KYUNGPOOK Math. J. 64(2024), 423-433
https://doi.org/10.5666/KMJ.2024.64.3.423
pISSN 1225-6951 eISSN 0454-8124

On Some Sums at the a-points of Derivatives of the Riemann Zeta-Function

Kamel Mazhouda*
University of Sousse, Higher Institute of Applied Sciences and Technology, 4003 Sousse, Tunisia, and, Univ. Polytechnique Hauts-De-France, INSA Hauts-De-France, FR CNRS 2037, CERAMATHS, F-59313 Valenciennes, France
e-mail : kamel.mazhouda@fsm.rnu.tn

Tomokazu Onozuka
Multiple Zeta Research Center, Kyushu University, 744, Motooka, Nishi-ku, Fukuoka 819-0395, Japan
e-mail : t-onozuka@math.kyushu-u.ac.jp

Abstract. Let $\zeta^{(k)}(s)$ be the k-th derivative of the Riemann zeta function and a be a complex number. The solutions of $\zeta^{(k)}(s)=a$ are called a-points. In this paper, we give an asymptotic formula for the sum

$$\sum_{1<\gamma_a^{(k)}<T} \zeta^{(j)}\left(\rho_a^{(k)}\right) \quad \text{as} \quad T \longrightarrow \infty,$$

where j and k are non-negative integers and $\rho_a^{(k)}$ denotes an a-point of the k-th derivative $\zeta^{(k)}(s)$ and $\gamma_a^{(k)}=\mathrm{Im}(\rho_a^{(k)})$.

1. Introduction

Let $\zeta(s)$ be the Riemann zeta function, $s=\sigma+it$ be a complex variable and a be a complex number. The zeros of $\zeta(s)-a$, which are denoted by $\rho_a=\beta_a+i\gamma_a$, are called a-points of $\zeta(s)$. First, we note that there is an a-point near any trivial zero $s=-2n$ for sufficiently large n and apart from these a-points, there are only finitely many other a-points in the half-plane $\sigma \le 0$ (see [4]). The a-points with $\beta_a \le 0$ are said to be trivial. All other a-points lie in a strip $0<\sigma<A$, where A depends on a, and are called the nontrivial a-points. These points satisfy a Riemann-von Mangoldt type formula, namely

$$N_a(T)=\sum_{0<\gamma_a<T;\ \beta_a>0} 1=\frac{T}{2\pi}\log\frac{T}{2\pi c_a e}+O(\log T), \tag{1.1}$$

* Corresponding Author.
Received September 30, 2023; revised June 3, 2024; accepted June 12, 2023.
2020 Mathematics Subject Classification: 11M06,11M26, 11M36.
Key words and phrases: Riemann zeta function, a-points, value distribution.

where

$$c_a = \begin{cases} 2 & \text{if} \quad a = 1, \\ 1 & \text{otherwise.} \end{cases}$$

This is the well-known Riemann-von Mangoldt formula when $a = 0$, which Bohr, Landau and Littlewood [1] generalized for all $a \in \mathbb{C}$. We observe that these asymptotics are essentially independent of a, that is,

$$N_a(T) \sim N(T), \quad T \to \infty,$$

where $N(T) = N_0(T)$ denotes the number of nontrivial zeros $\rho = \beta + i\gamma$ satisfying $0 < \gamma < T$. Levinson [8] showed that all but $O(N(T)/\log\log T)$ of the a-points with imaginary part in $T < t < 2T$ lie in $\left|\mathrm{Re}(s) - \frac{1}{2}\right| < \frac{(\log\log T)^2}{\log T}$. So the a-points are clustered around the line $\mathrm{Re}(s) = \frac{1}{2}$.

