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A Note on Noetherian Polynomial Modules

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Abstract. Let R be a commutative ring and let M be an R-module. In this note, we give a brief proof of the Hilbert basis theorem for Noetherian modules. This states that if R contains the identity and M is a Noetherian unitary R-module, then $M[X]$ is a Noetherian $R[X]$-module. We also show that if $M[X]$ is a Noetherian $R[X]$-module, then M is a Noetherian R-module and there exists an element $e \in R$ such that $em = m$ for all $m \in M$. Finally, we prove that if $M[X]$ is a Noetherian $R[X]$-module and $\text{ann}_R(M) = (0)$, then R has the identity and M is a unitary R-module.

1. Introduction

Let R be a commutative ring and let $R[X]$ be the polynomial ring over R. For an R-module M, let $M[X]$ be the set of polynomials in an indeterminate X with coefficients in M. Then $M[X]$ is an $R[X]$-module under the usual addition and the scalar multiplication as follows: For $f = \sum_{i=0}^{m} a_iX^i, g = \sum_{i=0}^{n} b_iX^i \in M[X]$ with $m \geq n$ and $h = \sum_{i=0}^{\ell} r_iX^i \in R[X]$,

$$f + g := \sum_{i=0}^{n}(a_i + b_i)X^i + \sum_{i=n+1}^{m} a_iX^i$$

and

$$hf := \sum_{i=0}^{\ell+m} c_iX^i,$$

where $c_i = \sum_{k=0}^{i} r_ka_{i-k}$ for all $i = 0, \ldots, \ell + m$ (cf. [2, Chapter 2, Exercise 6]). We call $M[X]$ the *polynomial* $R[X]$*-module*.

Let R be a commutative ring and let M be an R-module. Recall that M is a *Noetherian module* if it satisfies the ascending chain condition on R-submodules of M (or equivalently, every R-submodule of M is finitely generated); and R is a *Noetherian ring* if R is a Noetherian R-module. It is well known as Hilbert basis theorem for Noetherian modules that if R is a commutative ring with identity and M is a Noetherian unitary R-module, then $M[X]$ is a Noetherian $R[X]$-module [2, Chapter 7, Exercise 10]. When $M = R$, it recovers the well-known Hilbert basis

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theorem which states that if R is a Noetherian ring, then $R[X]$ is also a Noetherian ring [2, Chapter 7, Theorem 7.5] (or [4, Theorem 69]).

In this note, we study Hilbert basis theorem for Noetherian modules. We first give a brief proof of Hilbert basis theorem for Noetherian modules. We next consider the converse of Hilbert basis theorem for Noetherian modules. More precisely, we show that if $M[X]$ is a Noetherian $R[X]$-module, then M is a Noetherian R-module and there exists an element $e \in R$ such that $em = m$ for all $m \in M$. We also prove that if $M[X]$ is a Noetherian $R[X]$-module and $\text{ann}_R(M) = (0)$, then R contains the identity and M is a unitary R-module.

2. Main Results

We start this section with Hilbert basis theorem for Noetherian modules. While the result appears in [2, Chapter 7, Exercise 10], we insert a brief proof for the sake of reader's easy understanding.

Theorem 1. *Let R be a commutative ring with identity and let M be a unitary R-module. If M is a Noetherian R-module, then $M[X]$ is a Noetherian $R[X]$-module.*

Proof. Suppose to the contrary that $M[X]$ is not a Noetherian $R[X]$-module. Then there exists an $R[X]$-submodule N of $M[X]$ which is not finitely generated. Let f_1 be a nonzero element of least degree in N. Then $\langle f_1 \rangle \subsetneq N$. Let f_2 be an element of least degree in $N \setminus \langle f_1 \rangle$. Then $\langle f_1, f_2 \rangle \subsetneq N$. By repeating this process, for all integers $k \geq 1$, there exists an element f_{k+1} of least degree in $N \setminus \langle f_1, \dots, f_k \rangle$. For each integer $i \geq 1$, let a_i be the leading coefficient of f_i and let d_i be the degree of f_i. Then $d_i \leq d_{i+1}$ for all integers $i \geq 1$. Since M is a Noetherian R-module, the ascending chain $\langle a_1 \rangle \subseteq \langle a_1, a_2 \rangle \subseteq \cdots$ of R-submodules of M is stationary; so there exists an integer $\ell \geq 1$ such that $\langle a_1, \dots, a_\ell \rangle = \langle a_1, \dots, a_{\ell+1} \rangle$. Therefore $a_{\ell+1} \in \langle a_1, \dots, a_\ell \rangle$. Write $a_{\ell+1} = r_1 a_1 + \cdots + r_\ell a_\ell$ for some $r_1, \dots, r_\ell \in R$. Let $g = f_{\ell+1} - \big((r_1 X^{d_{\ell+1}-d_1}) f_1 + \cdots + (r_\ell X^{d_{\ell+1}-d_\ell}) f_\ell\big)$. Then $g \in N \setminus \langle f_1, \dots, f_\ell \rangle$ and $\deg(g) < \deg(f_{\ell+1})$. This is a contradiction to the choice of $f_{\ell+1}$. Thus $M[X]$ is a Noetherian $R[X]$-module. $\square$

