliver their ideas. Along with their verbal and visual communication in the group work, this practice also provided students with another way to participate in the group discussion by using written communication. Co-construction and writing in real time, both draw on mathematical, technological, and social aspects of virtual collaboration. They allow students to engage in an exchange of ideas and develop knowledge together. However, issues among students can occur when not all students have the opportunity to collaborate equally or their suggestions are not taken up by the group. In the next section, we present potential issues that can emerge in virtual collaboration.

Equitable Distribution of Opportunities to Collaborate

There are certain norms that need to be put in place prior to any collaboration experience. For example, it is important for groups to communicate in ways that allow all group members a chance to be seen and heard. Here we provide an example of a group, Bryn, Sam, and, Alex engaging in co-construction. While co-construction can provide all group members a chance to share their ideas and build knowledge together, the group needs to learn how to provide space for all voices to be heard.

Need for developing group norms for effective communication. In this example, a group of mathematics PSTs in the third author's classroom was working together to create a drawing that would provide the best explanation for the rotation kaleidoscope's mechanism. The team wanted to select a drawing to add to their final presentation so they made sense of the kaleidoscope while drawing various components. The episode starts with Bryn sharing their screen and asking, "If I draw in the middle then that's what happens, if I draw a circle that's what happens." Bryn shared her drawing with the group and sought feedback. When another student Sam asked, "What happens if you make a full circle?" the drawing student Bryn followed this request but shared, "I can't make a full circle" (Figure 14).

Figure 14. Bryn shared her screen to seek feedback from the group (left) and made a circle following the group's request (right)

While Bryn was able to hear and respond to Sam, the group did not spend time to unpack what Sam meant. Sam was referring to a different image but after Bryn's response decided to remain quiet and did not stress this point. Since Sam remained silent about her version of the circle, Bryn assumed they were in agreement and proceeded to the next question. The drawer, Bryn, shared another drawing and said, "The top left and bottom right copy a lot better than the other ones" She also shared that the top right and the bottom left matched. The group seemed to be okay with this suggestion as no student objected, and decided to move on. Another student Alex asked, "If you draw directly up and down in the middle where the axis would be?" The drawer Bryn followed this prompt and on the surface, it seems that all group members liked this image (Figure 15).

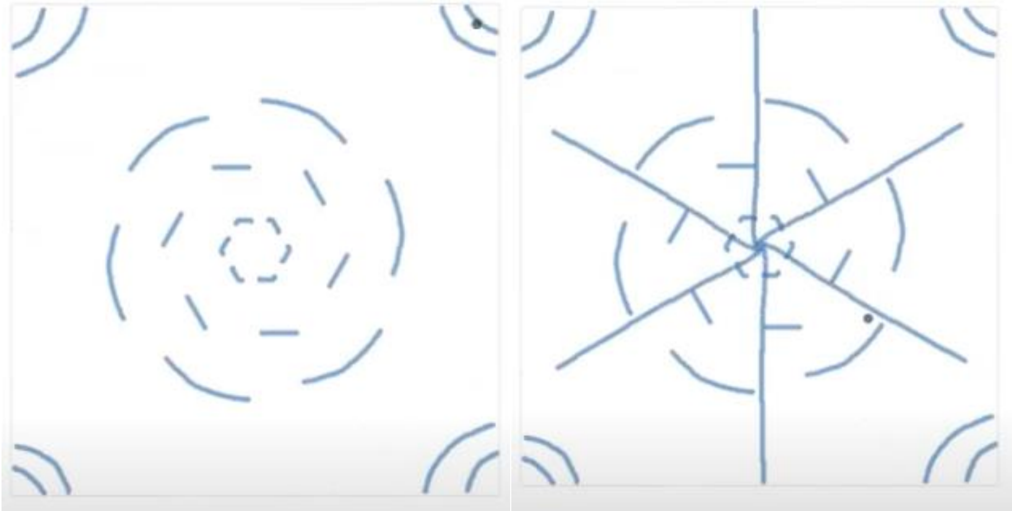


Figure 15. Bryn showed the group that the top left and bottom right match (left) and drew lines from the center following the group's request (right)

However, this was not the case. While the group seemed to agree on Figure 15 as a good fit for their final image to be added to their presentation, they did not use this image in the end. The group decided to design a new image because there was miscommunication while Figure 15 was being developed.

