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# Some Aspects of $\lambda$-$\triangle^m$-Statistical Convergence on Neutrosophic Normed Spaces

Reena Antal*
*Department of Mathematics, Chandigarh University, Mohali, India*
*e-mail* : reena.antal@gmail.com

Meenakshi Chawla
*Department of Applied Sciences, Chandigarh Group of Colleges, Jhanjeri, Mohali, India*
*e-mail* : chawlameenakshi7@gmail.com

Abstract. Neutrosophication is a useful tool for handling real-world problems with partially dependent, partially independent, and even independent components. By examining some properties related to $\lambda$-statistical convergence on neutrosophic normed spaces, we provide some functional tools that are helpful in situations of inconsistency and indeterminacy. Additionally, we establish some related results on $\lambda$-$\triangle^m$-statistical Cauchy sequences on neutrosophic normed spaces.

## 1. Introduction

Numerous research studies have been conducted and various classical theories have been developed to address real-life problems in science and technology. However, classical sets are inadequate in explaining uncertainties that occur in daily life situations. In modern logic, three-way decision situations, such as accepting/rejecting/pending cases, yes/no/not-applicable situations, win/lose/tie in sports, *etc.*, cannot be adequately explained by the theory of standard analysis. To solve this problem, non-standard analysis is employed. Neutrosophic sets are a valuable advancement of classical sets, fuzzy sets and intuitionistic fuzzy sets for non-standard analysis, introduced by Smarandache [25]. Neutrosophic sets can manage inconsistent, indeterminate, and imprecise data for problems where the fundamental rules of fuzzy set theory and intuitionistic fuzzy set theory are not sufficient.

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They are used to investigate the degrees of correctness, wrongness, and uncertainty of the elements in the set. Each element of a neutrosophic set has a truth value, a false value and an indeterminacy value, which falls within the non-standard unit interval. Because of this nature, neutrosophic sets are more adaptable, reasonable, and efficient tools for handling not only the free components of information but also partially independent and dependent components. Neutrosophic sets are suitable for real-life situations such as databases, image processing problems, control theory, medical diagnosis problems and decision-making problems. In a neutrosophic set, elements may have inconsistent information (*i.e.* the sum of the components >1) or incomplete information (i.e. the sum of the components <1) or consistent information (*i.e.* the sum of the components =1), and other interval-valued components (*i.e.* without any restriction on the sum of superior or inferior components).

**Definition 1.1.** [25] Let $U$ be a subset of $\mathbb{X}$ (which is a space of points) with $a \in \mathbb{X}$. Then set $U$ is called neutrosophic set(NS) with $\tau(a)$, $\upsilon(a)$ and $\eta(a)$ in $\mathbb{X}$ and expressed as

$$U = \{< a, \tau(a), \upsilon(a), \eta(a) > : \ a \in \mathbb{X} \, \text{and} \, \tau(a), \upsilon(a), \eta(a) \in I\}$$

where $\tau(a)$, $\upsilon(a)$ and $\eta(a)$ denotes truth membership function, indeterminacy membership function and falsity membership function respectively, such that $0^- \leq \tau(a) + \upsilon(a) + \eta(a) \leq 3^+$. Also $I = ]0^-, 1^+[$ represents a non-standard unit interval.

Wang *et al*. [27] and Ye [28] revised the existing definitions of the neutrosophic set using the interval [0,1] by introducing the single-valued neutrosophic set and simplified neutrosophic set respectively, that can be utilized in the applications of engineering and scientific areas.

Mahapatra and Bera [5] looked at the concept of neutrosophic soft linear spaces. Kirişci and Şimşek [15] introduced the idea of neutrosophic metric spaces and established their fundamental topological and geometric properties. In [16] proposed the following notion of neutrosophic normed spaces, which is an important consideration of neutrosophic metric spaces.

**Definition 1.2.** [16] A neutrosophic normed space(NNS) is a 4-tuple $(\mathbb{X}, \aleph, \circledast, \odot)$ consisting of a vector space $\mathbb{X}$, a normed space $\aleph = \{< \tau(a), \upsilon(a), \eta(a) > : a \in \mathbb{X}\}$ such that $\aleph : \mathbb{X} \times \mathbb{R}^+ \to [0,1]$, a continuous t-norm $\circledast$ and a continuous t-conorm $\odot$. For every $x, y \in \mathbb{X}$ and $s, t > 0$, we have:
(i) $0 \leq \tau(x,t), \upsilon(x,t), \eta(x,t) \leq 1$,
(ii) $\tau(x,t) + \upsilon(x,t) + \eta(x,t) \leq 3$,
(iii) $\tau(x,t) = 1$, $\upsilon(x,t) = 0$ and $\eta(x,t) = 0$ for $t > 0$ iff $x = 0$,
(iv) $\tau(x,t) = 0$, $\upsilon(x,t) = 1$ and $\eta(x,t) = 1$ for $t \leq 0$,
(v) $\tau(\alpha x, t) = \tau\left(x, \frac{t}{|\alpha|}\right)$, $\upsilon(\alpha x, t) = \upsilon\left(x, \frac{t}{|\alpha|}\right)$ and $\eta(\alpha x, t) = \eta\left(x, \frac{t}{|\alpha|}\right)$ for $\alpha \neq 0$,
(vi) $\tau(x, \circ)$ as continuous non-decreasing function,
(vii) $\tau(x,s) \circledast \tau(x,t) \leq \tau(x+y, s+t)$,
(viii) $\upsilon(x, \circ)$ as continuous non-increasing function,

(ix) $\upsilon(x,s) \odot \upsilon(y,t) \geq \upsilon(x+y, s+t)$,
(x) $\eta(x,\circ)$ as continuous non-increasing function,
(xi) $\eta(x,s) \odot \eta(y,t) \geq \eta(x+y, s+t)$,
(xii) $\lim_{t\to\infty} \tau(x,t) = 1$, $\lim_{t\to\infty} \upsilon(x,t) = 0$ and $\lim_{t\to\infty} \eta(x,t) = 0$.
The tuple $(\tau, \upsilon, \eta)$ is known as neutrosophic norm.

**Example 1.1.** [16] Let $(\mathbb{X}, \|\,.\,\|)$ be a normed space. For all $t > 0$ and $x \in \mathbb{X}$, take
(i) $\tau(x,t) = \frac{t}{t+\|x\|}$ , $\upsilon(x,t) = \frac{\|x\|}{t+\|x\|}$ and $\eta(x,t) = \frac{\|x\|}{t}$ when $t > \|x\|$,
(ii) $\tau(x,t) = 0$, $\upsilon(x,t) = 1$ and $\eta(x,t) = 1$ when $t \leq \|x\|$.
Also let $g \circledast h = gh$ and $g \odot h = g + h - gh$ for $g, h \in [0,1]$. The 4-tuple $(\mathbb{X}, \aleph, \circledast, \odot)$ is a NNS which satisfies above mentioned conditions.

A generalized version of intuitionistic fuzzy norms has been considered in neutrosophic normed spaces that helped to investigate fundamental properties such as convergence and completeness in these spaces. Omran and Elrawy [23] discussed the relationship between continuous operators with bounded operators in the structure of neutrosophic normed spaces. Khan and Khan [13] also contributed to this topic by studying various topological properties and characterizations of these spaces. Further, Kirişci and Şimşek [16]established the concept of convergence for sequences on neutrosophic normed spaces.

**Definition 1.3.** [16] Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm $(\tau, \upsilon, \eta)$. A sequence $x = \{x_k\}$ from $\mathbb{X}$ is called convergent to $x_0 \in \mathbb{X}$ with respect to neutrosophic norm $(\tau, \upsilon, \eta)$ if for every $\epsilon > 0$ and $t > 0$ we can find $k_0 \in \mathbb{N}$ provided $\tau(x_k - x_0, t) > 1 - \epsilon$, $\upsilon(x_k - x_0, t) < \epsilon$ and $\eta(x_k - x_0, t) < \epsilon$ for $k \geq k_0$. It is represented symbolically by $(\tau, \upsilon, \eta)$-$\lim_{k\to\infty} x_k = x_0$ or $x_k \xrightarrow{(\tau,\upsilon,\eta)} x_0$.

