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Advances in Load-Sharing Parameter Estimation for Reliability Systems

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시스템 신뢰도 계산을 위한 로드쉐어링 모수 추정에 관한 고찰

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This paper chronicles the evolution of load-sharing parameter estimation methodologies, with a particular focus on the significant contributions made by Kim and Kvam (2004) and Park (2012). Kim and Kvam's pioneering work underscored the inherent challenges in deriving closed-form solutions for load-share parameters, which necessitated the use of sophisticated numerical optimization techniques. Park's research, on the other hand, provided groundbreaking closed-form solutions and extended the theoretical framework to accommodate more general distributions of component lifetimes. This was achieved by incorporating EM-type methods for maximum likelihood estimation, which represented a significant advancement in the field.

Unlike previous efforts, this paper zeroes in on the specific characteristics and advantages of closed-form solutions for load-share parameters within reliability systems. Much like the basic Economic Order Quantity (EOQ) model enhances the understanding of real-life inventory systems dynamics, our analysis aims to thoroughly explore the conditions under which these closed-form solutions are valid. We investigate their stability, robustness, and applicability to various types of systems. Through this comprehensive study, we aspire to provide a deep understanding of the practical implications and potential benefits of these solutions. Building on previous advancements, our research further examines the robustness of these solutions in diverse reliability contexts, aiming to shed light on their practical relevance and utility in real-world applications.

Keywords : Load-sharing parameter estimation, Numerical optimization, Closed-form solutions, EM-type methods, Maximum likelihood estimation, Reliability engineering, Complex systems

1. Introduction

Load-sharing models are crucial in reliability engineering, particularly for systems where component failure rates are interdependent. Load-sharing parameter estimation is vital for accurately assessing the reliability of complex systems where the failure of one component influences the performance and failure rates of others. This is particularly important in industries such as aerospace, automotive, and manufacturing, where system reliability is crucial for safety, performance, and cost-effectiveness. Accurate estimation of load-sharing parameters enables engineers to design more reliable systems, optimize maintenance schedules, and reduce operational costs by preventing unexpected failures. Furthermore, understanding the behavior of load-sharing parameters under different conditions aids in the development of robust reliability models that can adapt to real-world scenarios, thereby enhanc-

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ing the predictive capabilities and resilience of engineering systems.

The seminal work by Kim and Kvam [5] laid the groundwork by addressing the complexity of estimating load-sharing parameters. However, they faced limitations in deriving closed-form solutions, resulting in a reliance on numerical optimization techniques. Building on the foundation laid by Kim and Kvam [5], recent advancements have further enriched the field of load-sharing reliability models. For instance, Singh et al. [8] applied both classical and Bayesian estimation methods to k-component load-sharing parallel systems, highlighting the impact of failure dependencies among components. Gurov and Utkin [3] introduced load-share reliability models with piecewise constant loads, offering a nuanced approach that better captures real-world system variability. Park [7] utilized the Expectation-Maximization algorithm to enhance parameter estimation accuracy in load-sharing systems. Ahmed [1] further contributed by presenting a Bayesian framework for reliability estimation using progressive Type-II censoring from a two-parameter bathtub-shaped lifetime model, showcasing the benefits of Bayesian methods in reliability assessments. Additionally, Ivanova and Kochetkova [4] explored the reliability characteristics of k-out-of-n systems, providing valuable insights into how load-sharing affects system reliability.

More recently, Chen and Hao [2] proposed a novel load allocation policy for reliability empprovement of load-sharing systems. In their policy, based on components' periodically inspected degradation states, the whole system load will be allocated to surviving components according to their current and predictive degradation states. They showed this kind of policy can lead to effective and robust reliability improvement compared to the traditional equal load allocation policy.

Collectively, these contributions have significantly advanced the understanding and methodology of load-sharing parameter estimation in reliability systems. The integration of classical, Bayesian, and Expectation-Maximization techniques has provided robust frameworks for addressing the complexities of interdependent failure rates and varying load-sharing conditions. These advancements not only enhance computational efficiency but also improve the accuracy and robustness of reliability assessments.