In [2], Conrey and Ghosh suggested the problem of estimating the average $\sum_{0<\gamma_0^{(k)}\le T} \zeta^{(j)}(\rho_0^{(k)})$ for non-negative integers j and k, where $\rho_0^{(k)} = \beta_0^{(k)} + i\gamma_0^{(k)}$ denote a zero of the k-th derivative $\zeta^{(k)}(s)$. One of the first result on this topic was given by Fujii [3]. He gave an asymptotic formula of the sum $\sum_{0<\gamma_0\le T} \zeta'(\rho_0) X^{\rho_0}$ for a rational number $X > 0$. The $k = 0$ case was treated by Kaptan, Karabulut and Yildirim in [5]. Garunkštis and Steuding in [4] gave a generalization of Fujii's asymptotic formula with $X = 1$ that if $T \longrightarrow \infty$, we have

$$\begin{aligned} \sum_{\substack{\rho_a;\ \text{nontrivial} \\ 0<\gamma_a\le T}} \zeta'(\rho_a) &= \left(\frac{1}{2} - a\right)\frac{T}{2\pi}\log^2\left(\frac{T}{2\pi}\right) + (C_0 - 1 + 2a)\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) \\ &\quad + (1 - C_0 - C_0^2 + 3C_1 - 2a)\frac{T}{2\pi} + E(T), \end{aligned} \tag{1.2}$$

In fact, in the proof of (1.2) Garunkštis and Steuding used the following formula established and corrected by Fujii in [3]

$$\begin{aligned} \sum_{0<\gamma\le T} \zeta'(\rho) &= \frac{T}{4\pi}\log^2\left(\frac{T}{2\pi}\right) + (C_0 - 1)\frac{T}{2\pi}\log\left(\frac{T}{2\pi}\right) \\ &\quad + (1 - C_0 - C_0^2 + 3C_1)\frac{T}{2\pi} + O\left(Te^{-C\sqrt{\log T}}\right), \end{aligned}$$

where the summation is over all nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$.

where C_n are the Stieltjes constants and

$$E(T) = \begin{cases} O\left(T^{\frac{1}{2}+\epsilon}\right) & \text{under the Riemann hypothesis,} \\ O\left(Te^{-C\sqrt{\log T}}\right) & \text{unconditionally,} \end{cases} \tag{1.3}$$

for any $\epsilon > 0$ and some constant C. Using formula (1.2), they concluded that the main term describes how the values $\zeta(1/2+it)$ approach the value a in the complex plane on average. In [4], Garunkštis and Steuding also proved that the set $\{(\zeta(1/2+it), \zeta'(1/2+it)) \mid t \in \mathbb{R}\}$ is not dense in $\mathbb{C}^2$. This result tells us a value distribution on the critical line. Note that Voronin [14] proved that the set $\{(\zeta(\sigma+it), \zeta'(\sigma+it), \ldots, \zeta^{(n-1)}(\sigma+it)) \mid t \in \mathbb{R}\}$ is dense in $\mathbb{C}^n$ for all positive integers n and every fixed $\sigma \in (1/2, 1)$.

Recently, Karabulut and Yildirim in [7] studied Conrey and Ghosh's average and proved that for fixed $j, k \in \mathbb{Z}_{\geq 0}$ and large T, we have

$$\sum_{0<\gamma_0^{(k)}\leq T} \zeta^{(j)}\left(\rho_0^{(k)}\right) = (-1)^j \left(\delta_{j,0} + B(j,k)\right) \frac{T}{2\pi} (\log T)^{j+1} + O_{j,k}\left(T \log^j T\right), \tag{1.4}$$

where $\delta_{j,0} = 1$ if $j = 0$ and 0 otherwise,

$$B(j,k) := -\frac{k+1}{j+1} - j! \sum_{r=1}^{k} \frac{e^{-z_r}}{z_r^{j+1}} P_j(z_r) + j! \sum_{r=1}^{k} \frac{1}{z_r^{j+1}}, \tag{1.5}$$

the sum over r being void in the case $k = 0$ and z_r $(r = 1, ..., k)$ being the zeros of $P_k(z) = \sum_{j=0}^{k} z^j/j!$.