**Remark 1.1.** In the previous example of the NNS $(\mathbb{X}, \aleph, \circledast, \odot)$, we have $x_k \xrightarrow{(\tau,\upsilon,\eta)} x_0$ if and only if $x_k \xrightarrow{\|\cdot\|} x_0$.

Kirişci and Şimşek[16] established the statistical convergence for sequences in the neutrosophic normed spaces using natural density. Although, natural density of set $A (A \subseteq \mathbb{N})$has given by $\delta(A) = \lim_{n\to\infty} \frac{1}{n} \mid \{a \leq n : a \in A\} \mid$, provided limit exists and $\mid . \mid$ designates the order of the enclosed set. Further, sequence $x = \{x_k\}$ is statistically convergent to $x_0$, if $A(\epsilon) = \{k \in \mathbb{N} : |x_k - x_0| > \epsilon\}$ has zero natural density (see [11]).

**Definition 1.4.**[16] Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm $(\tau, \upsilon, \eta)$. A sequence $x = \{x_k\}$ from $\mathbb{X}$ is called statistically convergent to $x_0 \in \mathbb{X}$ with respect to neutrosophic norm $(\tau, \upsilon, \eta)$ if for every $\epsilon > 0$ and $t > 0$, we have

$$\delta(\{k \in \mathbb{N} : \tau(x_k - x_0, t) \leq 1 - \epsilon \text{ or } \upsilon(x_k - x_0, t) \geq \epsilon,\ \eta(x_k - x_0, t) \geq \epsilon\}) = 0.$$

It is represented symbolically by $St_{(\tau,\upsilon,\eta)} - \lim_{k\to\infty} x_k = x_0$ or $x_k \xrightarrow{St_{(\tau,\upsilon,\eta)}} x_0$ .

Some noteworthy results related to statistical convergence on neutrosophic normed spaces in different directions have been studied ($c.f.$ [13, 17, 18]). In this paper, we have associated the theory of neutrosophic normed spaces with $\lambda$-statistical convergence of sequences. The $\lambda$-statistical convergence is a generalized form of sequence convergence presented by Mursaleen [22], using a non-decreasing sequence $\lambda = \{\lambda_n\}$ which tends to $\infty$ such that $\lambda_{n+1} \leq \lambda_n + 1$ and $\lambda_1 = 1$. Also, the generalized de la Vallée-Poussin mean has been given by $t_n(x) = \dfrac{1}{\lambda_n} \sum_{k \in I_n} x_k$, where $I_n = [1 + n - \lambda_n, n]$.

Throughout the paper we use $I_n$ for $[1 + n - \lambda_n, n]$.

**Definition 1.5.**[22] A sequence $x = \{x_k\}$ is called $\lambda$-statistically convergent to $x_0$ provided for every $\epsilon > 0$ satisfies

$$\lim_{n\to\infty} \frac{1}{\lambda_n} |\{\, k \in I_n : |x_k - x_0| \geq 1 - \epsilon \,\}| = 0,$$

or

$$\delta_\lambda(\{\, k \in I_n : |x_k - x_0| \geq 1 - \epsilon \,\}) = 0.$$

It is represented symbolically by $S_\lambda - \lim\limits_{n\to\infty} x_k = x_0$.

Kizmaz[19] discovered the difference sequence spaces conception by considering $Z(\triangle) = \{\, x = \{x_k\} : \{\triangle x_k\} \in Z \,\}$ with the spaces $Z = l_\infty$(spaces of all the bounded sequences), $c_0$(spaces of all the convergent sequences) and $c_0$(spaces of all the null sequences), where $\triangle x = \{\triangle x_k\} = \{\, x_k - x_{k+1} \,\}$, and $x = \{x_k\}$ is a real sequence for all $k \in \mathbb{N}$. The spaces $l_\infty(\triangle), c(\triangle)$ and $c_0(\triangle)$ are Banach spaces, due to the norm endowed by $\|x\|_\triangle = |x_1| + \sup_k |\triangle x_k|$. Moreover, the generalized difference sequence spaces were defined by Et and Çolak[8] considering $Z(\triangle^m) = \{x = \{x_k\} : \{\triangle^m x_k\} \in Z\}$, where $m$ be any fixed positive integer, for $Z = l_\infty, c, c_0$ and $\triangle^m x = \{\triangle^m x_k\} = \{\triangle^{m-1} x_k - \triangle^{m-1} x_{k+1}\}$ so that $\triangle^m x_k = \sum_{r=0}^{m} (-1)^r \binom{m}{r} x_{k+r}$. The $\triangle^m$-statistical convergence concept studied and established by Mikail and Nuray[9] with the help of statistical convergence.

**Definition 1.6.**[9] A sequence $x = \{x_k\}$ is called $\Delta^m$- statistically convergent to $x_0$ provided with every $\epsilon > 0$, we have

$$\delta\left(\{\, k \leq n : |\Delta^m x_k - x_0| \geq \epsilon \,\}\right) = 0.$$

It is represented symbolically by $St - \lim \Delta^m x_k = x_0$.

A lot of work related to convergence of difference sequences as fusion with different structures, has been done by various researchers [1, 10, 26, 4, 24, 21, 12, 2, 7, 3, 6, 14, 20] which leads us to investigate and explore $\lambda$-$\triangle^m$-statistical convergence with the theory of neutrosophic normed spaces.

## 2. Main Results

We first mention the conception of $\lambda$-$\triangle^m$-statistical convergence of the sequences on neutrosophic normed spaces (NNS) that will be helpful in studying the major results of work.

**Definition 2.1.** Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm $(\tau, \upsilon, \eta)$. A sequence $x = \{x_k\}$ from $\mathbb{X}$ is said to be $\lambda$-$\triangle^m$-statistically convergent to $x_0 \in \mathbb{X}$ with respect to neutrosophic norm $(\tau, \upsilon, \eta)$ if for every $t > 0$ and $\epsilon \in (0,1)$, satisfies

$$\delta_\lambda(\{k \in I_n : \tau(\triangle^m x_k - x_0; t) \leq 1-\epsilon \text{ or } \upsilon(\triangle^m x_k - x_0, t) \geq \epsilon, \eta(\triangle^m x_k - x_0, t) \geq \epsilon\}) = 0.$$

It is represented symbolically by $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$ or $\triangle^m x_k \xrightarrow{\lambda\text{-}St_{(\tau,\upsilon,\eta)}} x_0$.

**Example 2.1.** Let $(\mathbb{R}, \|\,.\,\|)$ be any normed space. For every $x \in \mathbb{X}$, we take $\tau(x,t) = \frac{t}{t+\|x\|}$ , $\upsilon(x,t) = \frac{\|x\|}{t+\|x\|}$ and $\eta(x,t) = \frac{p\|x\|}{t+p\|x\|}$ when $p \in \mathbb{R}$. Also, $g \circledast h = \min\{g,h\}$ and $g \odot h = \max\{g,h\}$ for $g, h \in [0,1]$.
Then, a 4-tuple $(\mathbb{X}, \aleph, \circledast, \odot)$ is a NNS.
Consider a sequence $x = \{x_k\}$ such that

$$\triangle^m x_k = \begin{cases} 2 & n - \sqrt{\lambda_n} + 1 \leq k \leq n \text{ } i.e. \text{ } k \in I_n \\ 0 & \text{otherwise} \end{cases}$$

For $t > 0$ and $\epsilon > 0$, we have

$$\begin{aligned} A(\epsilon,t) &= \{k \in I_n : \tau(\triangle^m x_k - 0, t) \leq 1-\epsilon \text{ or } \upsilon(\triangle^m x_k - 0, t) \geq \epsilon,\ \eta(\triangle^m x_k - 0, t) \geq \epsilon\} \\ &= \{k \in I_n : \frac{t}{t + |\triangle^m x_k|} \leq 1-\epsilon \text{ or } \frac{|\triangle^m x_k|}{t + |\triangle^m x_k|} \geq \epsilon,\ \frac{p|\triangle^m x_k|}{t + p|\triangle^m x_k|} \geq \epsilon\} \\ &= \{k \in I_n : |\triangle^m x_k| > 0\} \\ &= \{k \in I_n : |\triangle^m x_k| = 2\} \\ &= \{k \in I_n : k \in [n - \sqrt{\lambda_n} + 1 \leq k \leq n]\}. \end{aligned}$$

Now,

$$\frac{1}{\lambda_n}|A(\epsilon,t)| \leq \frac{\sqrt{\lambda_n}}{\lambda_n} \to 0 \text{ as } n \to \infty.$$

$\Rightarrow$ $\lim_{n\to\infty} \frac{1}{\lambda_n}|A(\epsilon, t| = 0$.
Thus, $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = 0$, *i.e.* $x = \{x_k\}$ is $\lambda$-$\triangle^m$-statistical convergent on $(\mathbb{X}, \aleph, \circledast, \odot)$.