Among these advancements, Park [6] made a particularly noteworthy contribution by proposing closed-form solutions for load-sharing parameters. He extended the methodology to accommodate more general distributions of component lifetimes, enhancing the model's applicability.

However, Park [6] did not investigate the characteristics of closed-form solutions for load-share parameters. In this study, we perform thorough investigations to provide deeper insights into the behavior, limitations, and practical applications of these solutions.

The remaining structure of this research is as follows. Chapter 2 discusses the development and application of closed-form solutions for load-sharing parameter estimation, examining methodologies proposed by various researchers and comparing their effectiveness in different reliability scenarios. Chapter 3 presents a detailed sensitivity analysis to understand how variations in load-sharing parameters impact system reliability, exploring the implications of parameter changes and providing insights into optimizing reliability through effective parameter management. Chapter 4 summarizes the key findings of the study, discussing their practical implications and contributions to the field of reliability engineering, emphasizing advancements in load-sharing parameter estimation and suggesting directions for future research.

2. Closed Form Solutions

2.1 Kim and Kvam [5] – The Numerical Solutions

Kim and Kvam [5] introduced a significant approach for reliability estimation in multi-component load-sharing systems. Their study focused on systems where the failure of one component increases the load on the remaining components, thereby affecting their failure rates.

Kim and Kvam's Approach

1. Load-Sharing Concept:

- In a multi-component system, as components fail one by one, the total load applied to the system is redistributed among the remaining surviving components. This is referred to as load-sharing.
- Kim and Kvam introduced unknown load-share parameters to describe the increased failure rates of surviving components upon each sequential failure of other components.

2. Mathematical Formulation:

- The system consists of k components with identical initial failure rates.

- Upon the first failure, the initial (nominal) failure rate θ of the surviving components changes to $r_1\ \theta$, upon the second failure to $r_2\ \theta$, and so on, until (k-1)th failure.
- The likelihood function for the i^{th} system and the likelihood based on n observed samples are presented in the following equations respectively:

$$L_i(\theta,\gamma \mid t_{i1},t_{i2},\cdots,t_{ik}) = k!\theta^k \prod_{j=1}^{k-1}\gamma_j \exp\left(-\theta\sum_{j=1}^{k}(k-j+1)\gamma_{j-1}t_{ij}\right)$$

$$L(\theta,\gamma \mid T) = (k!)^n(\theta)^{nk}\left(\prod_{j=1}^{k-1}\gamma_j\right)^n \exp\left(-\theta\sum_{i=1}^{n}\sum_{j=1}^{k}(k-j+1)\gamma_{j-1}t_{ij}\right),$$

where $\gamma_0 = 1$, $T=\left(t_{ij};1 \le i \le n, 1 \le j \le k\right)$, $\theta > 0$ and $\gamma > 0$.

3. Numerical Optimization:

Taking the logarithm of (2.1), differentiating with respect to θ, r_1, r_2, ..., r_{k-1}, denoting partial derivative with respect to θ as $L\theta = \partial \log L / \partial \theta$ and partial derivative of logL with respect to r_{j-1} as $L_{j-1} = \partial \log L / \partial r_{j-1}$ we have:

$$L_\theta = \frac{nk}{\theta} - \sum_{i=1}^{n}\sum_{j=1}^{k}(k-j+1)\gamma_{j-1}t_{ij} = 0 \quad \cdots \qquad (2.2)$$

$$L_{j-1} = \frac{n}{\gamma_{j-1}} - \theta\sum_{i=1}^{n}(k-j+1)t_{ij} = 0, \quad \text{for } j = 2,\cdots,k \qquad (2.3)$$

- Kim and Kvam could not derive closed-form solutions for the maximum likelihood estimators (MLEs) of the parameters θ, r_1, r_2, ..., r_{-1}.
- Instead, they proved the existence of a unique solution that maximizes the likelihood function and adopted a numerical optimization methodology, specifically the Gauss-Seidel method, to compute these parameter estimates.

Steps in Kim and Kvam's Numerical Method:

- Define the likelihood function for the system based on observed data.
- Prove the existence of a unique solution for the MLEs of the parameters.
- Apply the Gauss-Seidel iterative method to numerically estimate the parameters θ, r_1, r_2, ..., r_{-1}.

