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Distance-Regular Graphs with Valency k, Diameter $D \geq 3$ and at Most $Dk+1$ Vertices

Jongyook Park
Department of Mathematics, Kyungpook National University, Daegu, 41566, Republic of Korea
e-mail: jongyook@knu.ac.kr

Abstract. Let Γ be a distance-regular graph with valency k and diameter $D \geq 3$. It has been shown that for a fixed real number $\alpha > 2$, if Γ has at most αk vertices, then there are only finitely many such graphs, except for the cases where ($D = 3$ and Γ is imprimitive) and ($D = 4$ and Γ is antipodal and bipartite). And there is a classification for $\alpha \leq 3$. In this paper, we further study such distance-regular graphs for $\alpha > 3$.

Let $\beta \geq 3$ be an integer, and let Γ be a distance-regular graph with valency k, diameter $D \geq 3$ and at most $\beta k+1$ vertices. Note that if $D \geq \beta+1$, then Γ must have at least $\beta k+2$ vertices. Thus, the assumption that Γ has at most $\beta k+1$ vertices implies that $D \leq \beta$. We focus on the case where $D = \beta$ and provide a classification of distance-regular graphs having at most $Dk+1$ vertices.

1. Introduction

Let Γ be a distance-regular graph with valency k and diameter $D \geq 3$. Koolen and Park [3] showed that except in both cases when $D = 3$ and Γ is imprimitive, and $D = 4$ and Γ is antipodal and bipartite, if the number of vertices of Γ is bounded above by a fixed constant times the valency, then there are only finitely many such graphs. The exact statement of the result is given below.

Theorem 1. ([3, Theorem 1]) *Let $\alpha > 2$ be a real number. Then there are finitely many distance-regular graphs Γ with valency k, diameter $D \geq 3$ and v vertices satisfying $v \leq \alpha k$ unless one of the following holds:*

(*i*) $D = 3$ *and* Γ *is imprimitive,*
(*ii*) $D = 4$ *and* Γ *is antipodal and bipartite.*

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As a continuation of this work, Park [6] considered the situation where, for a fixed vertex, the number of vertices at distance $D-1$ or D is at most twice the valency. In particular, if $D=3$, then a distance-regular graph Γ has at most $3k+1$ vertices, and hence it provides a classification of Theorem 1 for $\alpha \leq 3$ (see [6, Theorem 1]).

In this paper, we further study Theorem 1 for $\alpha > 3$. Here, we observe the following: For an integer $\beta \geq 3$, if $D \geq \beta+1$, then a distance-regular graph with diameter D has at least $\beta k+2$ vertices (see [1, Proposition 5.1.1 (ii)]). So, if we assume that a distance-regular graph Γ has at most $\beta k+1$ vertices, then the diameter D satisfies $D \leq \beta$. This also says that the diameter D in Theorem 1 is bounded above by $\lceil \alpha \rceil$. We focus on the case where $D=\beta$ and provide a classification of distance-regular graphs having at most $Dk+1$ vertices.

This paper is organized as follows: in the follwing section, we give some definitions and preliminaries. In Section 3, we study distance-regular graphs with valency k, diameter $D \geq 3$ and v vertices satisfying $v \leq Dk+1$, and we provide a classification of such distance-regular graphs.

2. Definitions and Preliminaries

All the graphs considered in this paper are finite, undirected and simple. The reader is referred to [1] for more information. Let Γ be a connected graph with vertex set $V(\Gamma)$. The *distance* $d_\Gamma(x,y)$ between two vertices $x, y \in V(\Gamma)$ is the length of a shortest path between x and y in Γ. The *diameter* $D = D(\Gamma)$ of Γ is the maximum distance occuring in the graph Γ. For each $x \in V(\Gamma)$, let $\Gamma_i(x)$ be the set of vertices of Γ at distance i from x $(0 \leq i \leq D)$. In addition, define $\Gamma_{-1}(x) = \emptyset$ and $\Gamma_{D+1}(x) = \emptyset$. For the sake of simplicity, let $\Gamma(x) = \Gamma_1(x)$. In particular, Γ is *regular* with *valency* k if $k = |\Gamma(x)|$ holds for all $x \in V(\Gamma)$.