**Definition 2.2.** Let $(\mathbb{X}, \aleph, \circledast, \odot)$ be a NNS with neutrosophic norm $(\tau, \upsilon, \eta)$. A sequence $x = \{x_k\}$ from $\mathbb{X}$ is said to be $\lambda$-$\triangle^m$-statistically Cauchy with respect to neutrosophic norm $(\tau, \upsilon, \eta)$ for some non-negative number $r$ if for every $t > 0$ and $\epsilon \in (0,1)$, we can find $k_0 \in I_n$ such that

$$\begin{aligned} \delta_\lambda(\{k \in I_n : \tau(\triangle^m x_k - \triangle^m x_{k_0}; t) \leq 1-\epsilon \text{ or } \upsilon(\triangle^m x_k - \triangle^m x_{k_0}, t) \geq \epsilon,& \\ \eta(\triangle^m x_k - \triangle^m x_{k_0}, t) \geq \epsilon\}) = 0.& \end{aligned}$$

**Example 2.2.** Consider a real normed space $(\mathbb{X}, |\,.\,|)$. For every $t > 0$ and all

$x \in \mathbb{X}$, we take
(i) $\tau(x,t) = \frac{t}{t+|x|}$ , $\upsilon(x,t) = \frac{|x|}{t+|x|}$ and $\eta(x,t) = \frac{|x|}{t}$ when $t > |x|$,
(ii) $\tau(x,t) = 0$, $\upsilon(x,t) = 1$ and $\eta(x,t) = 1$ when $t \leq |x|$.
Also, $g \circledast h = gh$ and $g \odot h = g + h - gh$ for $g, h \in [0,1]$. Then, 4-tuple $(\mathbb{X}, \aleph, \circledast, \odot)$ is a NNS.
Consider a sequence $x = \{x_k\}$ such that

$$\triangle^m x_k = \begin{cases} \frac{1}{2^k} & 1 + n - \sqrt{\lambda_n} \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

For $t > 0$ and $\epsilon > 0$, choose $k_0$ with $2^{-k_0} < \epsilon$ we have

$$\begin{aligned} A(\epsilon,t) = \{ & k \in I_n : \tau(\triangle^m x_k - \triangle^m x_{k_0}, t) \leq 1 - \epsilon \\ & \text{or } \upsilon(\triangle^m x_k - \triangle^m x_{k_0}, t) \geq \epsilon,\ \eta(\triangle^m x_k - \triangle^m x_{k_0}, t) \geq \epsilon \} \\ = \{ & k \in I_n : \frac{t}{t + |\triangle^m x_k - \triangle^m x_{k_0}|} \leq 1 - \epsilon \\ & \text{or } \frac{|\triangle^m x_k - \triangle^m x_{k_0}|}{t + |\triangle^m x_k - \triangle^m x_{k_0}|} \geq \epsilon,\ \frac{|\triangle^m x_k - \triangle^m x_{k_0}|}{t} \geq \epsilon \} \\ = \{ & k \in I_n : |\triangle^m x_k - \triangle^m x_{k_0}| \geq \epsilon \} \\ = \{ & k \in I_n : k \in [1 + n - \sqrt{\lambda_n} \leq k \leq n] \} \end{aligned}$$

Now,

$$\frac{1}{\lambda_n} |A(\epsilon,t)| \leq \frac{\sqrt{\lambda_n}}{\lambda_n} \to 0 \text{ as } n \to \infty.$$

$\Rightarrow \lim_{n\to\infty} \frac{1}{\lambda_n} |A(\epsilon, t| = 0$.
Thus, $x = \{x_k\}$ is $\lambda$-$\triangle^m$-statistical Cauchy sequence on $(\mathbb{X}, \aleph, \circledast, \odot)$.

The next result can be obtained using above Definition 2.1.

**Lemma 2.1.** Consider $(\mathbb{X}, \aleph, \circledast, \odot)$ as a NNS with neutrosophic norm $(\tau, \upsilon, \eta)$. Then following statements are equivalent for the sequence $x = \{x_k\}$ from $\mathbb{X}$ for $\epsilon > 0$ and $t > 0$,
(i) $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$,
(ii) $\delta_\lambda(\{k \in I_n : \tau(\triangle^m x_k - x_0; t) \leq 1 - \epsilon \text{ or } \upsilon(\triangle^m x_k - x_0, t) \geq \epsilon, \eta(\triangle^m x_k - x_0, t) \geq \epsilon\}) = 0$,
(iii) $\delta_\lambda(\{k \in I_n : \tau(\triangle^m x_k - x_0; t) > 1 - \epsilon \,\text{and}\, \upsilon(\triangle^m x_k - x_0, t) < \epsilon, \eta(\triangle^m x_k - x_0, t) < \epsilon\}) = 1$,
(iv) $S_\lambda^{(\tau,\upsilon,\eta)} - \lim\ \tau(\triangle^m x_k - x_0, t) = 1$ and $S_\lambda^{(\tau,\upsilon,\eta)} - \lim\ \upsilon(\triangle^m x_k - x_0, t) = 0$.
Using above lemma and definitions we obtain our results on $\lambda$-$\triangle^m$-statistical convergence on NNS:

**Theorem 2.1.** *Let $x = \{x_k\}$ be any sequence from a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$. If $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$, then limit $x_0$ is unique.*

*Proof.* Assume, $S_\lambda^{(\tau,\upsilon,\eta)}-\lim \triangle^m x = x_0$ and $S_\lambda^{(\tau,\upsilon,\eta)}-\lim \triangle^m x = x_1$ and $x_0 \neq x_1$. For $t > 0$ and $\epsilon > 0$, take $\kappa > 0$ with $(1-\kappa) \circledast (1-\kappa) > 1-\epsilon$ and $\kappa \odot \kappa < \epsilon$. Define

$$A_{1,\tau}(\kappa,t) = \{k \in I_n : \tau(\triangle^m x_k - x_0, t/3) \leq 1-\kappa\},$$
$$A_{2,\tau}(\kappa,t) = \{k \in I_n : \tau(\triangle^m x_k - x_1, t/3) \leq 1-\kappa\},$$
$$A_{3,\upsilon}(\kappa,t) = \{k \in I_n : \upsilon(\triangle^m x_k - x_0, t/3) \geq \kappa\},$$
$$A_{4,\upsilon}(\kappa,t) = \{k \in I_n : \upsilon(\triangle^m x_k - x_1, t/3) \geq \kappa\},$$
$$A_{5,\eta}(\kappa,t) = \{k \in I_n : \eta \triangle^m x_k - x_0, t/3) \geq \kappa\},$$
$$A_{6,\eta}(\kappa,t) = \{k \in I_n : \eta(\triangle^m x_k - x_1, t/3) \geq \kappa\}.$$

Since $S_\lambda^{(\tau,\upsilon,\eta)}-\lim \triangle^m x = x_0$, then due to Definition 2.1 we get

$$\delta_\lambda(A_{1,\tau}(\kappa,t)) = \delta_\lambda(A_{3,\upsilon}(\kappa,t)) = \delta_\lambda(A_{5,\eta}(\kappa,t)) = 0.$$

