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Real Hypersurfaces in the Complex Projective Space with Pseudo Ricci-Bourguignon Solitions

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Abstract. First, we give a complete classification of pseudo Ricci-Bourguignon soliton on real hypersurfaces in the complex projective space $\mathbb{C}P^n = SU_{n+1}/S(U_1 \cdot U_n)$. Next, as an application, we give a complete classification of *gradient* pseudo Ricci-Bourguignon soliton on real hypersurfaces in the complex projective space $\mathbb{C}P^n$.

1. Introduction

From 21th Century, many authors have investigated real hypersurfaces in Hermitian symmetric spaces with rank 1 or rank 2 of compact type. For the case of rank 2, real hypersurfaces in the complex two-plane Grassmannians $G_2(\mathbb{C}^{n+2})$ or in the complex quadric Q^n were extensively studied by many authors (Lee-Suh [19],[20] and [21], Pérez [26], Pérez-Suh [27], Pérez-Suh-Watanabe [28], Suh [32], [33], [34] and [35], and Suh-Hwang-Woo [36]).

Motivated by the study of rank 2, in the class of Hermitian symmetric spaces with rank 1 of compact type, we can give the example of the complex projective space $\mathbb{C}P^n = SU_{n+1}/S(U_1 \cdot U_n)$ (see Kobayashi-Nomizu [17]). It is geometrically

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different from the case of rank 2, which has a Kähler structure and a Fubini-Study metric g of constant holomorphic sectional curvature 4 (see Cecil-Ryan [7], Djorić-Okumura [12], Romero [29], [30], and Smyth [31]).

Recently, Yamabe solitons and Ricci solitons on almost co-Kähler manifolds and three dimensional $N(k)$-contact manifolds have been investigated by Chaubey-De-Suh [9] and [11]. Moreover, the study of the Yamabe flow was initiated in the work of Hamilton [14], Morgan-Tian [22] and Perelman [25] as a geometric method to construct Yamabe metrics on Riemannian manifolds.

Let $g(t)$ be a Riemannian metric which is time dependent on a Riemannian manifold M. It is said to be evolved by the Yamabe flow if the metric g satisfies

$$\frac{\partial}{\partial t} g(t) = -\gamma g(t), \quad g(0) = g_0$$

on M, where γ denotes the scalar curvature on M. From such a view point, in this paper we want to give a complete classification of Yamabe solitons and gradient Yamabe solitons on Hopf real hypersurfaces in the complex projective space $\mathbb{C}P^n$.

On the other hand, it is well known that there exist two focal submanifolds of real hypersurfaces in Hermitian symmetric spaces of compact type and only one focal submanifold in Hermitian symmetric spaces of non-compact type (see Cecil and Ryan [7] and Helgason [13]). Since the complex projective space $\mathbb{C}P^n$ is a Hermitian symmetric space of compact type, any real hypersurface has two focal submanifolds (see Djorić-Okumura [12], Pérez [26]). Among them we consider two kinds of real hypersurfaces in $\mathbb{C}P^n$ with isometric Reeb flow or contact hypersurfaces. In $\mathbb{C}P^n$, Cecil-Ryan [7], and Okumura [23] gave a classification of real hypersurfaces with isometric Reeb flow as follows:

Theorem A. *Let M be a real hypersurface in the complex projective space $\mathbb{C}P^n$, $n \geq 3$. Then the Reeb flow on M is isometric if and only if M is an open part of a tube of radius $0 < r < \frac{\pi}{2}$ around a totally geodesic $\mathbb{C}P^k \subset \mathbb{C}P^n$ for some $k \in \{0, \cdots, n-1\}$ or a tube of radius $\frac{\pi}{2} - r$ over $\mathbb{C}P^\ell$, where $k + \ell = n - 1$.*

When a real hypersurface M in the complex projective space $\mathbb{C}P^n$ satisfies the formula $A\phi + \phi A = k\phi$, $k \neq 0$ and constant, we say that M is a *contact* real hypersurface in $\mathbb{C}P^n$. In the papers due to Blair [2] and Yano-Kon [39], they introduce the classification of contact real hypersurfaces in $\mathbb{C}P^n$ as follows:

Theorem B. *Let M be a connected orientable real hypersurface in the complex projective space $\mathbb{C}P^n$, $n \geq 3$. Then M is a contact real hypersurface if and only if M is congruent to an open part of a tube of radius $0 < r < \frac{\pi}{4}$ around an n-dimensional real projective space $\mathbb{R}P^n$ or a tube of radius $\frac{\pi}{4} - r$ over Q^{n-1}, where $0 < r < \frac{\pi}{4}$.*

Motivated by these results, in this paper we give some characterizations of real hypersurfaces in the complex projective space $\mathbb{C}P^n$ regarding a family of geometric flows. Indeed, we know that a solution of the Ricci flow equation $\frac{\partial}{\partial t}g(t) = -2\mathrm{Ric}(g(t))$ is given by

$$\frac{1}{2}(\mathcal{L}_V g)(X,Y) + \mathrm{Ric}(X,Y) = \Omega g(X,Y),$$

where Ω is a constant and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Chaubey-Suh-De [10], Jeong-Suh [15], Morgan-Tian [22], Perelman [25], Wang [37] and [38]). Then this solution (M,V,Ω,g) is said to be a *Ricci soliton* with potential vector field V and Ricci soliton constant Ω.

As a generalization of the Ricci flow, the Ricci-Bourguignon flow (see Bourguignon [3] and [4], Catino-Cremaschi-Djadli-Mantegazza-Mazzieri [6]) is given by

$$\frac{\partial}{\partial t}g(t) = -2(\mathrm{Ric}(g(t)) - \theta\gamma g(t)), \quad g(0) = g_0.$$

This family of geometric flows with $\theta = 0$ reduces to the Ricci flow $\frac{\partial}{\partial t}g(t) = -2\mathrm{Ric}(g(t))$, $g(0) = g_0$. If the constant $\theta = \frac{1}{2}$, it is said to be *Einstein flow*. Its critical point of the Einstein flow

$$\frac{\partial}{\partial t}g(t) = -2\big(\mathrm{Ric}(g(t)) - \frac{1}{2}\gamma g(t)\big), \quad g(0) = g_0,$$

implies that the Einstein gravitational tensor $\mathrm{Ric}(g(t)) - \frac{1}{2}\gamma g(t)$ vanishes. For a four-dimensional space time M^4, this is equivalent to the vanishing Ricci tensor by virtue of $d\gamma = 2\mathrm{div}(\mathrm{Ric})$. In this case, M^4 becomes vacuum. That is, $g(t) = g(0)$, the metric is constant along the time (see O'Neill [24]). For $\theta = \frac{1}{n}$, the tensor $\mathrm{Ric} - \frac{\gamma}{n}g$ is said to be *traceless Ricci tensor*, and for $\theta = \frac{1}{2(n-1)}$, it is said to be the *Schouten tensor*.

Now let us introduce a Ricci-Bourguignon soliton $(M,V,\Omega,\theta,\gamma,g)$ which is a solution of the Ricci-Bourguignon flow as follows:

$$\frac{1}{2}(\mathcal{L}_V g)(X,Y) + \mathrm{Ric}(X,Y) = (\Omega + \theta\gamma)g(X,Y),$$

for any tangent vector fields X and Y on M, where Ω is a soliton constant, θ any constant and γ the scalar curvature on M, and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Bourguignon [3], [4], and Morgan-Tian [22]). Then (M,g) is said to be a *Ricci-Bourguignon soliton* with potential vector field V and Ricci-Bourguignon soliton constant Ω.

If the Ricci operator Ric of a real hypersurface M in $\mathbb{C}P^n$ satisfies

$$\mathrm{Ric}(X) = aX + b\eta(X)\xi \tag{1.1}$$

for smooth functions a, b on M, then M is said to be *pseudo-Einstein*. Then we introduce a complete classification of pseudo-Einstein Hopf real hypersurfaces in the complex projective space $\mathbb{C}P^n$ due to Cecil-Ryan [7] as follows:

Theorem C. *Let M be a pseudo-Einstein real hypersurface in the complex projective space $\mathbb{C}P^n$, $n \geq 3$. Then M is locally congruent to one of the following:*

(i) *a geodesic hypersphere,*

(ii) *a tube of radius r around a totally geodesic $\mathbb{C}P^k$, $0 < k < n-1$, where $0 < r < \frac{\pi}{2}$ and $\cot^2 r = \frac{k}{n-k-1}$,*

(iii) *a tube of radius r around a complex quadric Q^{n-1} where $0 < r < \frac{\pi}{4}$ and $\cot^2 2r = n-2$.*

Let M be a Hopf hypersurface in the complex projective space $\mathbb{C}P^n$. Then we have

$$A\xi = \alpha\xi$$

for the shape operator A with the Reeb function $\alpha = g(A\xi, \xi)$ on M in $G_2(\mathbb{C}^{n+2})$. When we consider a tensor field J for any vector field X on M , which is a Kähler structure on the tangent space T_zM, $z \in M$, then JX is given by

$$JX = \phi X + \eta(X)N,$$

where $\phi X = (JX)^T$ is the tangential component of the vector field JX, $\eta(X) = g(\xi, X)$, $\xi = -JN$, and N denotes a unit normal vector field on M.