Let $\rho_a^{(k)} = \beta_a^{(k)} + i\gamma_a^{(k)}$ denote an a-point of $\zeta^{(k)}(s)$. Similar to the a-points of $\zeta(s)$, there is an a-point of $\zeta^{(k)}(s)$ near any trivial zero $s = -2n$ for sufficiently large n and apart from these a-points, there are only finitely many other a-points in the half-plane $\sigma \leq C$ for any $C < 0$ (see Lemma 2.3)

In this paper, we give an asymptotic formula for the sum

$$\sum_{1<\gamma_a^{(k)}<T} \zeta^{(j)}\left(\rho_a^{(k)}\right).$$

The basic idea of the proof is to interpret the sum of $\zeta^{(j)}(\rho_a^{(k)})$ as a sum of residues. By Cauchy's theorem, we have

$$\sum_{1<\gamma_a^{(k)}<T} f(\rho_a^{(k)}) = \frac{1}{2\pi i} \int_R f(s) \frac{\zeta^{(k+1)}(s)}{\zeta^{(k)}(s) - a} ds,$$

The Stieltjes constants are given by the Laurent series expansion of $\zeta(s)$ at $s = 1$,

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} (-1)^n \frac{C_n}{n!} (s-1)^n.$$

For example, $C_0 = \lim_{N\to+\infty}\left(\sum_{n=1}^{N} \frac{1}{n} - \log N\right)$.

where $f(s)$ is $\zeta^{(j)}(s)$ and R is the rectangle joining the points $b+i,\ b+iT,\ -b'+iT$ and $-b'+i$ with some constants $b, b' > 0$.

Our main result is stated in the following.

Theorem 1.1. Let j and k be non-negative integers and a be a complex number. For sufficiently large T, we have

$$\begin{aligned}\sum_{1<\gamma_a^{(k)}<T} \zeta^{(j)}\left(\rho_a^{(k)}\right) &= (-1)^j\left(\delta_{j,0} + a\delta_{k,0} + B(j,k)\right)\frac{T}{2\pi}(\log T)^{j+1} \\ &\quad + O_{j,k}\left(T(\log T)^j\right).\end{aligned} \tag{1.6}$$

Here and in the sequel, the implicit constant in the error terms may depend on a.

Remark. By Theorem 1.1, we can deduce the average value of $\zeta^{(j)}(\rho_a^{(k)})$, over the a-points $\rho_a^{(k)}$ of $\zeta^{(k)}(s)$ with $1 < \operatorname{Im}(\rho_a^{(k)}) < T$, i.e.,

$$\frac{1}{N_k(a,T)}\sum_{1<\gamma_a^{(k)}<T} \zeta^{(j)}(\rho_a^{(k)}),$$

where $N_k(a,T)$ is the number of terms in the above sum. Because of the asymptotic formula $N_k(a,T) \sim (T/2\pi)\log T$ (see [9]), the average is

$$(-1)^j\left(\delta_{j,0} + a\delta_{k,0} + B(j,k)\right)(\log T)^j.$$

So this tells us about the size of $\zeta^{(j)}(s)$ at certain points (namely the a-points of $\zeta^{(k)}(s)$).

2. Preliminary Lemmas and Equations

In this section, we prepare some lemmas and equations to prove Theorem 1.1. Let k be a positive integer. We start with some results (see [9]) about the a-points of k-th derivative of the Riemann zeta function (see also [13]). For $c > 1$, the following equation

$$\zeta^{(k)}(1-s) = (-1)^k 2(2\pi)^{-s}\Gamma(s)(\log s)^k\cos(\pi s/2)\zeta(s)\left(1 + O\left(\frac{1}{|\log s|}\right)\right) \tag{2.1}$$

holds in the region $\{s \in C;\ \sigma > c,\ |t| \geq 1\}$. Equation (2.1) (see [9, Theorem 2.2]) yields an a-point free region for $\zeta^{(k)}(s)$, that is, there exist real numbers $E_{1k}(a) \leq 0$ and $E_{2k}(a) \geq 1$ such that $\zeta^{(k)}(s) - a \neq 0$ for $\{s \in C;\ \sigma \leq E_{1k}(a),\ |t| \geq 1\}$ and $\{s \in C;\ \sigma \geq E_{2k}(a)\}$. Moreover, in [9, Theorem 2.3] the second author proved that there exists $N = N_k(a) \in N$ such that $\zeta^{(k)}(s) = a$ has just one root in $\mathcal{C}_n := \{s \in C;\ -2n-1 < \sigma < -2n+1,\ |t| < 1\}$ for each integer $n \geq N$.