We next consider the converse of Hilbert basis theorem for Noetherian modules. In order to study the converse of Hilbert basis theorem for Noetherian modules, we give a result which is required to prove the main theorem.

Lemma 2. *Let R be a commutative ring and let M be an R-module. If $M[X]$ is a Noetherian $R[X]$-module, then the following assertions hold.*

(1) M is a Noetherian R-module.

(2) For each $m \in M$, there exists an element $r \in R$ such that $rm = m$.

Proof. (1) Let N be an R-submodule of M. Then $N[X]$ is an $R[X]$-submodule of $M[X]$. Since $M[X]$ is a Noetherian $R[X]$-module, there exist $g_1, \dots, g_k \in N[X]$ such that $N[X] = \langle g_1, \dots, g_k \rangle$. Let $n \in N$. Then we obtain

$$n = f_1 g_1 + \cdots + f_k g_k + \ell_1 g_1 + \cdots + \ell_k g_k$$

for some $f_1, \ldots, f_k \in R[X]$ and $\ell_1, \ldots, \ell_k \in \mathbb{Z}$. For each $i = 1, \ldots, k$, write $f_i = \sum_{j=0}^{\alpha_i} r_{ij} X^j$ and $g_i = \sum_{j=0}^{\beta_i} n_{ij} X^j$. Then we have

$$n = r_{10}n_{10} + \cdots + r_{k0}n_{k0} + \ell_1 n_{10} + \cdots + \ell_k n_{k0};$$

so $N = \langle n_{10}, \ldots, n_{k0} \rangle$. Hence N is a finitely generated R-submodule of M. Thus M is a Noetherian R-module.

(2) Let $m \in M$. Then $\langle m \rangle \subseteq \langle m, mX \rangle \subseteq \langle m, mX, mX^2 \rangle \subseteq \cdots$ is an ascending chain of $R[X]$-submodules of $M[X]$. Since $M[X]$ is a Noetherian $R[X]$-module, there exists a nonnegative integer q such that $\langle m, \ldots, mX^q \rangle = \langle m, \ldots, mX^{q+1} \rangle$; so $mX^{q+1} \in \langle m, \ldots, mX^q \rangle$. Therefore we have

$$mX^{q+1} = f_0 m + \cdots + f_q(mX^q) + \ell_0 m + \cdots + \ell_q(mX^q)$$

for some $f_0, \ldots, f_q \in R[X]$ and $\ell_0, \ldots, \ell_q \in \mathbb{Z}$. For each $i = 0, \ldots, q$, write $f_i = \sum_{j=0}^{\alpha_i} r_{ij} X^j$. By comparing the coefficients of X^{q+1} in both sides, we obtain

$$\begin{aligned} m &= r_{0q+1}m + \cdots + r_{q1}m \\ &= (r_{0q+1} + \cdots + r_{q1})m. \end{aligned}$$

Note that $r_{0q+1} + \cdots + r_{q1} \in R$. Thus the proof is done. $\square$

Let R be a commutative ring and let M be an R-module. Then $\operatorname{ann}_R(M) := \{r \in R \mid rM = \{0\}\}$ is an ideal of R and is called the *annihilator* of M in R. We are now ready to give the main result in this note.

Theorem 3. *Let R be a commutative ring and let M be an R-module. If $M[X]$ is a Noetherian $R[X]$-module, then the following conditions hold.*

(1) There exists an element $e \in R$ such that $em = m$ for all $m \in M$.
(2) If $\operatorname{ann}_R(M) = (0)$, then R contains the identity and M is a unitary R-module.