In this paper we introduce a new notion named *generalized pseudo-anti commuting property* for the Ricci tensor of a real hypersurface M in the complex projective space $\mathbb{C}P^n$ as follows:

$$\mathrm{Ric}\,\phi + \phi\,\mathrm{Ric} = f\phi \tag{1.2}$$

for a smooth function f on M in $\mathbb{C}P^n$ (see Ki-Suh [16], and Yano-Kon [39]).

It is known that Einstein and pseudo-Einstein real hypersurfaces M in the complex projective space $\mathbb{C}P^n$ satisfy the condition of *generalized pseudo-anti commuting* Ricci tensor, that is, $\mathrm{Ric}\phi + \phi\mathrm{Ric} = f\phi$, where f denotes a smooth function on M in $\mathbb{C}P^n$ (see Besse [1], Cecil-Ryan [7] and Kon [18]). In Yano-Kon [39], real hypersurfaces of type (B) in the complex projective space $\mathbb{C}P^n$, which are characterized by $A\phi + \phi A = k\phi$, $k \neq 0$, also satisfy the formula of generalized pseudo-anti commuting Ricci tensor in (1.2).

Let us define a pseudo Ricci-Bourguignon soliton $(M, V, \eta, \Omega, \theta, \gamma, g)$ as follows:

$$\frac{1}{2}(\mathcal{L}_V g)(X,Y) + \mathrm{Ric}(X,Y) + \psi\eta(X)\eta(Y) = (\Omega + \theta\gamma)g(X,Y) \tag{1.3}$$

for any tangent vector fields X and Y on M, where Ω is said to be a pseudo Ricci-Bourguignon soliton constant, the functions θ and ψ are any constants and γ the scalar curvature on M, and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V.

When the function ψ identically vanishes, the pseudo Ricci-Bourguignon soliton $(M, V, \eta, \Omega, \theta, \gamma, g)$ is said to be a Ricci-Bourguignon soliton $(M, V, \Omega, \theta, \gamma, g)$. We also say that the pseudo Ricci-Bourguignon soliton is shrinking, steady, and expanding according to the pseudo Ricci-Bourguignon soliton constant function $\Omega > 0$, $\Omega = 0$, and $\Omega < 0$ respectively.

Now in this paper, by using the notion of generalized pseudo-anti commuting Ricci tensor (1.2), we give a theorem as follows:

Theorem 1. *Let M be a Hopf pseudo Ricci-Bourguignon soliton $(M, \xi, \eta, \Omega, \theta, \gamma, g)$ in the complex projective space $\mathbb{C}P^n$, $n \geq 3$. Then M is pseudo-Einstein and locally congruent to one of the following:*

(i) *a geodesic hypersphere, $\Omega + \theta\gamma = 2\{(n-1)\cot^2(r) + n\}$, and $\psi = 2n$,*

(ii) *a tube of radius r around a totally geodesic $\mathbb{C}P^k$, $0 < k < n-1$, where $0 < r < \frac{\pi}{2}$, $\cot^2 r = \frac{k}{n-k-1}$, $\Omega + \theta\gamma = 2n$, and $\psi = 2$.*

Let us denote by Df the gradient vector field of the function f on a real hypersurface M in the complex projective space $\mathbb{C}P^n$ defined by $g(Df, X) = g(\text{grad} f, X) = X(f)$ for any tangent vector field X on M. Now let us consider the gradient pseudo Ricci-Bourguignon soliton $(M, Df, \eta, \Omega, \theta, \gamma, g)$. It is a generalization of gradient Einstein soliton derived from a generalized Ricci potential for a Riemannian manifold (M, g) (see Catino-Mazzieri [5], Cernea-Guan [8]). It is defined by

$$\text{Hess}(f) + \text{Ric} + \psi\eta \otimes \eta = (\Omega + \theta\gamma)g,$$

where $\text{Hess}(f)$ is defined by $\text{Hess}(f) = \nabla Df$ and for any tangent vector fields X and Y on M

$$\text{Hess}(f)(X, Y) = XY(f) - (\nabla_X Y)f.$$

Then a gradient pseudo Ricci-Bourguignon soliton in $\mathbb{C}P^n$ can be defined by

$$\nabla_X Df + \text{Ric}(X) + \psi\eta(X)\xi = (\Omega + \theta\gamma)X$$

for any vector field X tangent to M in $\mathbb{C}P^n$. Then first by Theorem C and Theorem 1 we can assert a classification theorem of gradient pseudo Ricci-Bourguignon solitons in $\mathbb{C}P^n$ as follows:

Theorem 2. *Let M be a real hypersurface in $\mathbb{C}P^n$ with isometric Reeb flow, $n \geq 3$. If it admits the gradient pseudo Ricci-Bourguignon soliton $(M, Df, \eta, \Omega, \theta, \gamma, g)$, then M is pseudo-Einstein and locally congruent to one of the following*

(i) *a geodesic hypersphere, $\Omega + \theta\gamma = 2\{(n-1)\cot^2(r) + n\}$, and $\psi = 2n$,*

(ii) *a tube of radius r around a totally geodesic $\mathbb{C}P^k$, $0 < k < n-1$, where $0 < r < \frac{\pi}{2}$, $\cot^2 r = \frac{k}{n-k-1}$, $\Omega + \theta\gamma = 2n$, and $\psi = 2$.*

Next by virtue of Theorem B let us consider a contact real hypersurface in the complex projective space $\mathbb{C}P^n$. Then we can assert a classification of gradient pseudo Ricci-Bourguignon soliton in $\mathbb{C}P^n$ as follows:

Theorem 3. *Let M be a contact real hypersurface in the complex projective space $\mathbb{C}P^n$, $n \geq 3$. If it admits the gradient pseudo Ricci-Bourguignon soliton $(M, Df, \eta, \Omega, \theta, \gamma, g)$, then M is pseudo-Einstein and locally congruent to a tube of radius r around a complex quadric Q^{n-1} where $0 < r < \frac{\pi}{4}$ and $\cot^2(2r) = n-2$. Moreover, the soliton constants are given by $\Omega + \theta\gamma = 2n$ and $\psi = 2(2n-1)$.*

2. The Complex Projective Space

Let $(\bar{M}, g, J)$ be a Kähler manifold and $\bar{R}$ the Riemannian curvature tensor of $(\bar{M}, g)$. Since $\bar{\nabla} J = 0$, we immediately see that

$$\bar{R}(X, Y)JZ = J\bar{R}(X, Y)Z$$

holds for all $X, Y, Z \in T_p(\bar{M})$, $p \in \bar{M}$. From the curvature identities in Kobayashi and Nomizu [17] we also get

$$g(\bar{R}(X, Y)Z, W) = g(\bar{R}(JX, JY)Z, W) = g(\bar{R}(X, Y)JZ, JW).$$

Let $G_2^J(T\bar{M})$ be the Grassmann bundle over $\bar{M}$ consisting of all 2-dimensional J-invariant linear subspaces V of $T_p\bar{M}$, $p \in M$. Thus every $V \in G_2^J(T\bar{M})$ is a complex line in the corresponding tangent space of $\bar{M}$. The restriction of the section curvature function K to $G_2^J(T\bar{M})$ is called the *holomorphic sectional curvature function* on $\bar{M}$ and $K(V)$ is called the *holomorphic sectional curvature* of $\bar{M}$ with respect to $V \in G_2^J(T\bar{M})$.

A Kähler manifold M is said to have *constant holomorphic sectional curvature* if the holomorphic sectional curvature function is constant. Now we want to introduce the following.

Theorem 2.1. *A Kähler manifold $(\bar{M}, g, J)$ has constant holomorphic sectional curvature $c \in \mathbb{R}$ if and only if its Riemannian curvature tensor $\bar{R}$ is of the form*

$$\begin{aligned}\bar{R}(X, Y)Z = \frac{c}{4}\big\{&g(Y, Z)X - g(X, Z)Y \\ &+ g(JY, Z)JX - g(JX, Z)JY - 2g(JX, Y)JZ\big\}\end{aligned}$$

for any vector fields X, Y and Z on $\bar{M}$.