Let us recall that $\zeta^{(k)}(\overline{s}) = \overline{\zeta^{(k)}(s)}$, then if $a \notin R$, there exist infinitely many a-points, or infinitely many $\overline{a}$-points, of $\zeta^{(k)}(s)$ in $\{s \in C; 0 < t < 1\}$.

In the following lemma, we prove that equation (2.1) yields another a-point free region for $\zeta^{(k)}(s)$.

Lemma 2.2. *Let k be a non-negative integer. For any real number $C < 0$, there exists a constant $T_{k,C} > 0$ such that there are no a-points of $\zeta^{(k)}(s)$ in $\{s \in C;\ \sigma \leq C,\ |t| \geq T_{k,C}\}$.*

Proof. By Stirling's formula, for $|t| > 1$ and fixed $\sigma \geq 1 - C$, we have

$$|2(2\pi)^{-s}\Gamma(s)(\log s)^k \cos(\pi s/2)| \asymp |t|^{\sigma-1/2}|\log(1+|t|)|^k.$$

Moreover, one has

$$|\zeta(s)| \asymp 1$$

for fixed $\sigma \geq 1 - C$. Using the last estimates and (2.1), we get

$$|\zeta^{(k)}(1-s)| \asymp |t|^{\sigma-1/2}|\log(1+|t|)|^k$$

for $|t| > 1$ and fixed $\sigma \geq 1 - C$. Hence, there exists a constant $T_{k,C} > 0$ such that $|\zeta^{(k)}(s)| > |a|$ holds for all $E_{1k}(a) \leq \sigma \leq C$ and $|t| \geq T_{k,C}$. □

From Lemma 2.2, we deduce easily the following lemma.

Lemma 2.3. *Let k be a non-negative integer. For any real number $C < 0$, there are finitely many a-points of $\zeta^{(k)}(s)$ in*

$$\{s \in C;\ \sigma \leq C\} \setminus \left(\bigcup_{n \geq N_k(a)} \mathcal{C}_n \right).$$

For a positive integer k and a complex number a, we have (see [9, Theorem 1.1])

$$N_k(a,T) := \sum_{1<\gamma_a^{(k)}<T} 1 = \begin{cases} \dfrac{T}{2\pi}\log\dfrac{T}{2\pi} - \dfrac{T}{2\pi} + O(\log T) & \text{if} \quad a \neq 0 \\[2ex] \dfrac{T}{2\pi}\log\dfrac{T}{4\pi} - \dfrac{T}{2\pi} + O(\log T) & \text{if} \quad a = 0 \end{cases} \tag{2.2}$$

and for sufficiently large T, we also have

$$N_k(a, T+1) - N_k(a,T) \ll \log T. \tag{2.3}$$

Now, using [9, Lemma 2.6], for any constants σ_1, σ_2 and $s \in C$ with $\sigma_1 < \sigma < \sigma_2$ and large t, we have

$$\frac{\zeta^{(k+1)}(s)}{\zeta^{(k)}(s) - a} = \sum_{|\gamma_a^{(k)} - t|<1} \frac{1}{s - \rho_a^{(k)}} + O(\log t). \tag{2.4}$$

Lemma 4.1 in [9] states that, for a positive integer k and a sufficiently large $\sigma \geq E_{2k}$, we have

$$\frac{\zeta^{(k+1)}(s)}{\zeta^{(k)}(s)-a} \tag{2.5}$$

$$= \begin{cases} \displaystyle\sum_{l\geq 0;\ n_0,\dots,n_l\geq 2} \frac{(-1)^{k(l+1)}}{a^{l+1}}(\log n_0)^{k+1}(\log n_1 \dots \log n_l)^k \frac{1}{n_0^s \dots n_l^s} & (a \neq 0), \\ \displaystyle\sum_{l\geq 0;\ n_0\geq 2;\ n_1,\dots,n_l\geq 3} \left(\frac{-1}{(\log 2)^k}\right)^{l+1} (\log n_0)^{k+1}(\log n_1 \dots \log n_l)^k \frac{2^{(l+1)s}}{n_0^s \dots n_l^s} & (a = 0). \end{cases}$$

The right-hand side is complicated, so here we abbreviate it to $\sum_{d\geq 1}\alpha(d)d^{-s}$. When $k=0$, $\zeta'(s)/(\zeta(s)-a)$ also has a convergent Dirichlet series expansion (The case $k=0$ and $a\neq 1$ is mentioned in [11, (29)]). Note that $\alpha(1)=0$ holds if $k>0$ and $a\neq 0$.