Further $S_\lambda^{(\tau,\upsilon,\eta)}-\lim \triangle^m x = x_1$, due to Definition 2.1 we get

$$\delta_\lambda(A_{2,\tau}(\kappa,t)) = \delta_\lambda(A_{4,\upsilon}(\kappa,t)) = \delta_\lambda(A_{6,\eta}(\kappa,t)) = 0.$$

Consider

$$\begin{aligned} A_{\tau,\upsilon,\eta}(\kappa,t) = & (A_{1,\tau}(\kappa,t) \bigcup A_{2,\tau}(\kappa,t)) \bigcap (A_{3,\upsilon}(\kappa,t) \bigcup A_{4,\upsilon}(\kappa,t)) \\ & \bigcap (A_{5,\eta}(\kappa,t) \bigcup A_{6,\eta}(\kappa,t)). \end{aligned}$$

Clearly,

$$\delta_\lambda(A_{\tau,\upsilon,\eta}(\kappa,t)) = 0 \Leftrightarrow \delta_\lambda(I_n - A_{\tau,\upsilon,\eta}(\kappa,t)) = 1.$$

If $k \in I_n - A_{\tau,\upsilon,\eta}(\kappa,t)$ then either $k \in I_n - (A_{1,\tau}(\kappa,t) \bigcup A_{2,\tau}(\kappa,t))$ or $k \in I_n - (A_{3,\upsilon}(\kappa,t) \bigcup A_{4,\upsilon}(\kappa,t))$ or $k \in I_n - (A_{5,\eta}(\kappa,t) \bigcup A_{6,\eta}(\kappa,t))$.

If $k \in I_n - (A_{1,\tau}(\kappa,t) \bigcup A_{2,\tau}(\kappa,t))$ , then

$$\tau(x_0 - x_1, t) \geq \tau(\triangle^m x_k - x_0, t/2) \circledast \tau(\triangle^m x_k - x_1, t/2) > (1-\kappa) \circledast (1-\kappa) > 1-\epsilon.$$

As $\epsilon > 0$, we get $\tau(x_0 - x_1, t) = 1$ for all $t > 0$, then $x_0 = x_1$.

Also if $k \in I_n - (A_{3,\upsilon}(\kappa,t) \bigcup A_{4,\upsilon}(\kappa,t))$, then

$$\upsilon(x_0 - x_1, t) \leq \upsilon(\triangle^m x_k - x_0, t/2) \odot \upsilon(\triangle^m x_k - x_1, t/2) < \kappa \odot \kappa < \epsilon.$$

As $\epsilon > 0$, we get $\upsilon(x_0 - x_1, t) = 0$ for all $t > 0$, then $x_0 = x_1$.

Further if $k \in I_n - (A_{5,\eta}(\kappa,t) \bigcup A_{6,\eta}(\kappa,t))$, then

$$\eta(x_0 - x_1, t) \leq \eta(\triangle^m x_k - x_0, t/2) \odot \eta(\triangle^m x_k - x_1, t/2) < \kappa \odot \kappa < \epsilon.$$

As $\epsilon > 0$, we get $\eta(x_0 - x_1, t) = 0$ for all $t > 0$, then $x_0 = x_1$. Hence, limit is unique. $\square$

**Theorem 2.2.** *Consider* $(\mathbb{X}, \aleph, \circledast, \odot)$ *as a NNS with neutrosophic norm* $(\tau, \upsilon, \eta)$. *If* $(\tau, \upsilon, \eta) - \lim \triangle^m x = x_0$, *then* $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$. *But counter part does not hold.*

*Proof.* Assume $(\tau, \upsilon, \eta) - \lim \triangle^m x = x_0$. For given $\epsilon > 0$ and $t > 0$ we get $k_0 \in \mathbb{N}$ satisfying

$$\tau(\triangle^m x_k - x_0, t) > 1 - \epsilon, \upsilon(\triangle^m x_k - x_0, t) < \epsilon \text{ and } \eta(\triangle^m x_k - x_0, t) < \epsilon$$

for all $k \geq k_0$. This provides the set

$$\{ k \in I_n : \tau(\triangle^m x_k - x_0, t) \leq 1 - \epsilon \text{ or } \upsilon(\triangle^m x_k - x_0, t) \geq \epsilon, \eta(\triangle^m x_k - x_0, t) \geq \epsilon \},$$

have finite members. As $\lambda$-density of every finite set is zero. Then,

$$\delta_\lambda\{ k \in I_n : \tau(\triangle^m x_k - x_0, t) \leq 1 - \epsilon \text{ or } \upsilon(\triangle^m x_k - x_0, t) \geq \epsilon, \eta(\triangle^m x_k - x_0, t) \geq \epsilon \} = 0.$$

*i.e*,

$$S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0.$$

However, counter part of the above mentioned result fails to exist. That can be explained from the following example:

**Example 2.3.** Consider any real normed space $(\mathbb{X}, |\,.\,|)$. For every $t > 0$ and all $x \in \mathbb{X}$, we take
(i) $\tau(x,t) = \frac{t}{t+|x|}$ , $\upsilon(x,t) = \frac{|x|}{t+|x|}$ and $\upsilon(x,t) = \frac{|x|}{t}$ when $t > |x|$,
(ii) $\tau(x,t) = 0$, $\upsilon(x,t) = 1$ and $\upsilon(x,t) = 1$ when $t \leq |x|$.
Also, $g \circledast h = gh$ and $g \odot h = g + h - gh$ for $g, h \in [0,1]$. Then, 4-tuple $(\mathbb{X}, \aleph, \circledast, \odot)$ is a NNS.
Consider a sequence $x = \{x_k\}$ such that

$$\triangle^m x_k = \begin{cases} 1 & n - \sqrt{\lambda_n} + 1 \leq k \leq n \\ 0 & \text{otherwise} \end{cases}$$

For $t > 0$ and $\epsilon > 0$, we have

$$\begin{aligned} A(\epsilon, t) &= \{k \in I_n : \tau(\triangle^m x_k - 0, t) \leq 1 - \epsilon \text{ or } \upsilon(\triangle^m x_k - 0, t) \geq \epsilon, \eta(\triangle^m x_k - 0, t) \geq \epsilon\} \\ &= \{k \in I_n : \frac{t}{t + |\triangle^m x_k|} \leq 1 - \epsilon \text{ or } \frac{|\triangle^m x_k|}{t + |\triangle^m x_k|} \geq \epsilon, \frac{|\triangle^m x_k|}{t} \geq \epsilon\} \\ &= \{k \in I_n : |\triangle^m x_k| > 0\} \\ &= \{k \in I_n : |\triangle^m x_k| = 1\} \\ &= \{k \in I_n : k \in [n - \sqrt{\lambda_n} + 1 \leq k \leq n]\} \end{aligned}$$

Now,

$$\frac{1}{\lambda_n}|\,A(\epsilon,t)\,| \leq \frac{\sqrt{\lambda_n}}{\lambda_n} \to 0 \text{ as } n \to \infty.$$

$\Rightarrow \lim_{n\to\infty} \frac{1}{\lambda_n}|\,A(\epsilon,t\,| = 0.$

Thus, $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = 0$, *i.e.* $x = \{x_k\}$ is $\lambda$-$\triangle^m$-statistical convergent on $(\mathbb{X}, \aleph, \circledast, \odot)$.
Using above defined sequence, we get

$$\tau(\triangle^m x_k, t) = \begin{cases} \dfrac{t}{t+1} & n - \sqrt{\lambda_n} + 1 \leq k \leq n \\ \\ 1 & \text{otherwise} \end{cases}$$

$$i.e\ \tau(\triangle^m x_k, t) \leq 1,\ \forall\ k,$$

and

$$\upsilon(\triangle^m x_k, t) = \begin{cases} \dfrac{1}{t+1} & n - \sqrt{\lambda_n} + 1 \leq k \leq n \\ \\ 0 & \text{otherwise} \end{cases}$$

$$i.e\ \upsilon(\triangle^m x_k, t) \geq 0,\ \forall\ k,$$

and

$$\eta(\triangle^m x_k, t) = \begin{cases} \dfrac{1}{t} & n - \sqrt{\lambda_n} + 1 \leq k \leq n \\ \\ 0 & \text{otherwise} \end{cases}$$

$$i.e\ \eta(\triangle^m x_k, t) \geq 0,\ \forall\ k.$$

This implies $(\tau, \upsilon, \eta) - \lim \triangle^m x \neq 0$. □

Next, we will discuss some algebraic properties of $\lambda$-$\triangle^m$-statistical sequences in NNS as follows:

**Theorem 2.3.** *Let* $(\mathbb{X}, \aleph, \circledast, \odot)$ *be a NNS. Let* $x = \{x_k\}$ *and* $y = \{y_k\}$ *be any sequences from* $\mathbb{X}$. *Then*
*(i) If* $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$ *then* $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m ax = ax_0$ ; $a \in \mathbb{R}$,
*(ii) If* $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$ *and* $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m y = y_0$ *then* $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m (x + y) = x_0 + y_0$.