The complex vector space $\mathbb{C}^n$ ($n \in \mathbb{N}$) is in a canonical way an n-dimensional complex manifold. For $p \in \mathbb{C}^n$ denote by $\pi_p : T_p\mathbb{C}^n \to \mathbb{C}^n$ the canonical isomorphism. We define a Riemannian metric g on $\mathbb{C}^n$ by

$$g_p(u, v) = \langle \pi_p(u), \pi_p(v) \rangle$$

for all $u, v \in T_p\mathbb{C}^n$ and $p \in \mathbb{C}^n$, where $< \cdot, \cdot >$ is the real part of the standard Hermitian inner product on $\mathbb{C}^n$, that is,

$$< a, b >= \Re e \left(\sum_{\nu=1}^{n} a_\nu \bar{b}_\nu \right) \qquad (a, b \in \mathbb{C}^n).$$

The metric g is called the *canonical Riemannian metric* on $\mathbb{C}^n$. The complex structure J on $\mathbb{C}^n$ is given by the equation $\pi_p(Ju) = i\pi_p(u)$. It is easy to verify that $(\mathbb{C}^n, g, J)$ is a Kähler manifold. In fact, $(\mathbb{C}^n, g, J)$ a complex Euclidean space with vanishing constant holomorphic sectional curvature. The Kähler manifold $(\mathbb{C}^n, g, J)$ is known to be the *n-dimensional complex Euclidean space.*

We define an equivalence relation $\sim$ on $\mathbb{C}^{n+1} \setminus \{0\}$ by $z_1 \sim z_2$ if and only if there exists $\lambda \in \mathbb{C} \setminus \{0\}$ so that $z_2 = z_1\lambda$. We denote the quotient space $\mathbb{C}^{n+1} \setminus \{0\})/\sim$ by $\mathbb{C}P^n$. By construction, the points in $\mathbb{C}P^n$ are in one-to-one correspondence with the complex lines through $0 \in \mathbb{C}^{n+1}$. We equip $\mathbb{C}P^n$ with the quotient topology with respect to the canonical projection $\tau : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}P^n$. Then $\mathbb{C}P^n$ is a compact Hausdorff space and τ is a continuous map. There exists a unique complex manifold structure on $\mathbb{C}P^n$ so that τ is a holomorphic submersion. In this way $\mathbb{C}P^n$ becomes an n-dimensional complex manifold $(\mathbb{C}P^n, J)$. For $z \in \mathbb{C}^{n+1} \setminus \{0\}$ we also write $[z] = \tau(z) \in \mathbb{C}P^n$ (see Kobayashi and Nomizu [17]).

Let S^{2n+1} be the unit sphere in $\mathbb{C}^{n+1}$ and denote by π the restriction of τ to S^{2n+1}. We consider S^{2n+1} with the Riemannian metric induced from $\mathbb{C}^{n+1}$, which is the standard metric on S^{2n+1} turning it into a real space form with constant sectional curvature 1. The map $\pi : S^{2n+1} \to \mathbb{C}P^n$ is a surjective submersion whose fibers are 1-dimensional circles. There exists a unique Riemannian metric g on $\mathbb{C}P^n$ so that π becomes a Riemannian submersion. In such a way, the map $\pi : S^{2n+1} \to \mathbb{C}P^n$ is known as the *Hopf map* from S^{2n+1} onto $\mathbb{C}P^n$ and the Riemannian metric g is known as the *Fubini-Study metric* on $\mathbb{C}P^n$. The manifold $(\mathbb{C}P^n, J, g)$ is a Kähler manifold and called the n-dimensional *complex projective space.* The complex projective space $(\mathbb{C}P^n, J, g)$ is a complex space form with constant holomorphic sectional curvature 4.

By virtue of Theorem 2.1, the Riemannian curvature tensor $\bar{R}$ of $\mathbb{C}P^n$ can be given for any vector fields X, Y and Z in $T_p(\mathbb{C}P^n)$, $p \in \mathbb{C}P^n$ as follows:

$$\begin{aligned} \bar{R}(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y + g(JY, Z)JX \\ & - g(JX, Z)JY - 2g(JX, Y)JZ. \end{aligned} \tag{2.1}$$

3. Some General Equations

Let M be a real hypersurface in the complex projective space $\mathbb{C}P^n$ and denote by (ϕ, ξ, η, g) the induced almost contact metric structure. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. Then the vector field ξ is said to be the *Reeb* vector field on M in $\mathbb{C}P^n$. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \oplus \mathbb{R}\xi$, where $\mathcal{C} = \ker(\eta)$ is the maximal complex subbundle of TM. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and $\phi\xi = 0$.

In different way, the complex projective space $\mathbb{C}P^n$ is defined by using the fibration

$$\tilde{\pi} : S^{2n+1}(1) \to \mathbb{C}P^n, \quad p \to [p],$$

which is said to be a Riemannian submersion. Then naturally we can consider the following diagram for a real hypersurface in the complex projective space $\mathbb{C}P^n$ as follows:

$$\begin{tikzcd} M' = \tilde{\pi}^{-1}(M) \arrow[r, "\tilde{i}"] \arrow[d, "\pi"'] & S^{2n+1}(1) \subset \mathbb{C}^{n+1} \arrow[d, "\tilde{\pi}"'] \\ M \arrow[r, "i"] & \mathbb{C}P^n \end{tikzcd}$$

We now assume that M is a Hopf hypersurface. Then we have

$$A\xi = \alpha\xi,$$

where A denotes the shape operator of M in $\mathbb{C}P^n$ and the smooth function α is defined by $\alpha = g(A\xi, \xi)$ on M. When we consider the transformed vector field JX by the Kähler structure J on $\mathbb{C}P^n$ for any vector field X on M in $\mathbb{C}P^n$, we may write

$$JX = \phi X + \eta(X)N.$$

Then by using Kähler structure $\bar{\nabla} J = 0$, we get the following

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi \text{ and } \nabla_X \xi = \phi AX,$$

where $\bar{\nabla}$ and ∇ denote the Levi-Civita connections of $\bar{M}$ and M respectively.

Now we consider the equation of Codazzi

$$g((\nabla_X A)Y - (\nabla_Y A)X, Z) = \eta(X)g(\phi Y, Z) - \eta(Y)g(\phi X, Z) - 2\eta(Z)g(\phi X, Y).$$

By the equation of Gauss, the curvature tensor $R(X, Y)Z$ for a real hypersurface M in $\mathbb{C}P^n$ induced from the curvature tensor $\bar{R}$ in (2.1) of $\mathbb{C}P^n$ can be described in terms of the almost contact structure tensor ϕ and the shape operator A of M in $\mathbb{C}P^n$ as follows:

$$\begin{aligned} R(X, Y)Z = {} & g(Y, Z)X - g(X, Z)Y \\ & + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z \\ & + g(AY, Z)AX - g(AX, Z)AY \end{aligned} \tag{3.1}$$

for any vector fields $X, Y, Z \in T_zM$, $z \in M$. From this, contracting Y and Z on M in $\mathbb{C}P^n$, we get the Ricci operator of a real hypersurface M in $\mathbb{C}P^n$ as follows:

$$\mathrm{Ric}(X) = (2n+1)X - 3\eta(X)\xi + (\mathrm{Tr}A)AX - A^2X. \tag{3.2}$$

Then by contracting the Ricci operator in (3.2) the scalar curvature γ of M in $\mathbb{C}P^n$ is given by

$$\gamma = \sum_{i=1}^{2n-1} g(\mathrm{Ric}(e_i), e_i) = 4(n^2-1) + h^2 - \mathrm{Tr}A^2, \tag{3.3}$$

where the function h denotes the trace of the shape operator A of M in $\mathbb{C}P^n$.