This approach, while effective, required significant computational effort due to the iterative nature of the numerical optimization process.

2.2 Park [6] - Closed-Form Solution

In contrast to the numerical methods employed by Kim and Kvam [5], Park [6] managed to find closed-form solutions for the estimation of load-sharing parameters. This represented a significant advancement in the field by providing a more computationally efficient methodology.

Park's Approach

1. Extension of Load-Sharing Model:

- Park extended the load-sharing model to accommodate more general distributions of component lifetimes, rather than assuming identical initial failure rates for all components.

2. Closed-form MLEs:

- Park derived closed-form solutions for the MLEs of the parameters Θ, r_1 , r_2, ..., r_{-1}.
- The process involved taking the logarithm of the likelihood function, differentiating with respect to each parameter, and setting the derivatives equal to zero to find the maximum likelihood estimates.

Steps in Park's Solution Method:

1. Log-likelihood Function:

- Identical with Kim and Kvam [5]

2. Partial Derivatives:

- Identical with Kim and Kvam [5]

3. Solving Equations:

- From Eqs 2.2 and 2.2, denote $t_{\bullet j} = \sum_{i=1}^{n} t_{ij}$ to have

$$L_\theta = \frac{nk}{\theta} - \sum_{i=1}^{n}\sum_{j=1}^{k}(k-j+1)\gamma_{j-1}t_{\bullet j} = 0 \quad \cdots \qquad (2.4)$$

$$L_{j-1} = \frac{n}{\gamma_{j-1}} - \theta\sum_{i=1}^{n}(k-j+1)t_{\bullet j} = 0, \quad \text{for } j = 2,\cdots,k \qquad (2.5)$$

- From (2.5) rj-1 can be shown to have a simple form as

$$r_{j-1} = \frac{n}{\theta(k-j+1)t_{\bullet j}}, \quad \text{for } j = 2,\cdots,k \qquad (2.6)$$

- Since r_0=1, Eq (2.4) can be rewritten as

$$L_\theta = \frac{nk}{\theta} - kt_{\bullet 1} - \sum_{j=2}^{k}(k-j+1)\gamma_{j-1}t_{\bullet j} = 0 \quad \cdots \quad (2.7)$$

- From Eq(2.6) and Eq(2.7) one can easily solve the resulting set of equations simultaneously to obtain the closed-form expressions for θ and r_j as following:

$$\hat{\theta} = \frac{n}{kt_{\bullet 1}} = \frac{n}{k\sum_{i=1}^{n} t_{i1}} \quad \cdots\cdots \quad (2.8)$$

$$\hat{r}_{j-1} = \frac{kt_{\bullet 1}}{(k-j+1)t_{\bullet j}} = \frac{k\sum_{i=1}^{n} t_{i1}}{(k-j+1)\sum_{i=1}^{n} t_{ij}}, j = 2, \cdots, k \quad (2.9)$$

- The closed-form solutions for θ and r_j are derived explicitly, allowing direct computation without iterative methods.

Advantages of Park's Method:

1. **Efficiency:**
   - The closed-form solutions are computationally more efficient compared to iterative numerical methods.
2. **Robustness:**
   - The closed-form approach provides a more straightforward and potentially more accurate estimation process.

By comparing the numerical solutions of Kim and Kvam with the closed-form solutions of Park, we can appreciate the significant advancements made in the field of load-sharing parameter estimation. Park's method not only enhances computational efficiency but also improves the robustness of reliability assessments in systems with interdependent failure rates.

3. Senstivity Analysis

3.1 Simulation Data

To illustrate the effectiveness and accuracy of the closed-form solution we reused the simulation data presented in Kim and Kvam [5] which are failure times generated by the parameters (θ =0.1, r_1=1.5, r_2=3). The simulated data are listed in <Table 1> and the load-share parameter estimators are listed in <Table 2>.