For a connected graph Γ with diameter D, take any two vertices x and y at distance $i = d_\Gamma(x,y)$, and consider the numbers $c_i(x,y) = |\Gamma_{i-1}(x) \cap \Gamma(y)|$, $a_i(x,y) = |\Gamma_i(x) \cap \Gamma(y)|$ and $b_i(x,y) = |\Gamma_{i+1}(x) \cap \Gamma(y)|$ $(0 \leq i \leq D)$. We say that the *intersection numbers* c_i, a_i and b_i exist if the numbers $c_i(x,y)$, $a_i(x,y)$ and $b_i(x,y)$ respectively depend only on $i = d_\Gamma(x,y)$ and not on the choice of x and y with $d_\Gamma(x,y) = i$. Set $c_0 = b_D = 0$ and observe that $a_0 = 0$ and $c_1 = 1$. A connected graph Γ with diameter D is called a *distance-regular graph* if there exist intersection numbers c_i, a_i, b_i for all $i = 0, 1, \ldots, D$. Note that for a distance-regular graph Γ, the number $|\Gamma_i(x)|$ does not depend on the choice of $x \in V(x)$, and in this case the number $k_i = \frac{kb_1 \cdots b_{i-1}}{c_1 c_2 \cdots c_i}$ is equal to $|\Gamma_i(x)|$ for all $x \in V(x)$ and all $i = 0, 1, \ldots, D$.

For a connected graph Γ with diameter D, the *distance-i graph* Γ_i $(0 \leq i \leq D)$ is the graph whose vertices are those of Γ and edges are the 2-subsets of vertices at mutual distance i in Γ. In particular, $\Gamma_1 = \Gamma$. An *antipodal* graph is a connected graph Γ with diameter $D > 1$ for which its distance-D graph Γ_D is a disjoint union of complete graphs. A graph Γ is called *bipartite* if it has no odd cycle. Note that if Γ is a bipartite distance-regular graph with diameter D, then $a_1 = a_2 = \cdots = a_D = 0$.

For a connected graph Γ with diameter D, the *adjacency matrix* $A = A(\Gamma)$ is the matrix whose rows and columns are indexed by the vertices of Γ and the (x,y)-entry is 1 whenever two vertices x and y are adjacent, and 0 otherwise. The *eigenvalues* of Γ are the eigenvalues of A.

3. Distance-Regular Graphs Having at Most $Dk + 1$ Vertices

In this section, we study distance-regular graphs with valency k, diameter $D \geq 3$ and v vertices satisfying $v \leq Dk+1$, and we provide a classification of such distance-regular graphs. First, we show that the assumption $v \leq Dk+1$ implies that k_2 is small compared to the valency k or that the diameter D is small.

Lemma 2. *Let Γ be a distance-regular graph with valency k, diameter $D \geq 3$ and v vertices satisfying $v \leq Dk+1$. Then $k_2 \leq \frac{3}{2}k$ or $D \leq 4$.*

Proof: We assume that $k_2 > \frac{3}{2}k$. Then it is enough to prove that $D \leq 4$. Assume, on the contrary, that $D \geq 5$. Then, [1, Proposition 4.1.6 (*ii*)] says that $b_2 \geq c_3$, and this implies that $k_3 = k_2\frac{b_2}{c_3} \geq k_2 > \frac{3}{2}k$, i.e., $k_2 + k_3 > 3k$. From [1, Proposition 5.1.1 (*ii*)], we know that $k_i \geq k$ for all $4 \leq i \leq D-1$. But then we have that $v = 1 + k + (k_2 + k_3) + (k_4 + \cdots + k_{D-1}) + k_D > 1 + k + 3k + (D-4)k + 1 = Dk + 2$, where the inequality holds as $k_2 + k_3 > 3k$, $k_i \geq k$ for all $4 \leq i \leq D-1$ and $k_D \geq 1$. This contradicts that $v \leq Dk+1$. Thus, the diameter D must be at most 4. This finishes the proof. □

From the above lemma, we find that the assumption $v \leq Dk+1$ implies that $k_2 \leq \frac{3}{2}k$ or $D \in \{3,4\}$. The first case was addressed by Koolen and Park [4, Theorem 20], who classified distance-regular garphs with diameter $D \geq 3$ and $k_2 \leq \frac{3}{2}k$. The latter case is $D \in \{3,4\}$, and the classification for $D = 3$ is given in [6, Theorem 1]. So, we consider the case where $D = 4$ (under the assumption $k \geq 3$), and we prove that a ditance-regular graph with valency $k \geq 3$, diameter $D = 4$ and v vertices satisfying $v \leq Dk+1$ is a Hadamard graph.