Earlier in the episode when Sam asked Bryn to draw a circle, the drawer (Bryn) misinterpreted what drawing a circle meant. She drew the circle in Figure 14 and claimed that a circle could not be drawn. Sam did not correct Bryn at that time and decided to remain quiet. She later decided to exercise her agency by sharing her screen to show Figure 16. She had meant for a different drawing of a circle but chose to not explain herself earlier. Here, a missed opportunity for communication of ideas hindered the process of collaborative meaning-making.

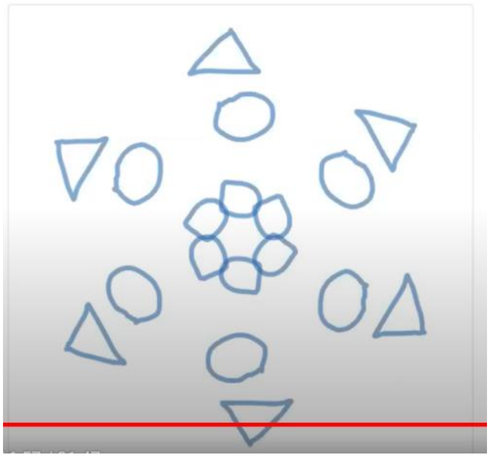


Figure 16. Sam shared her drawing to clarify what she meant by circles

This episode of miscommunication between group members indicates that drawing-in-response and co-construction can still take place when group members do not communicate effectively. However, this episode also stresses the need for developing group norms and for students to develop communication skills to not only listen when others are sharing but also take space to explain their points of view. This episode illuminates the need for instructors to enforce these norms to help students improve their communication skills, as well as guide students in how to advocate for their ideas and be a part of the decision-making process.

Use of images and text as a form of support to exercise agency. When working together to write an explanation for kaleidoscope #2, Bryn, Sam, and Alex started with an amended version of Figure 15, as shown in Figure 17. The team had developed this figure together, but due to a lack of communication, some students' ideas were missed. In this episode, we highlight how the group decided to change their figure and select a new one for their final presentation. In this episode, the reader will notice the students' use of images and text as a form of support to exercise their agency and share their ideas.

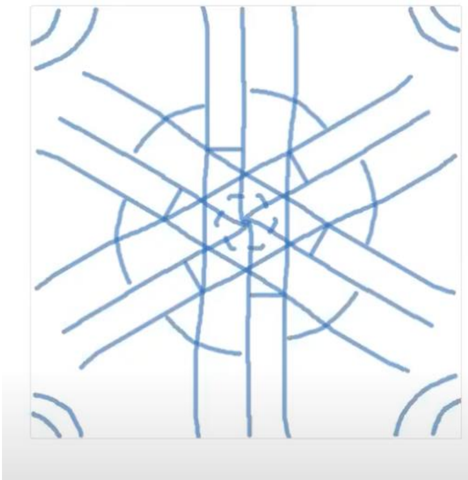


Figure 17. Initial image when writing an explanation for kaleidoscope #2

Starting with Figure 17, Bryn asked, "What do you think about full shapes though, you can't really make full shapes!" Sam who had originally requested that a circle be drawn, replied, "The screen that I'm on I can make circles". This student then shared their screen and displayed Figure 16. Using this figure Sam was able to explain her thinking. Sam shared that if a triangle (another full shape) is drawn, then the kaleidoscope seems to be rotating it (the triangle). This student was supporting the use of shapes such as triangles and circles instead of lines in the final figure because according to Sam these shapes explained the transformation in a better way than using lines. Sam was able to better express her thinking by using her drawing. This idea provided a chance for the group to discuss if using lines versus shapes would provide a clearer depiction of the transformation. Here they were making a decision to use the new image with shapes or the previous image with lines. The group decided to use a final image with shapes for their presentation.