We finish this section by the following estimate

$$\zeta^{(k)}(s) \ll |t|^{\mu(\sigma)+\epsilon}, \tag{2.6}$$

which holds as $|t|\to\infty$ for any small $\epsilon>0$, where the function $\mu(\sigma)$ satisfies the inequalities

$$\mu(\sigma) \leq \begin{cases} 0 & (\sigma \geq 1) \\ \frac{1-\sigma}{2} & (0<\sigma<1) \\ \frac{1}{2}-\sigma & (\sigma \leq 0). \end{cases}$$

3. Proof of Theorem 1.1

Let a be a complex number. We write $s=\sigma+it$, $\rho_a^{(k)}=\beta_a^{(k)}+i\gamma_a^{(k)}$ with real numbers $\sigma, t, \beta_a^{(k)}$ and $\gamma_a^{(k)}$. The case $a=0$ was already proved by Karabulut and Yildirim in [7], so here we assume $a\neq 0$. By the residue theorem, for a sufficiently large constant B and constant $b\in(1,9/8)$, we have

$$\sum_{\substack{1<\gamma_a^{(k)}<T \\ 1-b<\beta_a^{(k)}<B}} \zeta^{(j)}\left(\rho_a^{(k)}\right) = \frac{1}{2\pi i}\int_{\mathbf{R}} \zeta^{(j)}(s)\frac{\zeta^{(k+1)}(s)}{\zeta^{(k)}(s)-a}ds, \tag{3.1}$$

where the integration is taken over a rectangular contour in counterclockwise direction denoted by $\mathbf{R}$ with vertices $1-b+i$, $B+i$, $B+iT$, $1-b+iT$. Since there are finitely many a-points in $\{s\in\mathbb{C};\ \mathrm{Re}(s)\leq 1-b,\ \mathrm{Im}(s)\geq 1\}$, we have

$$\sum_{1<\gamma_a^{(k)}<T} \zeta^{(j)}\left(\rho_a^{(k)}\right) = \frac{1}{2\pi i}\int_{\mathbf{R}} \zeta^{(j)}(s)\frac{\zeta^{(k+1)}(s)}{\zeta^{(k)}(s)-a}ds + O(1).$$

Hence, we have

(3.2)

$$\begin{aligned}
&\sum_{1<\gamma_a^{(k)}<T} \zeta^{(j)}\left(\rho_a^{(k)}\right) \\
&= \frac{1}{2\pi i}\left\{\int_{1-b+i}^{B+i} + \int_{B+i}^{B+iT} + \int_{B+iT}^{1-b+iT} + \int_{1-b+iT}^{1-b+i}\right\} \zeta^{(j)}(s)\frac{\zeta^{(k+1)}(s)}{\zeta^{(k)}(s)-a}ds + O(1) \\
&=: \frac{1}{2\pi i}(I_1+I_2+I_3+I_4)+O(1).
\end{aligned}$$

The integral I_1 is independent of T, so we have $I_1 = O(1)$. Next, we consider I_2. Using (2.5), we get

$$\begin{aligned}
I_2 &= \int_{B+i}^{B+iT} \sum_{n=1}^{\infty} \frac{(-\log n)^j}{n^s} \sum_{d\geq 1} \frac{\alpha(d)}{d^s} ds \\
&= \sum_{n=1}^{\infty} (-\log n)^j \sum_{d\geq 1} \alpha(d) \int_{B+i}^{B+iT} (nd)^{-s} ds,
\end{aligned}$$

where we define $(-\log n)^j = 1$ when $n = 1$ and $j = 0$. The integral factor can be calculated as

$$\int_{B+i}^{B+iT} (nd)^{-s} ds = \begin{cases} iT - i & (nd = 1) \\ O\left((nd)^{-B}\right) & (nd > 1). \end{cases}$$