Putting $Z = \xi$ in the Codazzi equation, we get

$$g((\nabla_X A)Y - (\nabla_Y A)X, \xi) = -2g(\phi X, Y). \tag{3.4}$$

Since we have assumed that M is Hopf in $\mathbb{C}P^n$, differentiating $A\xi = \alpha\xi$ gives

$$(\nabla_X A)\xi = (X\alpha)\xi + \alpha\phi AX - A\phi AX.$$

From this, the left side of (3.4) becomes

$$\begin{aligned} &g((\nabla_X A)Y - (\nabla_Y A)X, \xi) \\ &= g((\nabla_X A)\xi, Y) - g((\nabla_Y A)\xi, X) \\ &= (X\alpha)\eta(Y) - (Y\alpha)\eta(X) + \alpha g((A\phi + \phi A)X, Y) - 2g(A\phi AX, Y). \end{aligned} \tag{3.5}$$

Putting $X = \xi$ in (3.4) and (3.5) and using the almost contact structure of (M, g), we have

$$Y\alpha = (\xi\alpha)\eta(Y).$$

Inserting this formula into (3.4) and (3.5) implies the following for any vector fields X and Y on M

$$0 = 2g(A\phi AX, Y) - \alpha g((\phi A + A\phi)X, Y) - 2g(\phi X, Y).$$

By virtue of this equation, we can assert the following

Lemma 3.1. *Let M be a Hopf real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. Then we have*

$$2A\phi AX = \alpha(A\phi + \phi A)X + 2\phi X$$

for any tangent vector field X on M.

By using the formulas given in section 3 we want to introduce an important lemma due to Okumura [23] and Yano-Kon [39] as follows:

Lemma 3.2. *Let M be a Hopf real hypersurface in $\mathbb{C}P^n$. Then the Reeb function α is constant. Moreover, if $X \in \mathcal{C}$ is a principal curvature vector of M with principal curvature λ, then $2\lambda \neq \alpha$ and ϕX is a principal curvature vector of M with principal curvature $\frac{\alpha\lambda+2}{2\lambda-\alpha}$. on M, where $\mathcal{C}$ denotes the orthogonal complement of the Reeb vector field ξ on M.*

Now by using (3.2) and (3.3), we introduce an important proposition due to Cecil-Ryan [7], Djorić-Okumura [12] as follows:

Proposition 3.3. *Let M be the tube of radius $0 < r < \frac{\pi}{2}$ around the totally geodesic $\mathbb{C}P^k$, $k \in \{1, \cdots, n-2\}$ in $\mathbb{C}P^n$, which is said to be of type (A_2). Then the following statements hold:*

(1) *M is a Hopf hypersurface.*

(2) *The principal curvatures and corresponding principal curvature spaces of M are given by*

principal curvature	*eigenspace*	*multiplicity*
$\lambda = \cot(r)$	T_λ	2ℓ
$\mu = -\tan(r)$	T_μ	$2k$
$\alpha = 2\cot(2r)$	$T_\alpha = \mathbb{R}JN$	1

where $\ell = n - k - 1$.

(3) *The shape operator A commutes with the structure tensor field ϕ as*

$$A\phi = \phi A.$$

(4) *The trace h of the shape operator A and its square h^2 becomes the following respectively*

$$h = (2\ell + 1)\cot(r) - (2k + 1)\tan(r),$$
$$h^2 = (2\ell + 1)^2\cot^2(r) + (2k + 1)^2\tan^2(r) - 2(2\ell + 1)(2k + 1).$$

(5) *The trace of the matrix A^2 is given by*

$$\mathrm{Tr}A^2 = (2\ell + 1)\cot^2(r) + (2k + 1)\tan^2(r) - 2.$$

(6) *The scalar curvature γ of the tube M is given by*

$$\gamma = 4(n - 1)n - 8k\ell + 2(2\ell + 1)\ell\cot^2(r) + 2(2k + 1)k\tan^2(r).$$

Remark 3.4. For $k = 0$, M is pseudo Einstein, that is, a geodesic hypersphere, which is said to be of type (A_1) such that

$$\mathrm{Ric}(X) = 2\{(n - 1)\cot^2(r) + n\}X - 2n\eta(X)\xi.$$

Now let M be a tube of radius r, $0 < r < \frac{\pi}{4}$, over the real projective space $\mathbb{R}P^n$, which is said to be of type (B) and a contact real hypersurface in the complex projective space $\mathbb{C}P^n$. It also can be regarded as a tube of radius $\frac{\pi}{4} - r$ over the complex quadric Q^{n-1}.

The tube of radius r around totally geodesic and totally real projective space $\mathbb{R}P^n$ has therefore three distinct constant principal curvatures $2\tan(2r)$, $-\cot(r)$, and $\tan(r)$. It also can be regarded as a tube of radius $\frac{\pi}{4} - r$ over a totally geodesic complex quadric Q^{n-1}. Then by (3.2) and (3.3), we want to give an important proposition due to Cecil-Ryan [7] as follows:

Proposition 3.5. *Let M be the tube of radius $0 < r < \frac{\pi}{4}$ around the complex quadric Q^{n-1} in $\mathbb{C}P^n$. Then the following statements hold:*

(1) *M is a Hopf hypersurface.*

(2) *The principal curvatures and corresponding principal curvature spaces of M are*

principal curvature	*eigenspace*	*multiplicity*
$\lambda = -\cot(\frac{\pi}{4} - r)$	T_λ	$n-1$
$\mu = \tan(\frac{\pi}{4} - r)$	T_μ	$n-1$
$\alpha = 2\cot(2r)$	$\mathbb{R}JN$	1

(3) *The shape operator A and the structure tensor field ϕ satisfy*

$$A\phi + \phi A = k\phi, \quad k \neq 0 : const.$$

(4) *The trace h of the shape operator A and its square h^2 becomes the following respectively*

$$h = \mathrm{Tr}A = 2\mathrm{cot}(2r) - 2(n-1)\mathrm{tan}(2r),$$
$$h^2 = 4\mathrm{cot}^2(2r) + 4(n-1)^2\mathrm{tan}^2(2r) - 8(n-1).$$

(5) *The trace of the matrix A^2 is given by*

$$\mathrm{Tr}A^2 = 4\mathrm{cot}^2(2r) + 4(n-1)\mathrm{tan}^2(2r).$$

(6) *The scalar curvature γ of the tube M is given by*

$$\gamma = 4(n-1)^2 + 4(n-1)(n-2)\mathrm{tan}^2(2r).$$

(7) *For $\cot^2(r) = n-2$, M is pseudo-Einstein such that*

$$\mathrm{Ric(X) = 2nX - 2(2n-1)}\eta(\mathrm{X})\xi.$$

4. Hopf Pseudo Ricci-Bourguignon Soliton Real Hypersurfaces

Let us introduce a pseudo Ricci-Bourguignon soliton $(M, \xi, \eta, \Omega, \theta, \gamma, g)$ which is a solution of the pseudo Ricci-Bourguignon flow defined by

$$\frac{\partial}{\partial t}g(t) = -2(\mathrm{Ric}(g(t)) - \theta\gamma g(t)) - 2\psi\eta(g(t))\otimes\eta(g(t)), \quad g(0) = g_0.$$

Then it is given by the following

$$\frac{1}{2}(\mathcal{L}_\xi g)(X,Y) + \mathrm{Ric}(X,Y) + \psi\eta(X)\eta(Y) = (\Omega + \theta\gamma)g(X,Y) \tag{4.1}$$

for any tangent vector fields X and Y on M, where Ω is a pseudo Ricci-Bourguignon soliton constant, ψ and θ any constants and γ the scalar curvature on M, and $\mathcal{L}_V$ denotes the Lie derivative along the direction of the vector field V (see Morgan-Tian [22]). Then by virtue of the Lie derivative, we have

$$\begin{aligned}(\mathcal{L}_\xi g)(X,Y) &= \xi(g(X,Y)) - g(\mathcal{L}_\xi X, Y) - g(X, \mathcal{L}_\xi Y)\\ &= g(\nabla_\xi X, Y) + g(X, \nabla_\xi Y) - g([\xi, X], Y) - g(X, [\xi, Y])\\ &= g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi)\\ &= g((\phi A - A\phi)X, Y).\end{aligned}$$

Then the formula (4.1) can be given by

$$\mathrm{Ric}(X) = \frac{1}{2}(A\phi - \phi A)X - \psi\eta(X)\xi + (\Omega + \theta\gamma)X. \tag{4.2}$$

From this, by applying the structure tensor ϕ to both sides, we get the following two formulas

$$\mathrm{Ric}(\phi X) = \frac{1}{2}(A\phi^2 - \phi A\phi)X - \psi\eta(\phi X)\xi + (\Omega + \theta\gamma)\phi X,$$

and

$$\phi\mathrm{Ric}(X) = \frac{1}{2}(\phi A\phi - \phi^2 A)X - \psi\eta(X)\phi\xi + (\Omega + \theta\gamma)\phi X.$$

By using the almost contact structure (ϕ, ξ, η, g) in the right side above, we know that the *generalized pseudo anti-commuting property* holds as follows:

$$\mathrm{Ric}(\phi X) + \phi\mathrm{Ric}(X) = 2(\Omega + \theta\gamma)\phi X. \tag{4.3}$$

Now we want to introduce an important proposition due to Ki-Suh [16], and Yano-Kon [39], which will be used in the proof of our Theorem 1 as follows:

Proposition 4.1 *Let M be a connected complete Hopf real hypersurface in the complex projective space $\mathbb{C}P^n$. If M satisfies the generalized pseudo-anti commuting property, then M is locally congruent to a geodesic hypersphere in the class of type (A_1), a pseudo-Einstein hypersurface in the class of type (A_2), or M is locally congruent to of type (B).*

Among real hypersurfaces of type (A_2) satisfying the generalized pseudo-anti commuting property (4.2) is only pseudo-Einstein. Then it is exactly the second case in Theorem C. That is M is locally congruent to a tube of radius r around a totally geodesic $\mathbb{C}P^k$, $0 < k < n-1$, where $0 < r < \frac{\pi}{2}$ and $\cot^2(r) = \frac{k}{n-k-1}$.