Once the group decided on using the drawing with shapes, Bryn the one who had drawn the original image with lines asked Sam (now the new drawer), "You should draw something in the corner, like a shape in the corner though, before you copy it [image] just because Alex put that the diagonal corners copied each other." Here Bryn was referring to Alex's note-taking and explanation in the final presentation. Alex had taken the role of the note-taker and had typed a description of the rotation in the final presentation. Alex had referred to the original figure drawn by Bryn (Figure 15) when writing the explanation for the kaleidoscope and had stated, "Diagonal corners copied each other." Bryn used this description to guide Sam in updating her figure. It was interesting to notice that the group came up with the explanation for kaleidoscope 2 (Figure 18) using Figure 17 but ended up using a different image for their final presentation. The group then used their write-up as a tool to ensure that the new figure demonstrated all the points that were alluded to in their text. Sam, drawing the new shape, agreed to Bryn's suggestions and made appropriate changes to the figure. Here the text provided authority to Bryn to guide Sam in changing her figure. The group submitted Figure 18 as their final drawing for Kaleidoscope #2.

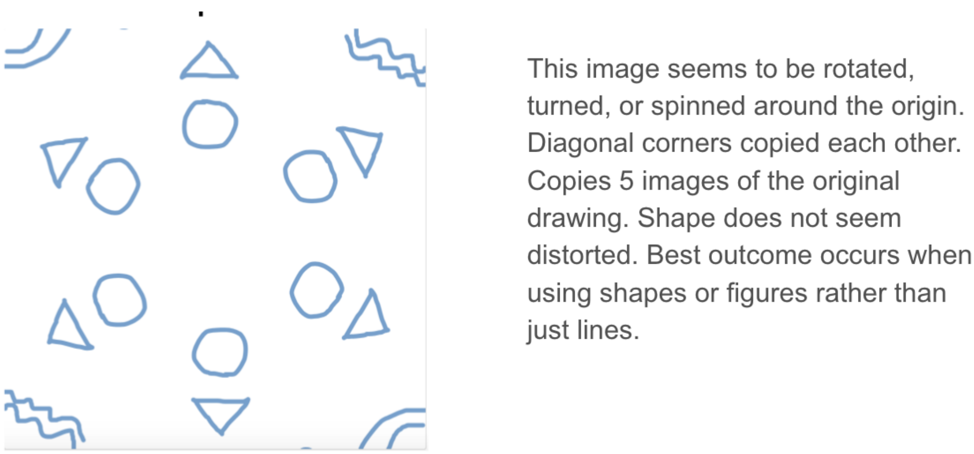


Figure 18. Final drawing and written description submitted by Bryn, Sam, and Alex for kaleidoscope #2

Through these episodes, we again see that all students in the group should have an equal opportunity to be seen and heard. They should be encouraged to exercise their own agency and to encourage others in the group to exercise their agency as well. Instructors planning collaborative tasks should develop and share norms that the groups can abide by. For example, in this episode, the instructor did not assign roles to the students but encouraged the students to select their own roles within the group. Bryn and Sam assumed drawing roles at different points during their collaboration, Alex assumed the role of mathematical authority to select transformation drawings as well as serving as the writer to document the transformations illustrated by their selected drawings. For this group, the members selected these roles by themselves. However, such random role selection may lead to some team members missing out on sharing their ideas.

IV. DISCUSSION

We explored a case of a technological environment and task design for how PSTs across three sites engaged in virtual collaboration during a geometry task and we identified three collaborative practices that emerged: drawing-in-response, co-construction, and writing in real time. Students' implementation of each practice showed that they socially interacted with peers, engaged in mathematical processes regarding transformations, and utilized technological tools in the online synchronous environment. In drawing-in-response, one student would respond to another student's verbal explanation by drawing figures in the collaborative space to confirm their meaning. As an extension, the co-construction of a figure consisted of the open sharing of questions, suggestions, and conjectures while one person drew in the space. The first two practices concerned drawing; writing-in-real time referred to students needing to write as part of the task to describe some mathematical behavior. Across all three practices, we explore possible problems that can occur as well, such as unequal distribution of opportunities to wield authority and agency. We found these practices to be meaningful for fostering virtual collaboration along social, mathematical, and technological dimensions.