By these estimates, we obtain

$$I_2 = O(T).$$

From (2.4), we have

$$I_3 = \sum_{|\gamma_a^{(k)}-T|<1} \int_{B+iT}^{1-b+iT} \frac{\zeta^{(j)}(s)}{s-\rho_a^{(k)}}\ ds + O\left(\int_{B+iT}^{1-b+iT} (\log T)\zeta^{(j)}(s) ds\right).$$

Now, we change the path of integration. If $\gamma_a^{(k)} < T$, we change the path to the upper semicircle with center $\rho_a^{(k)}$ and radius 1. If $\gamma_a^{(k)} > T$, we change the path to the lower semicircle with center $\rho_a^{(k)}$ and radius 1. Then, we have

$$\frac{1}{s-\rho_a^{(k)}} \ll 1$$

on the new path. This estimate and the bound (2.6) yields

$$I_3 = O\left(T^{b-\frac{1}{2}+\epsilon} \sum_{|\gamma_a^{(k)}-T|<1} 1\right) + O\left(T^{b-\frac{1}{2}+\epsilon}\log T\right).$$

In view of the number of a-points (2.3), we obtain

$$I_3 = O\left(T^{b-\frac{1}{2}+\epsilon}\log T\right).$$

This leads to $I_3 \ll T$ since $1 < b < 9/8$.

Finally, we estimate I_4. By (2.1) and Stirling's formula, for fixed $1 < b < 9/8$ and large $|t| > 2$, we have

$$\left|\zeta^{(k)}(1-b+it)\right| \asymp |t|^{b-1/2}\left|\log|t|\right|^k. \tag{3.3}$$

Therefore, there exists a constant A such that

$$\left|\frac{a}{\zeta^{(k)}(1-b+it)}\right| < 1$$

holds for any $|t| \geq A$. We divide the path of the integral into two parts

$$I_4 = \left(\int_{1-b+iT}^{1-b+iA} + \int_{1-b+iA}^{1-b+i}\right)\zeta^{(j)}(s)\frac{\zeta^{(k+1)}(s)}{\zeta^{(k)}(s)-a}ds.$$

Then, the second term is $O(1)$ since it is independent of T. Using

$$\frac{1}{\zeta^{(k)}(s)-a} = \sum_{n=0}^{M}\frac{a^n}{(\zeta^{(k)}(s))^{n+1}} + O\left(\frac{1}{\left|(\zeta^{(k)}(s))^{M+1}\right|}\right),$$

we get

$$\begin{aligned} I_4 \quad &= \quad -\sum_{n=0}^{M} a^n \int_{1-b+iA}^{1-b+iT}\frac{\zeta^{(j)}(s)\zeta^{(k+1)}(s)}{(\zeta^{(k)}(s))^{n+1}}ds \\ &\quad +O\left(\int_{1-b+iA}^{1-b+iT}\left|\frac{\zeta^{(j)}(s)\zeta^{(k+1)}(s)}{(\zeta^{(k)}(s))^{M+1}}\right|ds\right) + O(1). \end{aligned}$$

By (3.3), the integrand can be estimated as

$$\frac{\zeta^{(j)}(s)\zeta^{(k+1)}(s)}{(\zeta^{(k)}(s))^{n+1}} \asymp |t|^{(b-1/2)(1-n)}(\log t)^{-kn+j+1}. \tag{3.4}$$

Hence, each integral can be calculated as

$$\int_{1-b+iA}^{1-b+iT}\frac{\zeta^{(j)}(s)\zeta^{(k+1)}(s)}{(\zeta^{(k)}(s))^{n+1}}ds \ll T^{(b-1/2)(1-n)+1+\varepsilon}$$

for any small $\varepsilon > 0$. It follows from the last estimate that the sum for $n \geq 2$ is bounded as

$$\sum_{n=2}^{M} a^n \int_{1-b+iA}^{1-b+iT} \frac{\zeta^{(j)}(s)\zeta^{(k+1)}(s)}{(\zeta^{(k)}(s))^{n+1}} ds \ll T^{-(b-1/2)+1+\varepsilon} \ll T^{1/2}.$$