<Table 1> Failure Times for Load-Share Samples

n	t_{i1}	t_{i2}	t_{i3}
1	1.94	0.37	6.93
2	7.44	0.06	2.42
3	0.14	0.2	0.2
4	2.14	1.62	2.34
5	1.91	5.7	1.96
6	8.23	2.25	4.6
7	1.4	2.5	0.07
8	0.79	2.44	7.27
9	0.92	0.12	0.06
10	0.73	0.79	8.61
11	2.78	0.2	1.38
12	0.85	2.81	5.05
13	8.5	1.03	0.52
14	12.93	5.67	1.11
15	4.46	9.06	3.54
16	3.5	5.67	3.24
17	19.59	0.32	1.89
18	4.93	0.12	3.85
19	10.29	2.58	8.61
20	2.22	1.73	1.22

From the above failure-time data if we apply the closed form formula we get

$$\hat{\theta} = \frac{n}{kt_{\bullet 1}} = \frac{20}{3 \times 95.69} \approx 0.0697 \quad \cdots\cdots \quad (2.10)$$

$$\hat{r}_1 = \frac{kt_{\bullet 1}}{(k-j+1)t_{\bullet j}} = \frac{3 \times 95.69}{(3-2+1) \times 45.24} \approx 3.1727 \quad (2.11)$$

$$\hat{r}_2 = \frac{3 \times 95.69}{(3-3+1) \times 64.87} \approx 4.4253 \quad \cdots\cdots \quad (2.12)$$

<Table 2> Load-share Parameter Estimators

	r_1	r_2	Θ
Actual Parameter	1.5	3.0	0.1
MLE(Kim_Kvam)	1.7875	3.2393	0.1
MLE(Closed Form)	3.1727	4.4253	0.0697

The difference between closed-form MLE and numerical MLE (Kim-Kvam method) is primarily due to model assumptions, initial estimates, numerical stability, model fit, and data characteristics.

Differences Between Closed-form MLE and Numerical MLE:

1. Model Assumptions:

- Closed-form MLE (Park's Method): Assumes a specific mathematical form for the likelihood function, enabling the derivation of explicit solutions for the parameters. This method relies on the assumption that the underlying model accurately represents the real-world system.
- Numerical MLE (Kim and Kvam's Method): Utilizes numerical optimization techniques to find parameter estimates without requiring explicit closed-form expressions. This method can handle more complex models where closed-form solutions are not feasible, but it assumes that the iterative optimization process will converge to the true parameter values.

2. Initial Estimates:

- Closed-form MLE: Does not require initial parameter estimates as it provides direct solutions. This can be advantageous as it eliminates the potential bias introduced by selecting inappropriate initial values.
- Numerical MLE: Requires initial parameter estimates to start the iterative optimization process. The quality of the final estimates can be sensitive to the choice of these initial values, potentially leading to suboptimal solutions if the initial values are not chosen carefully.

3. Numerical Stability:

- Closed-form MLE: Provides explicit formulas for the parameter estimates, which are numerically stable and do not suffer from convergence issues. This makes the closed-form method more reliable for obtaining consistent parameter estimates.
- Numerical MLE: Involves iterative optimization, which can sometimes be unstable and sensitive to numerical errors. Convergence to the true parameter values is not always guaranteed, particularly in cases where the likelihood surface is complex with multiple local maxima.

4. Model Fit:

- Closed-form MLE: By providing direct solutions, the closed-form method ensures that the parameter estimates are consistent with the assumed mathematical model. However, this method might be less flexible in accommodating deviations from the model assumptions.
- Numerical MLE: Offers greater flexibility in fitting complex models to the data, as it does not require explicit closed-form solutions. This can result in better model fit in cases where the true underlying system behavior deviates from the assumptions of the closed-form model.

5. Data Characteristics:

- Closed-form MLE: May be more sensitive to deviations from the assumed data distribution, as it relies on specific mathematical forms. This can be a limitation in practical applications where real-world data may not perfectly adhere to theoretical distributions.
- Numerical MLE: Can better accommodate noisy or non-ideal data, as the iterative optimization process can adapt to the data characteristics. This makes the numerical method more robust in handling real-world data variability.

6. Comparison of Results:

In our analysis, we applied both methods to the simulated data. The closed-form MLE values obtained using Park's method and the numerical MLE values obtained using Kim and Kvam's method are listed in <Table 2>.