In summary, we found evidence-based practices for online synchronous classrooms that facilitated virtual collaboration in group work. In providing descriptive cases here, our goal was to present practices with sufficient detail to be relevant to the work of teacher educators and/or mathematics teachers, as well as the challenges that appear during virtual collaboration. Our intent is not to argue that virtual collaboration is superior to in-person collaboration. Rather, how do we as instructors facilitate collaboration if we already are or need to be in a virtual setting? We saw similar issues emerge in virtual collaboration here as in traditional real-life collaboration for students: unequal engagement and contributions, one person using a tool (Google Slides, Desmos) at a time, students' reluctance to intercede into someone's work or writing, etc.

Implications for Virtual Collaboration

Social collaboration. Issues of agency and authority (Levin et al., 2020) also became apparent in the virtual group work. While collaboration and authority sound like contrasts, we argue that individual students need some form of authority whether in terms of the content or socially in order to collaborate and for their contributions to be taken up by the group and built on. Some students may have more agency and authority in the group decision process. Our work shows however that writing in real time provides written records and artifacts that students with less agency can point to in vouching for their contribution and backing up their verbal communication. Through the issues and challenges we saw, there is a need to establish norms for collaboration in virtual spaces.

Technological collaboration. Our work has implications for identifying design features of technology and online environments that facilitate virtual collaboration. The screen-share feature in Zoom facilitated students to make sense of geometric transformations collectively on the Desmos applet in real time. The dynamic nature of the transformation tool on the shared screen supported students' communication visually and in real time when they made observations about its behavior and generated and tested their conjectures about the underlying transformation.

Our work also reveals some areas for improvement in the technological design of the task. These improvements include adding axes to the Desmos space so students can refer to them in describing what changes they see; this would add to the precision of the task. Our design of the task would also benefit from using color more strategically, to help students keep track of multiple copies of images created. We also found that the difficulty levels of the transformations mattered for engaging students in collaboration: Difficult transformations such as reflection and dilation were more group-worthy, as students talked together more to figure out their behaviors and test out conjectures with real time dynamic drawing. Interestingly, while the conveyance tools we used such as Google Slides do allow for multiple students to write at the same time, we found that one student would do the writing. Social norms about writing from in-person collaboration, where sometimes one student writes the group's answers on a physical document, persisted here.

Mathematical processes. Our analyses revealed that students engaged in mathematical processes through the three collaborative practices. In needing to create and select the best representation and written description of the four kaleidoscopes, students investigated the following questions about geometric properties such as how it changes the orientation of figures, where it creates copies of their drawing, located relative to the original, how it affects the size of figures, and what figures are invariant under certain transformation. We also observed how our task design and collaborative practices led students to engage in helpful mathematical practices in Common Core State Standards for Mathematics (CCSS-M) (CCSSI, 2018). The task feature of asking for the best visual representation of the transformation led groups to investigate its variants and invariants, in terms of where to create figures (e.g., location, position on the plane) and what types of figures (e.g., dots, lines, rays) to draw. The need to produce a written description also pushed groups to reflect on and refine their initially informal language to more precise and formal mathematical language to explain the transformations' behavior.

Limitations & Future Directions

One limitation of our work is it is not generalizable. We presented some cases of virtual collaboration, so we did not make any quantitative claims about the prevalence of the practices we found. We provide descriptive cases for instructors to see how student practices emerge but also potential issues that may occur. Another limitation is that our participants and data were limited to the United States, so we do not claim such practices will be common in other countries. Future work should explore virtual collaboration with a larger number of students in other countries.

The task design had some limitations as well - each "kaleidoscope" depicted multiple transformations (e.g., the translation kaleidoscope showed three translations; the rotation kaleidoscope showed five rotations). At times, the multiple transformations led to unintended observations. For example, the translations were all relative to horizontal and vertical vectors of the same length, so one group thought translations involved quadrants of the coordinate plane. This limitation does not affect our analysis, in that students' interactions with the technology, mathematics, and each other were still visible, even if their mathematical observations did not correspond with correct transformation behavior. We will adjust these specific task design features in any future iterations of this lesson and work. We plan to collect more empirical data on student collaboration, to assist teachers in how to support students.

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