Now geodesic hyperspheres and pseudo-Einstein real hypersurfaces are included in the class of type (A_1) and A_2 respectively. So by Theorem A, they are characterized by the commuting shape operator. That is, $A\phi = \phi A$. Accordingly, from the notion of pseudo Ricci-Bourguignon soliton $(M, \xi, \eta, \Omega, \theta, \gamma, g)$ of M, (4.1) becomes

$$\mathrm{Ric}(X) = (\Omega + \theta\gamma)X - \psi\eta(X)\xi.$$

This means that those hypersurfaces are pseudo-Einstein. Then by virtue of Theorem C there exist three kind of pseudo-Einstein real hypersurfaces in complex projective space $\mathbb{C}P^n$ such that

(i) a geodesic hypersphere,

(ii) a tube of radius r around a totally geodesic $\mathbb{C}P^k$, $0 < k < n-1$, where $0 < r < \frac{\pi}{2}$ and $\cot^2 r = \frac{k}{n-k-1}$,

(iii) a tube of radius r around a complex quadric Q^{n-1} where $0 < r < \frac{\pi}{4}$ and $\cot^2 2r = n-2$.

For the case (i) it can be easily verified that a geodesic hypersphere in Remark 3.4 satisfies the following

$$\mathrm{Ric}(X) = 2\{(n-1)\cot^2(r)X + n\}X - 2n\eta(X)\xi$$

for any vector fields X on M in $\mathbb{C}P^n$. Then $\Omega + \theta\gamma = 2n + 2(n-1)\cot^2(r)$, and $\psi = 2n$.

Now let us check the second case (ii) whether it satisfies a pseudo Ricci-Bourguignon soliton for $\cot^2 r = \frac{k}{n-k-1}$. Then for the Reeb vector field ξ we have the following

$$\begin{aligned}
\mathrm{Ric}(\xi) &= (a+b)\xi \\
&= \Big[2(n-1) + \{(2\ell+1)\cot(r) - (2k+1)\tan(r)\}(\cot(r) - \tan(r)) \\
&\quad - (\cot(r) - \tan(r))^2\Big]\xi \\
&= \Big\{2(n-1) + 2\ell\cot^2(r) - 2\ell - 2k + 2k\tan^2(r)\Big\}\xi,
\end{aligned}$$

where $\ell = n-k-1$ and $\cot^2(r) = \frac{k}{n-k-1}$. Then the coefficient $a+b$ is given by

$$a + b = g(\mathrm{Ric}(\xi), \xi) = 2n - 2.$$

Moreover, by using $\cot^2(r) = \frac{k}{n-k-1}$ and Proposition 3.3, for any vector fields $X \in T_\lambda$, $\lambda = \cot(r)$ and $Y \in T_\mu$, $\mu = \tan(r)$, we have the following formulas, respectively

$$\begin{aligned}
\mathrm{Ric}(X) = aX &= \{(2n+1) + h\lambda - \lambda^2\}X \\
&= \{(2n+1) + 2\ell\cot^2(r) - (2k+1)\}X \\
&= 2nX
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Ric}(Y) = aY &= \{(2n+1) - (2\ell+1) + 2k\tan^2(r)\}Y \\
&= 2nY.
\end{aligned}$$

Then the Ricci operator of pseudo-Einstein hypersurfaces satisfying the pseudo Ricci-Bourguignon soliton becomes

$$\mathrm{Ric}(X) = aX + b\eta(X)\xi = (\Omega + \theta\gamma)X - \psi\eta(X)\xi,$$

where soliton constants are given by $\Omega+\theta\gamma = a = 2n$ and $\psi = -b = 2$, respectively.

Finally, in the third case (iii) let us check that a tube of radius r around the complex quadric Q^{n-1} with $\cot^2(2r) = n-2$ in the complex projective space $\mathbb{C}P^n$ could satisfy the pseudo Ricci-Bourguignon soliton.

In order to do this, first we should check that this tube is pseudo-Einstein for $\cot^2(2r) = n-2$. In fact, it is characterized by $A\phi+\phi A = k\phi$, where $k\neq 0:$ constant. Moreover, by Proposition 3.5, the principal curvature are given by $\lambda = -\cot(\frac{\pi}{4}-r)$, $\mu = \tan(\frac{\pi}{4}-r)$ and $\alpha = 2\cot(2r)$. So $k = \lambda+\mu = -\frac{4}{\alpha}$. For any $X\in T_\lambda$ the vector field $\phi X\in T_\mu$. Then by (3.2) we know the following for any $X\in T_\lambda$

$$\mathrm{Ric}(X) = \{(2n+1) + h\lambda - \lambda^2\}X$$

and for any $Y\in T_\mu$

$$\mathrm{Ric}(Y) = \{(2n+1) + h\mu - \mu^2\}Y.$$

Then from $\cot^2(2r) = n-2$ we can verify the following

$$\begin{aligned}(h\lambda-\lambda^2)-(h\mu-\mu^2) &= (\lambda-\mu)(h-(\lambda+\mu))\\ &= (\lambda-\mu)(h+\frac{4}{\alpha})\\ &= 0,\end{aligned}$$

where by Proposition 3.5, we have used the following from $\cot^2(2r) = n-2$

$$\begin{aligned}h+\frac{4}{\alpha} &= 2\cot(2r) - 2(n-1)\tan(2r) + 2\tan(2r)\\ &= 2\cot(2r) - 2(n-2)\tan(2r)\\ &= 2\Big\{\frac{\cot^2(2r)-(n-2)}{\cot(2r)}\Big\} = 0.\end{aligned} \tag{4.4}$$

Moreover, $\mathrm{Ric}(\xi) = \{2(n-1) + (h\alpha-\alpha^2)\}\xi$. So for a pseudo-Einstein real hypersurface in $\mathbb{C}P^n$ we may put

$$\mathrm{Ric}(X) = aX + b\eta(X)\xi,$$

where by Proposition 3.5 and $\cot^2(2r) = n-2$, the constant $a+b$ is given by

$$\begin{aligned}a+b &= 2(n-1) + h\alpha - \alpha^2\\ &= 2(n-1) + \{2\cot(2r) - 2(n-1)\tan(2r)\}2\cot(2r) - (2\cot(2r))^2\\ &= -2(n-1)\end{aligned}$$

and by the property of contact hypersurfaces, we know that $\lambda+\mu=-\frac{4}{\alpha}$. So by virtue of (4.4), it follows that $h=\lambda+\mu$. Then the constant a is given by

$$a=(2n+1)+h\lambda-\lambda^2=(2n+1)+\lambda\mu=2n.$$

By the above two constants a and $a+b$, another constant b becomes

$$b=-2(2n-1).$$

Then if the third case satisfies the pseudo Ricci-Bourguignon soliton, we can assert the following

$$\begin{aligned}\mathrm{Ric}(X)&=2nX-2(2n-1)\eta(X)\xi\\&=\frac{1}{2}(A\phi-\phi A)X+(\Omega+\theta\gamma)X-\psi\eta(X)\xi\end{aligned}\tag{4.5}$$

where the soliton constants Ω,θ and ψ are given by $\Omega+\theta\gamma=2n$ and $\psi=2(2n-1)$. Then for $X\in T_\lambda$ and $\phi X\in T_\mu$ the formula (4.5) is given by

$$\mathrm{Ric}(X)=2nX=\frac{1}{2}(\mu-\lambda)\phi X+2nX.$$

This means that $\lambda=\mu$. That is, $-\cot(\frac{\pi}{4}-r)=\tan(\frac{\pi}{4}-r)$, which gives a contradiction. So there does not exist a real hypersurface of type (B) which satisfy the pseudo Ricci-Bourguignon soliton.