Similarly, one has

$$\int_{1-b+iA}^{1-b+iT} \left| \frac{\zeta^{(j)}(s)\zeta^{(k+1)}(s)}{(\zeta^{(k)}(s))^{M+1}} \right| ds \ll T^{1/2}.$$

Therefore, we get

$$\begin{aligned} I_4 &= -\int_{1-b+iA}^{1-b+iT} \frac{\zeta^{(j)}(s)\zeta^{(k+1)}(s)}{\zeta^{(k)}(s)} ds - a \int_{1-b+iA}^{1-b+iT} \frac{\zeta^{(j)}(s)\zeta^{(k+1)}(s)}{(\zeta^{(k)}(s))^2} ds + O\left(T^{1/2}\right) \\ &=: -K_1 - aK_2 + O\left(T^{1/2}\right). \end{aligned}$$

Karabulut and Yildirim [7] already studied K_1, and they gave an estimate

$$K_1 = -2\pi i \left\{ \delta_{j,0} \frac{T}{2\pi} \log \frac{T}{2\pi} + (-1)^j B(j,k) \frac{T}{2\pi} \left(\log \frac{T}{2\pi} \right)^{j+1} + O\left(T(\log T)^j\right) \right\}.$$

It remains to evaluate K_2. From (3.4), for $k \geq 1$, we have

$$K_2 \ll \int_{1-b+iA}^{1-b+iT} |\log t|^j \, |ds| \ll T(\log T)^j,$$

so hereafter we consider the case $k = 0$. In this case, we use Conrey and Ghosh's result (see [2, (16)])

$$(-1)^m \zeta^{(m)}(s) = \chi(s)(1 + O(1/|t|)) \left(\ell - \left(\frac{d}{ds} \right) \right)^m \zeta(1-s) \tag{3.5}$$

for $\sigma \leq 1/2$ and $|t| \gg 1$, where $\chi(s) := 2(2\pi)^{s-1}\Gamma(1-s)\sin(\pi s/2)$ and $\ell := \log(|t|/2\pi)$. Substituting this equation into the integrand of K_2 with $k = 0$, we have

$$(-1)^{j+1} \frac{(\ell - (s/ds))^j \, \zeta(1-s) \, (\ell - (d/ds)) \, \zeta(1-s)}{(\zeta(1-s))^2} \left(1 + O\left(\frac{1}{|t|} \right) \right).$$

Since the path of the integral satisfies $\mathrm{Re}(s) = 1 - b$ with $1 < b < 9/8$, $\zeta(1-s) \asymp 1$ and $\zeta^{(j)}(1-s) \ll 1$ hold for any non-negative integer j. Therefore, we have

$$\begin{aligned} K_2 &= \int_{1-b+iA}^{1-b+iT} \left((-\ell)^{j+1} + O\left((\log|t|)^j \right) \right) ds \\ &= (-1)^{j+1} iT(\log T)^{j+1} + O\left(T(\log T)^j\right). \end{aligned}$$

Combining K_1 and K_2, we obtain

$$I_4 = (-1)^j i \left(\delta_{j,0} + a\delta_{k,0} + B(j,k)\right) T \left(\log T\right)^{j+1} + O\left(T(\log T)^j\right).$$

From estimates of I_1, I_2, I_3 and I_4, we finally obtain Theorem 1.1.

4. Concluding Remarks

In this section we present some problems that will be considered in a sequel to this note.

- Let $L(s,\chi)$ be the Dirichlet L-function associated with a primitive character $\chi \mod q$. We believe that we can extend Theorem 1.1 to higher derivatives of $L(s,\chi)$. To do so, we first extend Karabulut and Yildirim's result given by equation (1.4) (see [7]) using the same argument as in [6].
- The a-points of an L-function $L(s)$ are the roots of the equation $L(s) = a$. We refer to Steuding book [12, chapter 7] and Selberg paper [10] for some results about a-points of L-functions from the Selberg class. Therefore, it is an interesting problem to extend Theorem 1.1 to other classes of L-functions (the Selberg class with some further conditions) and its higher derivatives.

Acknowledgement. The authors thank the anonymous referee for his careful reading of the manuscript and constructive suggestions which improved the paper.

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