Theorem 3. *Let Γ be a distance-regular graph with valency $k \geq 3$, diameter $D = 4$ and v vertices satisfying $v \leq 4k+1$. Then Γ is a Hadamard graph.*

Proof: By [1, Proposition 5.1.1 (*ii*)], we know that $k_2 \geq k$, and this implies that $k_3 + k_4 \leq 2k$. Then [6, Theorem 2] implies that $k_D = k_4 = 1$ (as $k > 2$ and $D > 3$). Note that $k_4 = 1$ implies that $c_4 = k$, $a_4 = 0$, $b_3 = 1$ and $k_3 = k$.

We first consider the case where Γ is bipartite. Then $a_1 = a_2 = a_3 = 0$, and this implies that $b_1 = k-1 = c_3$ and $c_2 = \frac{k}{2} = b_2$. Thus, the graph Γ has an intersection array $\{k, k-1, \frac{k}{2}, 1; 1, \frac{k}{2}, k-1, k\}$, and hence Γ is a Hadamard graph.

Now, we assume that Γ is not bipartite. By [2, Theorem 2], we know that the intersection number c_2 is at most $\frac{k}{3}$, i.e., $k_2 = \frac{kb_1}{c_2} \geq \frac{kb_1}{k/3} = 3b_1$. Since $k_2 \geq 3b_1$, we have that $4k+1 \geq v = 1+k+k_2+k_3+k_4 = 1+k+k_2+k+1 = 2+2k+k_2 \geq 2+2k+3b_1$, and hence $b_1 \leq \frac{2}{3}k - \frac{1}{3}$ and $a_1 = k-1-b_1 \geq \frac{k}{3} - \frac{2}{3}$. Note that the second largest eigenvalue θ_1 of Γ is at least $\frac{a_1+\sqrt{a_1^2+4k}}{2} > a_1 + 1$ (see, for example, [5, Theorem 1.2]).

We consider the standard sequence $u_0(\theta_1), u_1(\theta_1), \ldots, u_4(\theta_1)$ of Γ corresponding to θ_1. Since $a_1 \geq \frac{k}{3} - \frac{2}{3}$, we have that $\theta_1 > a_1 + 1 \geq \frac{k+1}{3}$. Thus, $u_1(\theta_1) = \frac{\theta_1}{k} > \frac{k+1}{3k}$. By [1, Theorem 4.1.4], we find an upper bound on the multiplicity m_1 of θ_1 as follows:

$$m_1 = \frac{v}{\sum_{i=0}^{4}[u_i(\theta_1)]^2 k_i} \leq \frac{v}{[u_1(\theta_1)]^2 k} < \frac{4k+1}{[(k+1)/3k]^2 k} = \frac{36k^2 + 9k}{(k+1)^2} < 36.$$

So, $m_1 \leq 35$. By [1, Theorem 5.3.2], we find an upper bound on the valency k as follows:

$$k \leq \frac{(m_1 - 1)(m_1 + 2)}{2} \leq 629.$$

Thus, $v \leq 4k + 1 = 2517$. All feasible intersection arrays of non-bipartite distance-regular graphs with diameter 4 having at most 4096 vertices are contained in [1, p.419–425]. We checked the feasible intersection arrays and found that there is no non-bipartite distance-regular graph with valency $k \geq 3$, diameter 4 and at most $4k + 1$ vertices. This finishes the proof. □

Remark 1. *A Hadamard graph has an intersection array* $\{k, k-1, \frac{k}{2}, 1; 1, \frac{k}{2}, k-1, k\}$ *for some even integer* $k \geq 2$. *Its diameter is* 4, *and it has* $4k$ *vertices. If* $k = 2$, *then it is the* 8*-gon. If* $k = 4$, *then it is the* 4*-cube. If* $k \geq 6$, *then it satisfies that* $k_2 > \frac{3}{2}k$.