Then summing up the above discussion for (iii), together with the cases (i) and (ii), we can assert our Main Theorem 1 in the introduction.

5. Gradient Pseudo Ricci-Bourguignon Soliton on Isometric Reeb Flow in $\mathbb{C}P^n$

In this section let M be a tube of radius r, $0<r<\frac{\pi}{2}$, over a totally geodesic $\mathbb{C}P^k$, $k\in\{0,1,\cdots,n-2,n-1\}$ in $\mathbb{C}P^n$, which is said to be of type (A_1) or of type (A_2). In Theorem A, we have mentioned that the Reeb flow on M in $\mathbb{C}P^n$ is isometric if and only if M is locally congruent to a totally geodesic $\mathbb{C}P^k$ in $\mathbb{C}P^n$ for $k\in\{0,1,\cdots,n-1\}$. Then for $k=0$ or $k=n-1$ we say that M is a geodesic hypersphere which is said to be of type (A_1) and it has with two distinct principal curvatures. For $k\in\{1,\cdots,n-2\}$, M is locally congruent to a tube over $\mathbb{C}P^k$ in $\mathbb{C}P^n$. Moreover, it is said to be of type (A_2) and has with three distinct constant principal curvatures.

Then the shape operator of M in the complex projective space $\mathbb{C}P^n$ with isometric Reeb flow can be expressed as

$$A=\mathrm{diag}\big(\alpha,\underbrace{\cot(r),\cdots,\cot(r)}_{2\ell},\underbrace{-\tan(r),\cdots,-\tan(r)}_{2k}\big)$$

for three constant principal curvatures $\alpha = 2\cot(2r)$, $\cot(r)$ and $-\tan(r)$ with multiplicities 1, 2ℓ and $2k$ respectively, where $\ell = n - k - 1$.

Then, by putting $X = \xi$ in (3.2), and using $A\xi = \alpha\xi$, we have the following

$$\begin{aligned}\mathrm{Ric}(\xi) &= (2n+1)\xi - 3\xi + hA\xi - A^2\xi \\ &= 2(n-1)\xi + (h\alpha - \alpha^2)\xi \\ &= \kappa\xi,\end{aligned}$$

where we have put $\kappa = 2(n-1) + h\alpha - \alpha^2$. So by Proposition 3.3, the constant κ is given by

$$\begin{aligned}\kappa &= 2(n-1) + (h\alpha - \alpha^2) \\ &= 2(n-1) + \{(2\ell+1)\cot(r) - (2k+1)\tan(r)\}2\cot(2r) - (2\cot(2r))^2 \\ &= 2(n-1) + 2\{\ell\cot^2(r) + k\tan^2(r) - (k+\ell)\} \\ &= 2\ell\cot^2(r) + 2k\tan^2(r).\end{aligned}$$

Then by taking the covariant derivative we get the following two formulas

$$(\nabla_X \mathrm{Ric})\xi = \kappa\phi AX - \mathrm{Ric}(\phi AX),$$

and

$$(\nabla_\xi \mathrm{Ric})X = h(\nabla_\xi A)X - (\nabla_\xi A^2)X.$$

Since M admits the gradient pseudo Ricci-Bourguignon soliton $(M, Df, \eta, \Omega, \theta, \gamma, g)$, we could consider the soliton vector field W as $W = Df$ for any smooth function on M. In the introduction we have noted that $\mathrm{Hess}(f)$ is defined by $\mathrm{Hess}(f) = \nabla Df$ for any tangent vector fields X and Y on M in such a way that

$$\mathrm{Hess}(f)(X,Y) = g(\nabla_X Df, Y).$$

Then the gradient pseudo Ricci-Bourguignon soliton $(M, Df, \eta, \Omega, \theta, \gamma, g)$ can be given by

$$\nabla_X Df + \mathrm{Ric}(X) + \psi\eta(X)\xi = (\Omega + \theta\gamma)X \tag{5.1}$$

for any tangent vector field X on M. Then by covariant differentiation, it gives

$$\begin{aligned}&\nabla_X\nabla_Y Df + (\nabla_X \mathrm{Ric})(Y) + \mathrm{Ric}(\nabla_X Y) \\ &\quad + \psi(\nabla_X\eta)(Y)\xi + \psi\eta(\nabla_X Y)\xi + \psi\eta(Y)\phi AX \\ &= (\Omega + \theta\gamma)\nabla_X Y\end{aligned}$$

for any vector field X and Y tangent to M in $\mathbb{C}P^n$. From this, together with the above two formulas for the derivative of Ricci operator and the constant scalar curvature γ for the isomeric Reeb flow, it follows that

$$\begin{aligned}R(\xi, Y)Df &= \nabla_\xi\nabla_Y Df - \nabla_Y\nabla_\xi Df - \nabla_{[\xi,Y]}Df \\ &= (\nabla_Y \mathrm{Ric})\xi - (\nabla_\xi \mathrm{Ric})Y + \psi\phi AY \\ &= (\kappa + \psi)\phi AY - \mathrm{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y.\end{aligned} \tag{5.2}$$

Then from (3.1) we have the following for a real hypersurface M in $\mathbb{C}P^n$ with isometric Reeb flow

$$R(\xi,Y)Df = g(Y,Df)\xi - g(\xi,Df)Y + g(AY,Df)A\xi - g(A\xi,Df)AY. \tag{5.3}$$

From this, let us take a vector field $Y \in T_\lambda$, $\lambda = \cot(r)$. Moreover, we can decompose the tangent space $T\mathbb{C}P^n$ as

$$T\mathbb{C}P^n = T_\lambda \oplus T_\mu \oplus T_\alpha \oplus \mathbb{R}N,$$

where $\lambda = \cot(r)$, $\mu = -\tan(r)$ and $\alpha = 2\cot(2r)$. If M is of type (A_1), that is, a geodesic hypersphere in $\mathbb{C}P^n$, it can be decomposed as

$$T\mathbb{C}P^n = T_\lambda \oplus T_\alpha \oplus \mathbb{R}N,$$

or otherwise

$$T\mathbb{C}P^n = T_\mu \oplus T_\alpha \oplus \mathbb{R}N.$$

Then for $Y \in T_\lambda$ (5.3) gives

$$\begin{aligned} R(\xi,Y)Df &= g(Y,Df)\xi - g(\xi,Df)Y + \alpha\lambda g(Y,Df)\xi - \alpha\lambda g(\xi,Df)Y \\ &= (1+\alpha\lambda)\{g(Y,Df)\xi - g(\xi,Df)Y\}. \end{aligned} \tag{5.4}$$

Then by taking the inner product of (5.4) with the Reeb vector field ξ and using (5.2), it follows that $(1+\alpha\lambda)g(Y,Df) = \cot^2(r)g(Y,Df) = 0$. But $\cot^2(r)\neq 0$ for the radius $0 < r < \frac{\pi}{2}$ of isometric Reeb flow M in $\mathbb{C}P^n$. It means the following for any $Y \in T_\lambda$

$$g(Y,Df) = 0. \tag{5.5}$$

From this, together with Proposition 3.3 and (5.1), we get the result (i) in our Theorem 2 in the introduction.

Now let us check (5.3) for $Y \in T_\mu$, $\mu = -\tan(r)$. Then (5.3) gives

$$R(\xi,Y)Df = g(Y,Df)\xi - g(\xi,Df)Y + \alpha\mu g(Y,Df)\xi - \alpha\mu g(\xi,Df)Y. \tag{5.6}$$

Then by taking the inner product (5.6) with the Reeb vector field ξ and $Y \in T_\mu$ respectively and using (5.2), we get

$$(1+\alpha\mu)g(Y,Df) = 0 \quad \text{and} \quad (1+\alpha\mu)g(\xi,Df) = 0, \tag{5.7}$$

where $g(R(\xi,Y)Df,\xi) = 0$ and the left side $g(R(\xi,Y)Df,Y) = 0$ is given by virtue of the following formulas

$$g(\phi AY,Y) = \mu g(\phi Y,Y) = 0,$$

$$\mathrm{Ric}(\phi AY)=\mu\{(2n+1)+\mu h-\mu^2\}\phi Y,$$

and

$$g((\nabla_\xi A)Y,Y)=-\mu g(\nabla_\xi Y,Y)=0.$$

Since $1+\alpha\mu=1+(\cot(r)-\tan(r))(-\tan(r))=\tan^2(r)\neq 0$ for $0<r<\frac{\pi}{2}$ for isometric Reeb flow M in $\mathbb{C}P^n$, (5.7) implies that

$$g(Y,Df)=0 \text{ and } g(\xi,Df)=0 \tag{5.8}$$

for any $Y\in T_\mu$, $\mu=-\tan r$. For a geodesic hypersphere of type (A_1) in $\mathbb{C}P^n$ it holds either $g(Y,Df)=0$ for $Y{\in}T_\lambda=\mathcal{C}$ or for $Y{\in}T_\mu=\mathcal{C}$ from the above decomposition, where $\mathcal{C}$ denotes the orthogonal complement of the Reeb vector field ξ in the tangent space TM of M in $\mathbb{C}P^n$. Of course, it also holds $g(\xi,Df)=0$ for a geodesic hypersphere in $\mathbb{C}P^n$.