*Proof.* (i) Assume $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$ .Then, for the fixed $\epsilon > 0$ and any $t > 0$, we can take

$$A(\epsilon,t) = \{k \in I_n : \tau(\triangle^m x_k - x_0, t) \leq 1-\epsilon \text{ or } \upsilon(\triangle^m x_k - x_0, t) \geq \epsilon, \eta(\triangle^m x_k - x_0, t) \geq \epsilon\}.$$

Which provides
$$\delta_\lambda(A(\epsilon,t))=0 \text{ so that } \delta_\lambda([A(\epsilon,t)]^c)=1.$$
Let $k\in[A(\epsilon,t)]^c$ and $a\neq 0$, then
$$\begin{aligned}\tau(\triangle^m(ax_k)-ax_0,t)=\tau(a(\triangle^m x_k-x_0),t)&=\tau\left(\triangle^m x_k-x_0,\frac{t}{|a|}\right)\\&\geq \tau(\triangle^m x_k-x_0,t)\circledast\tau\left(0,\frac{t}{|a|}-t\right)\\&=\tau(\triangle^m x_k-x_0,t)\circledast 1\\&>1-\epsilon,\end{aligned}$$
and
$$\begin{aligned}\upsilon(\triangle^m(ax_k)-ax_0,t)=\upsilon(a(\triangle^m x_k-x_0),t)&=\upsilon\left(\triangle^m x_k-x_0,\frac{t}{|a|}\right)\\&\leq \upsilon(\triangle^m x_k-x_0,t)\odot\upsilon\left(0,\frac{t}{|a|}-t\right)\\&\leq\upsilon(\triangle^m x_k-x_0,t)\odot 0\\&<\epsilon,\end{aligned}$$
and
$$\begin{aligned}\eta(\triangle^m(ax_k)-ax_0,t)=\eta(a(\triangle^m x_k-x_0),t)&=\eta\left(\triangle^m x_k-x_0,\frac{t}{|a|}\right)\\&\leq \eta(\triangle^m x_k-x_0,t)\odot\eta\left(0,\frac{t}{|a|}-t\right)\\&\leq\eta(\triangle^m x_k-x_0,t)\odot 0\\&<\epsilon.\end{aligned}$$
Therefore, $\delta_\lambda([A(\epsilon,t)]^c)=1$. Hence, $S_\lambda^{(\tau,\upsilon,\eta)}-\lim ax=ax_0,\ a\neq 0$.

When $a=0$, we get
$$\tau(0\triangle^m x_k,t)>1-\epsilon,\upsilon(0\triangle^m x_k,t)<\epsilon\,\text{and}\,\eta(0\triangle^m x_k,t)<\epsilon.$$
Hence, $S_\lambda^{(\tau,\upsilon,\eta)}-\lim\triangle^m ax=ax_0,\ a\in\mathbb{R}$.

(ii) As $S_\lambda^{(\tau,\upsilon,\eta)}-\lim\triangle^m x=x_0$ and $S_\lambda^{(\tau,\upsilon,\eta)}-\lim\triangle^m y=y_0$. Then, for $t>0$ and $\epsilon>0$, take $\kappa>0$ with $(1-\kappa)\circledast(1-\kappa)>1-\epsilon$ and $\kappa\odot\kappa<\epsilon$.
Define sets for the given sequences $x=\{x_k\}$ and $y=\{y_k\}$ sets

$A_x(\kappa,t)=\{k\in I_n:\tau(\triangle^m x_k-x_0,\frac{t}{2})\leq 1-\kappa\,\text{or}\,\upsilon(\triangle^m x_k-x_0,\frac{t}{2})\geq\kappa,\eta(\triangle^m x_k-x_0,\frac{t}{2})\geq\kappa\}$, and

$A_y(\kappa,t)=\{k\in I_n:\tau(\triangle^m y_k-y_0,\frac{t}{2})\leq 1-\kappa \,\text{or}\, \upsilon(\triangle^m y_k-y_0,\frac{t}{2})\geq\kappa,\eta(\triangle^m y_k-y_0,\frac{t}{2})\geq\kappa\}.$

We have, $\delta_\lambda(A_x(\kappa,t))=\delta_\lambda(A_y(\kappa,t))=0$.

Consider $A(\kappa,t)=A_x(\kappa,t)\cap A_y(\kappa,t)$, then $\delta_\lambda(A(\kappa,t))=0$ *i.e* $\delta([A(\kappa,t)]^c)=1$.
For all $k\in[A(\kappa,t)]^c$,

$$\begin{aligned}\tau(\triangle^m(x_k+y_k)-(x_0+y_0),t)&=\tau(\triangle^m x_k-x_0+\triangle^m y_{kr}-y_0,t)\\&\geq\tau(\triangle^m x_k-x_0,t/2)\circledast\tau(\triangle^m y_k-y_0,t/2)\\&\geq(1-\kappa)\circledast(1-\kappa)\\&>1-\epsilon,\end{aligned}$$

and

$$\begin{aligned}\upsilon(\triangle^m(x_k+y_k)-(x_0+y_0),t)&=\upsilon(\triangle^m x_k-x_0+\Lambda y_k-y_0,t)\\&\leq\upsilon(\triangle^m x_k-x_0,t/2)\odot\upsilon(\triangle^m y_k-y_0,t/2)\\&\leq\kappa\odot\kappa\\&<\epsilon,\end{aligned}$$

and

$$\begin{aligned}\eta(\triangle^m(x_k+y_k)-(x_0+y_0),t)&=\eta(\triangle^m x_k-x_0+\Lambda y_k-y_0,t)\\&\leq\eta(\triangle^m x_k-x_0,t/2)\odot\eta(\triangle^m y_k-y_0,t/2)\\&\leq\kappa\odot\kappa\\&<\epsilon.\end{aligned}$$

$$\Rightarrow S_\lambda^{(\tau,\upsilon,\eta)}-\lim\triangle^m(x+y)=x_0+y_0.$$

□

**Theorem 2.4.** *Consider* $(\mathbb{X},\aleph,\circledast,\odot)$ *as a NNS. A sequence* $x=\{x_k\}$ *from* $\mathbb{X}$ *is* $S_\lambda^{(\tau,\upsilon,\eta)}-\lim\triangle^m x=x_0$ *if and only if set* $J=\{j_1<j_2<j_3.....\}\subseteq I_n$ *exists with* $\delta_\lambda(J)=1$ *and* $(\tau,\upsilon,\eta)_\lambda-\lim\triangle^m x_{j_n}=x_0$.