Proof. (1) Suppose that $M[X]$ is a Noetherian $R[X]$-module. Then by Lemma 2(1), M is a Noetherian R-module; so $M = \langle m_1, \ldots, m_n \rangle$ for some $m_1, \ldots, m_n \in M$. By Lemma 2(2), for each $i = 1, \ldots, n$, there exists an element $r_i \in R$ such that $r_i m_i = m_i$. Let $e_1 = r_1 + r_2 - r_1 r_2$ and for each $i \in \{2, \ldots, n-1\}$, let $e_i = e_{i-1} + r_{i+1} - e_{i-1} r_{i+1}$. Then $e_1 m_i = m_i$ for all $i = 1, 2$. Suppose that there exists an index $k \in \{1, \ldots, n-2\}$ such that $e_k m_i = m_i$ for all $i \in \{1, \ldots, k+1\}$. Then we have

$$\begin{aligned} e_{k+1}m_i &= (e_k + r_{k+2} - e_k r_{k+2})m_i \\ &= e_k m_i + r_{k+2}m_i - e_k r_{k+2} m_i \\ &= m_i \end{aligned}$$

for all $i = 1, \ldots, k+1$. Also, we obtain

$$\begin{aligned} e_{k+1}m_{k+2} &= (e_k + r_{k+2} - e_k r_{k+2})m_{k+2} \\ &= e_k m_{k+2} + r_{k+2}m_{k+2} - e_k r_{k+2} m_{k+2} \\ &= m_{k+2}. \end{aligned}$$

Therefore $e_{k+1}m_i = m_i$ for all $i = 1, \dots, k+2$. Hence by the induction, $e_{n-1}m_i = m_i$ for all $i = 1, \dots, n$. Let $m \in M$. Then we have

$$m = s_1m_1 + \cdots + s_nm_n + \ell_1m_1 + \cdots + \ell_nm_n$$

for some $s_1, \dots, s_n \in R$ and $\ell_1, \dots, \ell_n \in \mathbb{Z}$. Thus we obtain

$$\begin{aligned} e_{n-1}m &= e_{n-1}(s_1m_1 + \cdots + s_nm_n + \ell_1m_1 + \cdots + \ell_nm_n) \\ &= s_1e_{n-1}m_1 + \cdots + s_ne_{n-1}m_n + \ell_1e_{n-1}m_1 + \cdots + \ell_ne_{n-1}m_n \\ &= s_1m_1 + \cdots + s_nm_n + \ell_1m_1 + \cdots + \ell_nm_n \\ &= m. \end{aligned}$$

(2) Let a be any element of R. By (1), there exists an element $e \in R$ such that $em = m$ for all $m \in M$; so $aem = am$ for all $m \in M$, which indicates that $(ae - a)m = 0$ for all $m \in M$. Therefore $(ae - a)M = \{0\}$. Since $\mathrm{ann}_R(M) = (0)$, $ae - a = 0$. Hence $ae = a$. Thus e is the identity of R, which also shows that M is a unitary R-module. □

As an immediate consequence of Theorems 1 and 3(2) and Lemma 2(1), we have

Corollary 4. *Let R be a commutative ring and let M be an R-module with* $\mathrm{ann}_R(M) = (0)$*. Then the following conditions are equivalent.*

(1) R contains the identity and M is a Noetherian unitary R-module.
(2) $M[X]$ is a Noetherian $R[X]$-module.

By applying $M = R$ to Theorem 3(1), we regain

Corollary 5. ([3, Theorem]) *Let R be a commutative ring. If $R[X]$ is a Noetherian ring, then R contains the identity.*

We next give two examples which show that the converse of Theorem 3(2) is not true in general and the annihilator condition in Theorem 3(2) is essential.

Example 6. Let $M = \{0, 2, 4\}$ be a $\mathbb{Z}_6$-submodule of $\mathbb{Z}_6$. Then $\mathbb{Z}_6$ contains the identity and M is a unitary $\mathbb{Z}_6$-module. Note that $\mathbb{Z}_6[X]$ is a Noetherian $\mathbb{Z}_6[X]$-module by Theorem 1 (or [2, Chapter 7, Exercise 10]) and $M[X]$ is a $\mathbb{Z}_6[X]$-submodule of $\mathbb{Z}_6[X]$. Hence $M[X]$ is a Noetherian $\mathbb{Z}_6[X]$-module. However, $\mathrm{ann}_{\mathbb{Z}_6}(M) = \{0, 3\}$.