Summing up (5.5), (5.8) and the above documents, the gradient of the smooth function f is identically vanishing, that is, $g(Y,Df)=0$ for any tangent vector field $Y\in T_zM$, $z\in M$. Consequently, we can conclude that the gradient pseudo Ricci Bourguignon soliton $(M,Df,\eta,\Omega,\theta,\gamma,g)$ is trivial. That is, $Df=0$, the potential function f is constant on M. Then it means that the gradient pseudo Ricci-Bourguignon soliton (5.1) becomes pseudo-Einstein. That is, by Proposition 3.3, we get $\mathrm{Ric}(X)=(\Omega+\theta\gamma)X-\psi\eta(X)\xi$, where $\Omega+\theta\gamma=2n$ and $\psi=2$.

Consequently, by virtue of Theorem 1 and Theorem C, we give a complete proof of our Theorem 2 in the Introduction.

6. Gradient Pseudo Ricci-Bourguignon Soliton on Contact Real Hypersurfaces in $\mathbb{C}P^n$

In this section, we want to give a property for gradient pseudo Ricci-Bourguignon soliton on a contact real hypersurface M in the complex projective space $\mathbb{C}P^n$. Then by Theorem B the scalar curvature γ is constant. The gradient pseudo Ricci-Bourguignon soliton $(M,Df,\eta,\Omega,\theta,\gamma,g)$ gives the following for any tangent vector field X on M in $\mathbb{C}P^n$

$$\nabla_X Df+\mathrm{Ric}(X)+\psi\eta(X)=(\Omega+\theta\gamma)X. \tag{6.1}$$

Then by differentiating (6.1), the curvature tensor of $Df=\mathrm{grad}f$ is given by the following

$$
\begin{aligned}
(6.2) \qquad R(X,Y)Df =& \nabla_X\nabla_Y Df - \nabla_Y\nabla_X Df - \nabla_{[X,Y]}Df \\
=& -(\nabla_X \mathrm{Ric})Y - \mathrm{Ric}(\nabla_X Y) - \psi(\nabla_X\eta)(Y)\xi - \psi\eta(\nabla_X Y)\xi \\
& - \psi\eta(Y)\nabla_X\xi + (\Omega+\theta\gamma)\nabla_X Y \\
& + (\nabla_Y \mathrm{Ric})X + \mathrm{Ric}(\nabla_Y X) + \psi(\nabla_Y\eta)(X)\xi + \psi\eta(\nabla_Y X)\xi \\
& + \psi\eta(X)\nabla_Y\xi - (\Omega+\theta\gamma)\nabla_Y X \\
& + \mathrm{Ric}([X,Y]) - (\Omega+\theta\gamma)[X,Y] + \psi\eta([X,Y])\xi \\
=& (\nabla_Y \mathrm{Ric})X - (\nabla_X \mathrm{Ric})Y - \psi(\nabla_X\eta)(Y)\xi + \psi(\nabla_Y\eta)(X)\xi \\
& - \psi\eta(Y)\nabla_X\xi + \psi\eta(X)\nabla_Y\xi
\end{aligned}
$$

where we have used the Ricci soliton constant θ and gradient pseudo Ricci-Bourguignon soliton constant Ω, and the scalar curvature γ is constant on a contact real hypersurface M in $\mathbb{C}P^n$ in Proposition 3.5.

Now let us assume that M is a contact real hypersurface in $\mathbb{C}P^n$, which is characterized by

$$A\phi + \phi A = k\phi, \text{where } k\neq 0 : \text{constant}.$$

Then it is Hopf and the Ricci operator is given by

$$\mathrm{Ric}(X) = (2n+1)X - 3\eta(X)\xi + hAX - A^2X$$

for any tangent vector field X on M. From this, let us put $X=\xi$. Then M being Hopf and $A\xi = \alpha\xi$ implies

$$\mathrm{Ric}(\xi) = \ell\xi,$$

where $\ell = 2(n-1) + h\alpha - \alpha^2$ is constant, and the mean curvature $h = \mathrm{Tr}A$ is constant for a contact hypersurface M in $\mathbb{C}P^n$. Then by taking covariant derivative to the Ricci operator, we have

$$(\nabla_X \mathrm{Ric})\xi = \nabla_X(\mathrm{Ric}(\xi)) - \mathrm{Ric}(\nabla_X\xi) = \ell\phi AX - \mathrm{Ric}(\phi AX),$$

and

$$
\begin{aligned}
(\nabla_\xi \mathrm{Ric})(X) =& \nabla_\xi(\mathrm{Ric}X) - \mathrm{Ric}(\nabla_\xi X) \\
=& h(\nabla_\xi A)X - (\nabla_\xi A^2)X.
\end{aligned}
$$

From (6.2), together with above formula, by putting $X=\xi$ we have the following for a contact hypersurface M in $\mathbb{C}P^n$

$$
\begin{aligned}
(6.3) \qquad R(\xi,Y)Df =& (\nabla_Y \mathrm{Ric})\xi - (\nabla_\xi \mathrm{Ric})Y \\
& - \psi(\nabla_\xi\eta)(Y)\xi + \psi(\nabla_Y\eta)(\xi)\xi - \psi\eta(Y)\nabla_\xi\xi + \psi\eta(\xi)\nabla_Y\xi \\
=& (\ell+\psi)\phi AY - \mathrm{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y.
\end{aligned}
$$

Then the diagonalization of the shape operator A of the contact real hypersurface in complex projective space $\mathbb{C}P^n$ is given by

$$A = \mathrm{diag}\Big(2\cot(2r), \underbrace{-\cot(\frac{\pi}{4}-r), \cdots, -\cot(\frac{\pi}{4}-r)}_{n-1}, \underbrace{\tan(\frac{\pi}{4}-r), \cdots, \tan(\frac{\pi}{4}-r)}_{n-1}\Big).$$

Here by Proposition 3.5 the principal curvatures are given by $\alpha = 2\cot(2r)$, $\lambda = -\cot(\frac{\pi}{4}-r)$ and $\mu = \tan(\frac{\pi}{4}-r)$ with multiplicities 1, $n-1$ and $n-1$ respectively. All of these principal curvatures satisfy

$$\kappa = \lambda + \mu = -\cot(\frac{\pi}{4}-r) + \tan(\frac{\pi}{4}-r) = -2\tan(2r) = -\frac{4}{\alpha}.$$

On the other hand, the curvature tensor $R(X,Y)Z$ of M induced from the curvature tensor $\bar{R}(X,Y)Z$ of the complex projective space $\mathbb{C}P^n$ gives

$$\begin{aligned} R(\xi,Y)Df =& g(Y,Df)\xi - g(\xi,Df)Y \\ &+ g(AY,Df)A\xi - g(A\xi,Df)AY \\ =& (1+\alpha\lambda)\{g(Y,Df)\xi - g(\xi,Df)Y\} \end{aligned} \tag{6.4}$$

for any $Y \in T_\lambda$, $\lambda = -\cot(\frac{\pi}{4}-r)$ for a contact real hypersurface M in the complex projective space $\mathbb{C}P^n$. Consequently, (6.3) and (6.4) give

$$(\ell+\psi)\phi AY - \mathrm{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y = (1+\alpha\lambda)\{g(Y,Df)\xi - g(\xi,Df)Y\}.$$

From this, by taking the inner product with the Reeb vector field ξ, we have

$$(1+\alpha\lambda)g(Y,Df) = 0. \tag{6.5}$$

Then for any $Y \in T_\lambda$ in (6.5) it follows that

$$g(Y,Df) = 0, \tag{6.6}$$

where we have noted that $1+\alpha\lambda = 1 + 2\cot(2r)(-\cot(\frac{\pi}{4}-r)) \neq 0$. Because if we assume that $1 = 2\cot(2r)\cot(\frac{\pi}{4}-r)$, then $\tan(2r) = 2\cot(\frac{\pi}{4}-r)$. Then it follows that

$$(\cos(r)-\sin(r))\sin(r)\cos(r) = (\cos(r)+\sin(r))^2(\cos(r)-\sin(r)),$$

which gives $\sin(r)\cos(r) = -1$. This gives us a contradiction for $0 < r < \frac{\pi}{4}$. Accordingly, the gradient vector field Df is orthogonal to the eigenspace T_λ, that is, $g(Y,Df) = 0$ for any $Y \in T_\lambda$.