The results indicate that the numerical MLE values are closer to the true parameter values compared to the closed-form MLE values. This difference can be attributed to the factors discussed above. The numerical method's iterative approach allows it to better capture the complexity and noise in the data, resulting in parameter estimates that are more aligned with the true values. In contrast, the closed-form method, while computationally efficient, may introduce some bias due to the strict assumptions of the underlying model.

The findings suggest that while closed-form solutions offer significant computational advantages and can provide quick and reliable estimates under ideal conditions, numerical optimization techniques remain valuable, especially when dealing with complex and noisy real-world data. Therefore, the choice between closed-form and numerical methods should be guided by the specific characteristics of the data and the system being analyzed. A comprehensive approach that combines both methods may offer the best balance between computational efficiency and estimation accuracy.

3.2 Sensitivity Analysis of Load-Share Parameters on the Average System Reliability

Sensitivity analysis is very useful for understanding the impact of specific parameters on the model's results. Through sensitivity analysis, we can determine the importance of specific parameters and evaluate the reliability of the model.

Setting up the base model : The base model uses the estimated values of $\hat{\theta}$, $\widehat{\gamma_1}$, $\widehat{\gamma_2}$.

Defining Parameter Ranges

$\theta : 0.0697 \pm 30\% \, (i.e., 0.0488 \sim 0.0906)$

$r_1 : 3.1727 \pm 30\% \, (i.e., 2.2209 \sim 4.1245)$

$r_2 : 4.4253 \pm 30\% \, (i.e., 3.9828 \sim 5.7529)$

Analysis Scenario : In our sensitivity analysis, we generated scenarios by varying each parameter within their defined ranges. Specifically, we considered each parameter at three levels: minimum, median, and maximum values, resulting in the following approach:

<Table 3> 3 Levels of Parameter Estimates

Parameter Values	r_1	r_2	Θ
Minimum (30% below)	2.2209	3.9828	0.0488
Median (estimated)	3.1727	4.4253	0.0697
Maximum (30% above)	4.1245	5.7529	0.0906

<Table 4> lists all 27 scenarios generated by varying each parameter at 30% below, at, and 30% above their estimated values. We can use this table to perform the sensitivity analysis and analyze how changes in each parameter affect the model's reliability where the reliability function R(t) is defined as follows:

$$R_i(t) = \exp(-\theta \bullet (t_{i1} + r_1 t_{i2} + r_2 t_{i3})) \quad \cdots \quad (2.13)$$

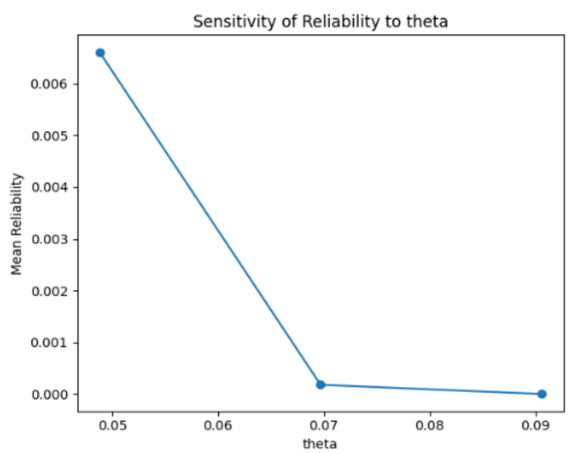
The mean reliability is then obtained by averaging the reliability values of all samples (in our case, n=20):

$$\overline{R}(t) = \frac{1}{n}\sum_{i=1}^{n} R_i(t) \quad \cdots \quad (2.14)$$

It can be observed from Figure1 that as the θ value increases, reliability decreases sharply. When the θ value increases from 0.0488 to 0.0697, reliability decreases significantly, and even when it increases from 0.0697 to 0.0906, reliability approaches nearly zero. As the system's failure rate increases, reliability decreases very rapidly. Therefore, reducing the failure rate is crucial for maintaining the system's reliability.