Now, we are ready to give a classification of distance-regular graphs with valency k, diameter $D \geq 3$ and v vertices satisfying $v \leq Dk + 1$

Theorem 4. *Let* Γ *be a distance-regular graph with valency* k, *diameter* $D \geq 3$ *and* v *vertices satisfying* $v \leq Dk + 1$. *Then* $k = 2$ *or* $D \leq 4$. *In particular, if* $k \geq 3$, *then* $D \leq 4$ *and one of the follwing holds:*

(*i*) $D = 3$ *and* Γ *is bipartite,*
(*ii*) $D = 3$ *and* Γ *is a Taylor graph,*
(*iii*) $D = 3$ *and* Γ *is the Johnson graph* $J(7, 3)$,
(*iv*) $D = 3$ *and* Γ *is the halved* 7*-cube,*
(*v*) $D = 4$ *and* Γ *is a Hadamard graph.*

Proof: By Lemma 2, we know that $k_2 \leq \frac{3}{2}k$ or $D \leq 4$. If $k_2 \leq \frac{3}{2}k$, then by [4, Theorem 20], we also know that $k = 2$ or $D \leq 4$. Thus, $k = 2$ or $D \leq 4$ holds.

If $k = 2$, then the graph Γ is a polygon and it has either Dk or $Dk + 1$ vertices.

Now, we assume that $k \geq 3$ and $D \leq 4$. If $D = 3$, then the graph Γ has at most $3k + 1$ vertices. And then by [6, Theorem 1] we know that Γ is one of $(i) - (iv)$. So, we may assume that $D = 4$. Then by Theorem 3, we know that Γ is a Hadamard graph. This finishes the proof. □

Note that if we set $D = 3$ in Theorem 4, then we obtain the same result as [6, Theorem 1]. And we obtain the following result as an immediate consequence of Theorem 4.

Corollary 5. *Let $\beta \geq 3$ be an integer. Let Γ be a distance-regular graph with valency k, diameter $D \geq 3$ and v vertices satisfying $v \leq \beta k + 1$. Then, one of the following holds:*

(i) $k = 2$ *and* Γ *is a polygon,*
(ii) $D = 3$ *and* Γ *is bipartite,*
(iii) $D = 3$ *and* Γ *is a Taylor graph,*
(iv) $D = 3$ *and* Γ *is the Johnson graph* $J(7, 3)$*,*
(v) $D = 3$ *and* Γ *is the halved 7-cube,*
(vi) $D = 4$ *and* Γ *is a Hadamard graph.*
(vii) $3 \leq D < \beta$ *and* $Dk + 2 \leq v \leq \beta k + 1$.

Proof: Recall that $D \leq \beta$ holds, otherwise Γ has at least $\beta k + 2$ vertices.

If $D = \beta$, i.e., $v \leq Dk + 1$, then by Theorem 4 one of $(i) - (vi)$ holds. So, we may assume that $D < \beta$.

If $v \leq Dk + 1 (< \beta k + 1)$, then again by Theorem 4 one of $(i) - (vi)$ holds.

Now, we assume that $v > Dk + 1$, i.e., $Dk + 2 \leq v \leq \beta k + 1$. In this case, we have (vii). This finishes the proof. □

Remark 2. (1) *In the case* (vii)*, we do not give an upper bound on the valency k. However, from Theorem 1, we know that there are only finitely many such distance-regular graphs Γ except* ($D = 3$ *and* Γ *is imprimitive*) *and* ($D = 4$ *and* Γ *is antipodal and bipartite*)*.*

(2) [6, Theorem 1] *already provides a classification for the case where $\beta = 3$ in Theorem 5. However, a classification for the case where $\beta = 4$ is not known yet. One can think about the case where $D = 3$ and $3k + 2 \leq v \leq 4k + 1$. Combined with Theorem 3, this will give a classification of distance-regular graphs having at most $4k + 1$ vertices.*

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