Next, we consider for $Y \in T_\mu$, $\mu = \tan(\frac{\pi}{4}-r)$ in Proposition 3.5. Then using these properties in (6.3) and (6.4) implies the following

$$(\ell+\psi)\phi AY - \mathrm{Ric}(\phi AY) - h(\nabla_\xi A)Y + (\nabla_\xi A^2)Y = (1+\alpha\mu)\{g(Y,Df)\xi - g(\xi,Df)Y\}.$$

From this, by taking the inner product with the Reeb vector field ξ, we get

$$g(Y, Df) = 0 \quad \text{for any} \quad Y \in T_\mu, \tag{6.7}$$

where $1 + \alpha\mu \neq 0$. If we assume that $1 + \alpha\mu = 0$, then by Proposition 3.5, we get $1 + 2\cot(2r)\tan(\frac{\pi}{4} - r) = 0$. Then it gives $-\tan(2r) = 2\frac{\cos(r)-\sin(r)}{\cos(r)+\sin(r)}$. Since $\tan(2r) = \frac{\sin(2r)}{\cos(2r)}$, we get the following

$$\begin{aligned}(\cos(r) + \sin(r))\sin(r)\cos(r) =& - (\cos(r) - \sin(r))(\cos^2(r) - \sin^2(r))\\ =& - (\cos(r) - \sin(r))^2(\cos(r) + \sin(r)).\end{aligned}$$

From $\cos(r) + \sin(r) \neq 0$ we get $\sin(r)\cos(r) = 1$, which gives also a contradiction for $0 < r < \frac{\pi}{4}$.

Finally, let us take the inner product the above formula with $Y \in T_\mu$, and use $AY = \mu Y$, $A\phi Y = \lambda\phi Y$ for a contact hypersurface in $\mathbb{C}P^n$, we have

$$\begin{aligned}-(1 + \alpha\mu)g(\xi, Df) =&(\ell + \psi)g(\phi AY, Y) - g(\mathrm{Ric}(\phi AY), Y)\\ &- hg((\nabla_\xi A)Y, Y) + g((\nabla_\xi A^2)Y, Y)\\ =&0,\end{aligned}$$

where in the second equality we have used the following formulas

$$\begin{aligned}\mathrm{Ric}(\phi AY) =&(2n + 1)\phi AY + hA\phi AY - A^2\phi AY\\ =&\mu\{(2n + 1) + \lambda h - \lambda^2\}\phi Y,\end{aligned}$$

$$\begin{aligned}g((\nabla_\xi A)Y, Y) =&g(\nabla_\xi(AY) - A\nabla_\xi Y, Y)\\ =&g(\mu\nabla_\xi Y - A\nabla_\xi Y, Y) = 0.\end{aligned}$$

and

$$\begin{aligned}g((\nabla_\xi A^2)Y, Y) =&g(\nabla_\xi(A^2Y) - A^2\nabla_\xi Y, Y)\\ =&g(\mu^2\nabla_\xi Y - A^2\nabla_\xi Y, Y)\\ =&\mu^2 g(\nabla_\xi Y, Y) - \mu^2 g(\nabla_\xi Y, Y) = 0.\end{aligned}$$

From this, together with $1 + \alpha\mu \neq 0$, we can assert that

$$g(\xi, Df) = 0. \tag{6.8}$$

Consequently, from (6.6), (6.7) and (6.8) it follows that the gradient vector field Df is identically vanishing on the tangent space $T_xM = T_\lambda \oplus T_\mu \oplus T_\alpha$, $x \in M$. Then $Df = 0$ in (6.1) means that M is pseudo-Einstein $\mathrm{Ric}(X) = (\Omega + \theta\gamma)X - \psi\eta(X)\xi$, $x \in M$. Since $\lambda + \mu = -\frac{4}{\alpha}$, we get the following

Lemma 6.1 *Let M be a contact real hypersurface in $\mathbb{C}P^n$, $n \geq 3$. If M satisfies gradient pseudo Ricci-Bourguignon soliton, then M is pseudo-Einstein and*

$$h = \lambda + \mu.$$

Proof. By the above arguments, we get that M is pseudo-Einstein. Then Theorem C gives $\cot^2(2r) = n-2$. From this it follows that

$$\begin{aligned} h+\frac{4}{\alpha} =& 2\cot(2r) - 2(n-1)\tan(2r) + 2\tan(2r) \\ =& 2\cot(2r) - 2(n-2)\tan(2r) \\ =& 2\frac{\cot^2(2r)-(n-2)}{\cot(2r)} \\ =& 0. \end{aligned}$$

From this, together with $\lambda+\mu = -\frac{4}{\alpha}$, it becomes $h = \lambda+\mu$. This completes the proof of our lemma. □

Then if we put $\mathrm{Ric}(X) = aX + b\eta(X)\xi$, then the constants a and b can be calculated as follows:

Proposition 6.2 *Let M be a contact real hypersurface in $\mathbb{C}P^n$, $n{\geq}3$. If M satisfies gradient pseudo Ricci-Bourguignon soliton, then M is pseudo-Einstein and the soliton constants are given by*

$$a = \Omega + \theta\gamma = 2n, \quad \textit{and} \quad b = -\psi = -2(n-1).$$

Proof. Since M is pseudo-Einstein, we may put $\mathrm{Ric}(X) = aX + b\eta(X)\xi$. Then from (3.2) it follows that

$$\begin{aligned} a+b = g(\mathrm{Ric}(\xi),\xi) &= 2(n-1) + h\alpha - \alpha^2 \\ &= 2(n-1) + \{2\cot(2r) - 2(n-1)\tan(2r)\}(2\cot(2r)) - (2\cot(2r))^2 \\ &= 2(n-1) - 4(n-1) = -2(n-1). \end{aligned}$$

Next for any vector field $X{\in}T_\lambda$, (3.2) implies the following

$$\mathrm{Ric}(X) = (2n+1)X + hAX - A^2X = \{(2n+1) + h\lambda - \lambda^2\}X.$$

Then by using Lemma 6.1 it follows that

$$\begin{aligned} a = g(\mathrm{Ric}(X),X) &= (2n+1) + h\lambda - \lambda^2 \\ &= (2n+1) + (\lambda+\mu)\lambda - \lambda^2 \\ &= (2n+1) + \lambda\mu = (2n+1) - 1 = 2n. \end{aligned}$$

Then the other constant $b = -2(n-1) - 2n = -2(2n-1)$. So from the pseudo-Einstein property

$$\mathrm{Ric}(X) = aX + b\eta(X)\xi = (\Omega+\theta\gamma)X - \psi\eta(X)\xi$$

we get the above assertion. □

Then summing up the above discussion, together with Lemma 6.1 and Proposition 6.2, we give a complete proof of our Main Theorem 3 in the introduction.

Remark 6.3. The metric g of a Riemannian manifold M of dimension $n \geq 3$ is said to be a *gradient η-Einstein soliton* [5] if there exists a smooth function f on M such that

$$\mathrm{Ric}(X) + \nabla^2 f + \psi\eta(X)\xi = (\Omega + \frac{1}{2}\gamma)X,$$

where γ denotes the scalar curvature of M and Ω and ψ are η-Einstein gradient soliton constants on M. Here $\nabla^2 f$ denotes the Hessian operator of g and f the Einstein potential function of the η-gradient Einstein soliton. So this soliton is an example of gradient pseudo Ricci-Bourguignon soliton.

Remark 6.4. Let M be a contact real hypersurface in $\mathbb{C}P^n$, $n \geq 3$, with gradient η-Einstein soliton. Then Lemma 6.1 implies that M is pseudo-Einstein. So by Theorem C it satisfies $\cot^2(2r) = n - 2$. From this and Proposition 3.5 implies that the scalar curvature is given by

$$\begin{aligned}\gamma &= 4(n-1)^2 + 4(n-1)(n-2)\tan^2(2r) \\ &= 4n(n-1).\end{aligned}$$

Moreover, by the definition of gradient η-Einstein soliton, the soliton constant θ in Remark 6.3 is given by $\theta = \frac{1}{2}$. Then by Proposition 6.2, it gives

$$\Omega = 2n + \frac{1}{2}\gamma = 2n + 2n(n-1) = 2n^2 > 0.$$

This means that the gradient η-Einstein soliton becomes shrinking.

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