Example 7. Let $R = \{0, 2, 4, 6, 8, 10\}$ be a subring of $\mathbb{Z}_{12}$ and let $M = \{0, 4, 8\}$ be an R-submodule of $\mathbb{Z}_{12}$. Then R is a commutative ring without identity and M is a nonunitary R-module with $\mathrm{ann}_R(M) = \{0, 6\}$. Also, $4m = m$ for all $m \in M$. Finally, $M[X]$ is a Noetherian $R[X]$-module. (To see this, suppose to the contrary that $M[X]$ is not a Noetherian $R[X]$-module. Then there exists an $R[X]$-submodule N of $M[X]$ which is not finitely generated. Let f_1 be a nonzero element of least degree in N. Then $\langle f_1 \rangle \subsetneq N$; so there exists an element f_2 of least degree in $N \setminus \langle f_1 \rangle$. For all $i = 1, 2$, let a_i be the leading coefficient of f_i and let d_i be the degree of f_i. Then $d_1 \leq d_2$. If $a_1 = a_2 = 4$ or $a_1 = a_2 = 8$, then we let $g = f_2 - (4X^{d_2 - d_1})f_1$. Then $g \in N \setminus \langle f_1 \rangle$ and $\deg(g) < \deg(f_2)$. This is a contradiction to the choice of

f_2. If $a_1 = 4$ and $a_2 = 8$, then we let $g = f_2 - (2X^{d_2-d_1})f_1$. Then $g \in N \setminus \langle f_1 \rangle$ and $\deg(g) < \deg(f_2)$. This is impossible because of the choice of f_2. Similarly, if $a_1 = 8$ and $a_2 = 4$, then we let $g = f_2 - (2X^{d_2-d_1})f_1$. Then $g \in N \setminus \langle f_1 \rangle$ and $\deg(g) < \deg(f_2)$. This is also absurd because of the minimality of the degree of f_2 in $N \setminus \langle f_1 \rangle$. Hence $M[X]$ is a Noetherian $R[X]$-module.)

Let R be a commutative ring and let M be an R-module. Then the *Nagata's idealization* of M in R (or the *trivial extension* of R by M) is a commutative ring

$$R(+)M := \{(r, m) \mid r \in R \text{ and } m \in M\}$$

with usual addition and multiplication defined by $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + r_2m_1)$ for all $(r_1, m_1), (r_2, m_2) \in R(+)M$. It is routine to check that $(R(+)M)[X]$ is isomorphic to $R[X](+)M[X]$ (cf. [1, Corollary 4.6(1)]). It was shown that if R contains the identity and M is a unitary R-module, then $R(+)M$ is a Noetherian ring if and only if R is a Noetherian ring and M is finitely generated [1, Theorem 4.8] (or [5, Corollary 3.9]).

Corollary 8. *Let R be a commutative ring and let M be an R-module. If $R[X](+)M[X]$ is a Noetherian ring, then R contains the identity and M is a unitary R-module.*

Proof. Suppose that $R[X](+)M[X]$ is a Noetherian ring. Since $R[X](+)M[X]$ is isomorphic to $(R(+)M)[X]$, $(R(+)M)[X]$ is a Noetherian ring. Hence by Corollary 5, $R(+)M$ contains the identity.

Let (a, n) be the identity of $R(+)M$. For each $r \in R$, $(a, n)(r, 0) = (r, 0)$; so $ar = r$. Thus a is the identity of R. Let $m \in M$. Then $(a, n)(0, m) = (0, m)$; so $am = m$. Thus M is a unitary R-module. □

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References

[1] D. D. Anderson and M. Winders, *Idealization of a module*, J. Commut. Algebra **1**(2009), 3–56.

[2] M. F. Atiyah and I. G. MacDonald, *Introduction to Commutative Algebra*, Addison-Wesley Series in Math., Westview Press, 1969.

[3] R. W. Gilmer, *If $R[X]$ is Noetherian, R contains an identity*, Amer. Math. Monthly, **74**(1967), 700.

[4] I. Kaplansky, *Commutative Rings*, Polygonal Publishing House, Washington, New Jersey, 1994.

[5] J. W. Lim and D. Y. Oh, *S-Noetherian properties on amalgamated algebras along an ideal*, J. Pure Appl. Algebra, **218**(2014), 1075–1080.