*Proof. Necessary part*:

Consider $S_\lambda^{(\tau,\upsilon,\eta)}-\lim\triangle^m x=x_0$ . For $t>0$ and $\kappa\in\mathbb{N}$, we consider

$A(\kappa,t)=\{k\in I_n:\tau(\triangle^m x_k-x_0,t)>1-\frac{1}{\kappa}$ and $\upsilon(\triangle^m x_k-x_0,t)<\frac{1}{\kappa},\eta(\triangle^m x_k-x_0,t)<\frac{1}{\kappa}\}$,

and

$K(\kappa,t)=\{k\in I_n:\tau(\triangle^m x_k-x_0,t)\leq 1-\frac{1}{\kappa}$ or $\upsilon(\triangle^m x_k-x_0,t)\geq\frac{1}{\kappa},\eta(\triangle^m x_k-x_0,t)\geq\frac{1}{\kappa}\}$.

Since $S_{\lambda}^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$, then $\delta_\lambda(K(\kappa,t)) = 0$. Moreover, $A(\kappa,t) \supset A(\kappa+1,t)$, and

$$\delta_\lambda(A(\kappa,t)) = 1. \tag{2.1}$$

Next, for any $k \in A(\kappa,t)$, we have $(\tau,\upsilon,\eta)_\lambda - \lim \triangle^m x = x_0$.

We prove this part by contradiction. If for any $k \in A(\kappa,t)$ we have $\mu > 0$ and $k_0 \in \mathbb{N}$ satisfying

$\tau(\triangle^m x_k - x_0, t) \leq 1-\mu$ or $\upsilon(\triangle^m x_k - x_0, t) \geq \mu$, $\eta(\triangle^m x_k - x_0, t) \geq \mu$, for all $k \geq k_0$,

This implies that

$\tau(\triangle^m x_k - x_0, t) > 1-\mu$ and $\upsilon(\triangle^m x_k - x_0, t) < \mu$, $\eta(\triangle^m x_k - x_0, t) < \mu$, for all $k < k_0$.

Therefore,

$\delta_\lambda\{k \in I_n : \tau(\triangle^m x_k - x_0, t) > 1-\mu \text{ and } \upsilon(\triangle^m x_k - x_0, t) < \mu, \eta(\triangle^m x_k - x_0, t) < \mu\} = 0.$

As $\alpha > \frac{1}{\kappa}$, we've $\delta_\lambda(A(\kappa,t)) = 0$, which leads contradiction to (2.1). Then, we get set $A(\kappa,t)$ with $\delta_\lambda(A(\kappa,t)) = 1$. Hence $x = \{x_k\}$ is $\lambda$-$\triangle^m$-statistical convergent to $x_0$.

*Sufficient Part*:
Suppose there exists a subset $J = \{j_1 < j_2 < j_3 < \dots\} \subseteq \mathbb{N}$ such that $\delta_\lambda(J) = 1$ and $(\tau,\upsilon,\eta)_\lambda - \lim \triangle^m y_{j_n} = x_0$.
*i.e.* $\exists\ N_0 \in \mathbb{N}$ for every $\epsilon > 0$ and any $t > 0$ satisfying

$\tau(\triangle^m x_k - x_0, t) > 1-\epsilon$, $\upsilon(\triangle^m x_k - x_0, t) < \epsilon$ and $\eta(\triangle^m x_k - x_0, t) < \epsilon$ ; $k \geq N_0$.

Take

$K(\epsilon,t) = \{k \in I_n : \tau(\triangle^m x_k - x_0, t) \leq 1-\epsilon \text{ or } \upsilon(\triangle^m x_k - x_0, t) \geq \epsilon, \eta(\triangle^m x_k - x_0, t) \geq \epsilon\}.$

Then,

$$K(\epsilon,t) \subseteq I_n - \{j_{N_0+1}, j_{N_0+2}, \dots.\}.$$

Since $\delta_\lambda(J) = 1$ then we get $\delta_\lambda(K(\epsilon,t)) \leq 0$. Therefore, $S_{\lambda}^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$. □

**Theorem 2.5.** *Consider $(\mathbb{X}, \aleph, \circledast, \odot)$ as a NNS. Then $S_{\lambda}^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$ if and only if there exists a sequence $y = \{y_k\}$ with $(\tau,\upsilon,\eta)_\lambda - \lim \triangle^m y = x_0$ and $\delta_\lambda(\{k \in I_n : \triangle^m x = \triangle^m y\}) = 1$.*

*Proof. Necessary part*:

Consider $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$. By Theorem 2.4, we get a set $J \subseteq I_n$ with $\delta_\lambda(J) = 1$ and $(\tau,\upsilon,\eta)_\lambda - \lim \triangle^m x_{j_n} = x_0$.
Consider a sequence $y = \{y_k\}$ such that

$$\triangle^m y_k = \begin{cases} \triangle^m x_k & k \in J \\ x_0 & \text{otherwise} \end{cases}$$

Then $y = \{y_k\}$ serve our purpose.

*Sufficient part*:
Consider $x = \{x_k\}$ and $y = \{y_k\}$ from $\mathbb{X}$ with $(\tau,\upsilon,\eta)_\lambda - \lim \triangle^m y = x_0$ and $\delta_\lambda(\{k \in I_n : \triangle^m x = \triangle^m y\}) = 1$. Then for any $t > 0$ and every $\epsilon > 0$, we've

$$\{k \in I_n : \tau(\triangle^m y_k - x_0, t) \le 1-\epsilon \text{ or } \upsilon(\triangle^m y_k - x_0, t) \ge \epsilon, \eta(\triangle^m y_k - x_0, t) \ge \epsilon\} \subseteq A \cup B$$

where $A = \{k \in I_n : \tau(\triangle^m x_k - x_0, t) \le 1-\epsilon$ or $\upsilon(\triangle^m x_k - x_0, t) \ge \epsilon$, $\eta(\triangle^m x_k - x_0, t) \ge \epsilon\}$,

$$B = \{k \in I_n : \triangle^m y_k \ne \triangle^m x_k\}.$$

Since $(\tau,\upsilon,\eta)_\lambda - \lim \triangle^m x = x_0$ then above defined set $A$ has at most finitely many elements. Also $\delta_\lambda(B) = 0$ as $\delta_\lambda(B^c) = 1$ where $B^c = \{k \in I_n : \triangle^m y_k = \triangle^m x_k\}$. Therefore

$$\delta_\lambda(\{k \in I_n : \tau(\triangle^m x_k - x_0, t) \le 1-\epsilon \,\text{or}\, \upsilon(\triangle^m x_k - x_0, t) \ge \epsilon, \eta(\triangle^m x_k - x_0, t) \ge \epsilon\}) = 0.$$

Hence $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$. □

**Theorem 2.6.** *Let $x = \{x_k\}$ be a sequence from a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$. Then $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$ if and only if there are sequences $y = \{y_k\}$ and $z = \{z_k\}$ from $\mathbb{X}$ with $\triangle^m x_k = \triangle^m y_k + \triangle^m z_k$ for all $k \in I_n$ where $(\tau,\upsilon,\eta)_\lambda - \lim \triangle^m y = x_0$ and $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m z = x_0$.*

*Proof. Necessary part*:
Let $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$. By Theorem 2.4 we get a set $J = \{k_q : q = 1, 2, 3, ....\} \subseteq \mathbb{N}$ with $\delta_\lambda(J) = 1$ and $(\tau,\upsilon,\eta)_\lambda - \lim\limits_{k_q \to \infty} \triangle^m y_{k_q} = x_0$.
Consider the sequences $y = \{y_k\}$ and $z = \{z_k\}$

$$\triangle^m y_k = \begin{cases} \triangle^m z_k & k \in J \\ x_0 & \text{otherwise} \end{cases}$$

and

$$\triangle^m x_k = \begin{cases} 0 & k \in J \\ \triangle^m y_{jk} - x_0 & \text{otherwise} \end{cases}$$

which gives the required result.

*Sufficient Part*:
If two such sequences $y = \{y_k\}$ and $z = \{z_k\}$ exists in $\mathbb{X}$ with the required properties, then the result follows using Theorem 2.2 and Theorem 2.3. □

**Theorem 2.7.** *Consider* $(\mathbb{X}, \aleph, \circledast, \odot)$ *as a NNS with norm* $(\tau, \upsilon, \eta)$. *Then* $S^{(\tau,\upsilon,\eta)}(\triangle^m) \subseteq S_\lambda^{(\tau,\upsilon,\eta)}(\triangle^m)$ *if and only if* $\lim_{k\to\infty} \inf \frac{\lambda_n}{n} > 0$.

*Proof.* For given $\epsilon > 0$ and $t > 0$ we have

$$\begin{aligned}&\{k \leq n : \tau(\triangle^m x_k - x_0; t) \leq 1 - \epsilon \text{ or } \upsilon(\triangle^m x_k - x_0; t) \geq \epsilon, \eta(\triangle^m x_k - x_0; t) \geq \epsilon\}\\ &\quad \supseteq \{k \in I_n : \tau(\triangle^m x_k - x_0; t) \leq 1 - \epsilon \ \text{ or } \ \upsilon(\triangle^m x_k - x_0; t) \geq \epsilon, \eta(\triangle^m x_k - x_0; t) \geq \epsilon\}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{n}|\{k \leq n : \tau(\triangle^m x_k - x_0; t) \leq 1 - \epsilon \,\text{or}\, \upsilon(\triangle^m x_k - x_0; t) \geq \epsilon, \eta(\triangle^m x_k - x_0; t) \geq \epsilon\}|\\ &\geq \frac{1}{\lambda_n}|\{k \in I_n : \tau(\triangle^m x_k - x_0; t) \leq 1 - \epsilon \,\text{or}\, \upsilon(\triangle^m x_k - x_0; t) \geq \epsilon, \eta(\triangle^m x_k - x_0; t) \geq \epsilon\}|\\ &\geq \frac{\lambda_n}{n}\frac{1}{\lambda_n}|\{k \in I_n : \tau(\triangle^m x_k - x_0; t) \leq 1 - \epsilon \,\text{or}\, \upsilon(\triangle^m x_k - x_0; t) \geq \epsilon, \eta(\triangle^m x_k - x_0; t) \geq \epsilon\}|\end{aligned}$$

Take limit as $n \to \infty$ then we get $S^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$ (As $\lim_{k\to\infty} \inf \frac{\lambda_n}{n} > 0$).
Hence $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$.

Conversely,
Suppose that $\lim_{k\to\infty} \inf \frac{\lambda_n}{n} = 0$. We can take a sub-sequence $\{n_j\}$ such that $\frac{\lambda_{n_j}}{n_j} < \frac{1}{j}$.
Consider a sequence $x = \{x_k\}$ such that

$$\triangle^m y_k = \begin{cases} 1 & k \in I_{n_j} \\ 0 & \text{otherwise.} \end{cases}$$

Then take $t > 0$ and $\epsilon \in (0, 1)$ such that $1 \notin B(0, \epsilon, t)$. Also, to each $n \in \mathbb{N}$ we get $n_j \in \mathbb{N}$ such that $n_j \leq n$ for $j > 0$.

$$\frac{1}{n}|\{k \leq n : \tau(\triangle^m x_k; t) \leq 1 - \epsilon \ \text{ or } \ \upsilon(\triangle^m x_k; t) \geq \epsilon,\ \eta(\triangle^m x_k; t) \geq \epsilon\}| < \frac{1}{j}.$$

Then $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = 0$. For $k \notin I_{n_j}$ we get

$$\lim_{j\to\infty} \frac{1}{\lambda_{n_j}}|\{k \in I_{n_j} : \tau(\triangle^m x_k; t) \leq 1 - \epsilon \ \text{ or } \ \upsilon(\triangle^m x_k; t) \geq \epsilon,\ \eta(\triangle^m x_k; t) \geq \epsilon\}| = 1.$$

$$\lim_{n\to\infty} \frac{1}{\lambda_n}|\{k \in I_n : \tau(\triangle^m x_k - 1; t) \leq 1 - \epsilon \,\text{or}\, \upsilon(\triangle^m x_k - 1; t) \geq \epsilon, \eta(\triangle^m x_k - 1; t) \geq \epsilon\}| = 1.$$

This implies that $x \notin S_\lambda^{(\tau,\upsilon,\eta)}(\triangle^m)$. $\square$

Next we establish the result related to Cauchy criterion for $\lambda$-$\triangle^m$-statistical convergent sequences in NNS.

**Theorem 2.8.** *A sequence $x = \{x_k\}$ from a NNS $(\mathbb{X}, \aleph, \circledast, \odot)$ is $\lambda$-$\triangle^m$-statistical convergent corresponding to $(\tau, \upsilon, \eta)$ if and only if it is $\lambda$-$\triangle^m$-statistical Cauchy corresponding to $(\tau, \upsilon, \eta)$.*

*Proof. Necessary part*: Consider $S_\lambda^{(\tau,\upsilon,\eta)} - \lim \triangle^m x = x_0$. Then, for any $t > 0$ and $\epsilon > 0$, take $\kappa > 0$ with $(1-\kappa)\circledast(1-\kappa) > 1-\epsilon$ and $\kappa\odot\kappa < \epsilon$. Consider $A(\kappa, t) = \{k \in I_n : \tau(\triangle^m x_k - x_0, t/2) \leq 1-\kappa$ or $\upsilon(\triangle^m x_k - x_0, t/2) \geq \kappa$, $\eta(\triangle^m x_k - x_0, t/2) \geq \kappa\}$. $\therefore \delta_\lambda(A(\kappa, t)) = 0$ and $\delta_\lambda([A(\kappa, t)]^c) = 1$.

Let $B(\epsilon, t) = \{k \in I_n : \tau(\triangle^m x_k - \Lambda x_s, t) \leq 1 - \epsilon \,\text{or}\, \upsilon(\triangle^m x_k - \triangle^m x_s, t) \geq \epsilon, \eta(\triangle^m x_k - \triangle^m x_s, t) \geq \epsilon\}$.
Here, for the result we show that $B(\epsilon, t) \subset A(\kappa, t)$. As $k \in B(\epsilon, t) - A(\kappa, t)$
$\Rightarrow \tau(\triangle^m x_k - x_0, t/2) \leq 1-\kappa$ or $\upsilon(\triangle^m x_k - x_0, t/2) \geq \kappa$, $\eta(\triangle^m x_k - x_0, t/2) \geq \kappa$.

$$\begin{aligned} 1-\epsilon \geq \tau(\triangle^m x_k - \triangle^m x_s, t) &\geq \tau(\triangle^m x_k - x_0, t/2) \circledast \tau(\triangle^m x_s - x_0, t/2) \\ &> (1-\kappa) \circledast (1-\kappa) \\ &> 1-\epsilon, \end{aligned}$$

$$\begin{aligned} \epsilon \leq \upsilon(\triangle^m x_k - \triangle^m x_s, t) &\leq \upsilon(\triangle^m x_k - x_0, t/2) \odot \upsilon(\triangle^m x_s - x_0, t/2) \\ &< \kappa \odot \kappa \\ &< \epsilon, \end{aligned}$$

and

$$\begin{aligned} \epsilon \leq \eta(\triangle^m x_k - \triangle^m x_s, t) &\leq \eta(\triangle^m x_k - x_0, t/2) \odot \eta(\triangle^m x_s - x_0, t/2) \\ &< \kappa \odot \kappa \\ &< \epsilon, \end{aligned}$$

which is not possible. This implies that $B(\epsilon, t) \subset A(\kappa, t)$ and $\delta_\lambda(B(\epsilon, t)) = 0$ *i.e.* $\lambda$-$\triangle^m$-statistical Cauchy corresponding to $(\tau, \upsilon, \eta)$.

*Sufficient part*:
Let $x = \{x_k\}$ be $\lambda$-$\triangle^m$-statistical Cauchy corresponding to $(\tau, \upsilon, \eta)$ but not $\lambda$-$\triangle^m$-statistical convergent corresponding to $(\tau, \upsilon, \eta)$. Then, for any $t > 0$ and $\epsilon > 0$, we have $\delta_\lambda(C(\epsilon, t)) = 0$ where

$$\begin{aligned} C(\epsilon, t) = \{k \in I_n : \tau(\triangle^m x_k - \triangle^m x_{k_0}, t) \leq 1 - \epsilon \,\text{or}\, \upsilon(\triangle^m x_k - \triangle^m x_{k_0}, t) \geq \epsilon, \\ \eta(\triangle^m x_k - \triangle^m x_{k_0}, t) \geq \epsilon\}. \end{aligned}$$

Take $\kappa > 0$ with $(1-\kappa) \circledast (1-\kappa) > 1-\epsilon$ and $\kappa \odot \kappa < \epsilon$.
Let $D(\kappa, t) = \{k \in I_n : \tau(\triangle^m x_k - x_0, t/2) > 1-\kappa$ or $\upsilon(\triangle^m x_k - x_0, t/2) < \kappa\}$.

Now for $k \in D(\epsilon, t)$ we get

$$\begin{aligned}\tau(\triangle^m x_k - \triangle^m x_{k_0}, t) &\geq \tau(\triangle^m x_k - x_0, t/2) \circledast \tau(\triangle^m x_{k_0} - \xi, t/2)\\ &> (1-\kappa) \circledast (1-\kappa)\\ &> 1-\epsilon,\end{aligned}$$

$$\begin{aligned}\upsilon(\triangle^m x_k - \triangle^m x_{k_0}, t) &\leq \upsilon(\triangle^m x_k - \xi, t/2) \odot \upsilon(\triangle^m x_{k_0} - \xi, t/2)\\ &< \kappa \odot \kappa\\ &< \epsilon,\end{aligned}$$

and

$$\begin{aligned}\eta(\triangle^m x_k - \triangle^m x_{k_0}, t) &\leq \eta(\triangle^m x_k - \xi, t/2) \odot \eta(\triangle^m x_{k_0} - \xi, t/2)\\ &< \kappa \odot \kappa\\ &< \epsilon.\end{aligned}$$

Since $x = \{x_k\}$ is not $\lambda$-$\triangle^m$-statistical convergent corresponding to $(\tau, \upsilon, \eta)$. Therefore, $\delta_\lambda([C(\epsilon, t)]^c) = 0$ *i.e.* $\delta_\lambda(C(\epsilon, t)) = 1$ , which results contradiction for $x = \{x_k\}$, assumed to be $\lambda$-$\triangle^m$-statistical Cauchy. Thus, $x = \{x_k\}$ converges $\lambda$-$\triangle^m$-statistically corresponding to $(\tau, \upsilon, \eta)$. □

### 3. Conclusions

In this paper, we have introduced the convergence structure, called $\lambda$-$\triangle^m$-statistical convergence, on neutrosophic normed spaces for difference sequences. Neutrosophic sets are efficient tools for handling indeterminate and inconsistent data. The theory of generalized statistical convergence acts as a powerful mathematical technique for dealing with convergence problems. The computational methods and techniques may not always be sufficient to provide the best results alone, although merging two or more can lead to improved solutions. The introduction of $\lambda$-$\triangle^m$-statistical convergence in this structure is significant because it provides a new mathematical tool for practically addressing convergence problems. Moreover, this concept can be further explored in KM fuzzy metric spaces and KM fuzzy normed spaces.

## References

[1] A. Alotaibi and M. Mursaleen, *Generalized statistical convergence of difference sequences*, Adv. Differ. Equ., **2013(1)**(2013), 1–5.

[2] H. Altınok. *Statistical convergence of order $\beta$ for generalized difference sequences of fuzzy numbers*, J. Intell. Fuzzy Syst., **26(2)**(2014), 847–856.

[3] R. Antal, M. Chawla, V. Kumar, and B. Hazarika, *On $\triangle^m$-statistical convergence double sequences in intuitionistic fuzzy normed spaces*, Proyecciones (Antofagasta, On line), **41(3)**(2022), 697–713.

[4] P. Baliarsingh, U. Kadak, and M. Mursaleen, *On statistical convergence of difference sequences of fractional order and related korovkin type approximation theorems*, Quaest. Math., **41(8)**(2018), 1117–1133.

[5] T. Bera and N. K. Mahapatra, *On neutrosophic soft linear spaces*, Fuzzy Inf. Eng., **9(3)**(2017), 299–324.

[6] N. Demir and H. Gumus, *Rough statistical convergence for difference sequences*, Kragujevac. J. Math., **46(5)**(2022), 733–742.

[7] A. Esi, *Generalized difference sequence spaces defined by Orlicz functions*, Gen. Math., **17(2)**(2009), 53–66.

[8] M. Et and R. Çolak, *On some generalized difference sequence spaces*, Soochow J. Math., **21(4)**(1995), 377–386.

[9] M. Et and F. Nuray, *$\triangle^m$-statistical convergence*, Indian J. Pure Appl. Math., **32(6)**(2001), 961–969.

[10] M. Et and H. Şengül, *On ($\triangle^m$, I)-lacunary statistical convergence of order $\alpha$*, J. Math. Anal., **7(5)**(2016), 78–84.

[11] J. A. Fridy. *On statistical convergence*, Analysis, **5**(1985), 301–313.

[12] B. Hazarika, *Lacunary generalized difference statistical convergence in random 2-normed spaces*, Proyecciones, **31(4)**(2012), 373–390.

[13] V. A. Khan and M. D. Khan, *Some topological character of neutrosophic normed spaces*, Neutrosophic Sets Syst., **47**(2021), 397–410.

[14] V. A. Khan, S. K. Rahaman, B. Hazarika, and M. Alam, *Rough lacunary statistical convergence in neutrosophic normed spaces*, J. Intell. Fuzzy Syst., **45(5)**(2023), 7335-7351.

[15] M. Kirişci and N. Şimşek, *Neutrosophic metric spaces*, Math. Sci., **14**(2020), 241–248.

[16] M. Kirişçi and N. Şimşek, *Neutrosophic normed spaces and statistical convergence*, J. Anal., **28(4)**(2020), 1059–1073.

[17] Ö. Kişi, *Ideal convergence of sequences in neutrosophic normed spaces*, J. Intell. Fuzzy Syst., **41(2)**(2021), 2581–2590.

[18] Ö. Kişi and V. Gürdal, *On triple difference sequences of real numbers in neutrosophic normed spaces*, Commun. Adv. Math. Sci., **5(1)**(2022), 35–45.

[19] H. Kizmaz, *On certain sequence spaces*, Canad. Math. Bull., **24(2)**(1981), 169–176.

[20] V. Kumar, I. R. Ganaie, and A. Sharma, *On $S\lambda$-summability in neutrosophic soft normed linear spaces*, Neutrosophic Sets Syst., **58(1)**(2023), 556–571.

[21] S. A. Mohiuddine and B. Hazarika, *Some classes of ideal convergent sequences and generalized difference matrix operator*, Filomat, **31(6)**(2017), 1827–1834.

[22] M. Mursaleen, *$\lambda$-statistical convergence*, Math. Slovaca, **50(1)**(2000), 111–115.

[23] S. Omran and A. Elrawy, *Continuous and bounded operators on neutrosophic normed spaces*, Neutrosophic Sets Syst., **46**(2021), 276–289.

[24] M. Sen and M. Et, *Lacunary statistical and lacunary strongly convergence of generalized difference sequences in intuitionistic fuzzy normed linear spaces*, Bol. Soc. Paran. Mat., **38(1)**(2020), 117–129.

[25] F. Smarandache, *Neutrosophic set-a generalization of the intuitionistic fuzzy set*, IEEE International Conference on Granular Computing, Atlanta, GA, USA, (2006), 38–42.

[26] B. C. Tripathy and A. Baruah, *Lacunary statically convergent and lacunary strongly convergent generalized difference sequences of fuzzy real numbers*, Kyungpook Math. J., **50(4)**(2010), 565–574.

[27] H. Wang, F. Smarandache, Y. Zhang, and R. Sunderraman, *Single valued neutrosophic sets*, Infinite study, 2010.

[28] J. Ye, *A multicriteria decision-making method using aggregation operators for simplified neutrosophic sets*, J. Intell. Fuzzy Syst., **26(5)**(2014), 2459–2466.