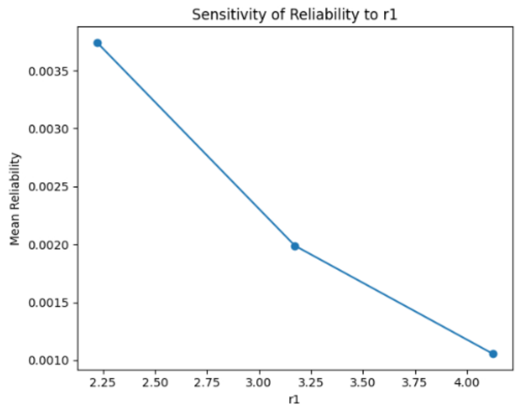
<Table 4> Parameter Table

Scenarios	Θ	r_1	r_2
1	0.0488	2.2209	3.0977
2	0.0488	2.2209	4.4253
3	0.0488	2.2209	5.7529
4	0.0488	3.1727	3.0977
5	0.0488	3.1727	4.4253
6	0.0488	3.1727	5.7529
7	0.0488	4.1245	3.0977
8	0.0488	4.1245	4.4253
9	0.0488	4.1245	5.7529
10	0.0697	2.2209	3.0977
11	0.0697	2.2209	4.4253
12	0.0697	2.2209	5.7529
13	0.0697	3.1727	3.0977
14	0.0697	3.1727	4.4253
15	0.0697	3.1727	5.7529
16	0.0697	4.1245	3.0977
17	0.0697	4.1245	4.4253
18	0.0697	4.1245	5.7529
19	0.0906	2.2209	3.0977
20	0.0906	2.2209	4.4253
21	0.0906	2.2209	5.7529
22	0.0906	3.1727	3.0977
23	0.0906	3.1727	4.4253
24	0.0906	3.1727	5.7529
25	0.0906	4.1245	3.0977
26	0.0906	4.1245	4.4253
27	0.0906	4.1245	5.7529



<Figure 1> Parameter Θ on Mean Reliability

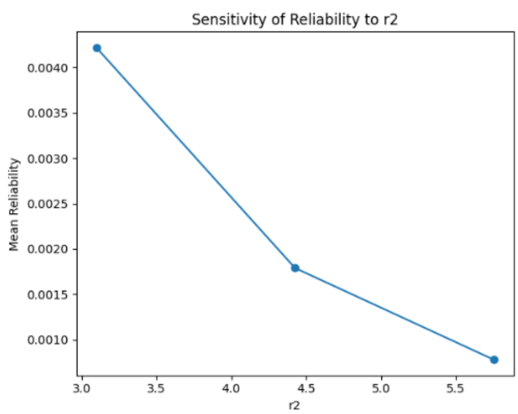
This emphasizes the importance of maintenance strategies and preventive management.



<Figure 2> Parameter r_1 on Mean Reliability

In <Figure 2>, as the r_1 value increases, reliability decreases. However, compared to θ, the rate of decrease is relatively moderate. This means that while the r_1 value affects reliability, it is not as impactful as θ. Therefore, managing the load share parameter r_1 after the first failure is also important. As the r_1 value increases, reliability decreases, so it is necessary to optimize the distribution of load the system experiences after the first failure to maintain reliability.

In <Figure 3>, as the r_2 value increases, reliability tends to decrease, showing a pattern similar to r_1. However, the decrease is slightly more pronounced compared to r_1. The load share parameter r_2 after the second failure also has a significant impact on reliability. As the r_2 value increases, reliability decreases, so it is crucial to effectively manage the distribution of load the system experiences after the second failure.



<Figure 3> Parameter r_2 on Mean Reliability

Reducing the system's failure rate is crucial for enhancing reliability, which can be achieved through preventive maintenance and regular inspections. Additionally, it is necessary to manage load share parameters effectively, as they impact reliability, by optimizing load distribution after failures using load balancing techniques and appropriate redesign. A comprehensive maintenance strategy that considers both failure rates and load share parameters is essential for improving system reliability and reducing operational costs.

4. Conclusion

This paper has delved into the evolution and advancement of load-sharing parameter estimation methodologies within reliability systems, specifically focusing on the transition from numerical optimization techniques to closed-form solutions. Building upon the seminal works of Kim and Kvam (2004) and Park (2012), this study explored the characteristics, robustness, and practical applications of these closed-form solutions in various reliability contexts.

Kim and Kvam's pioneering efforts highlighted the inherent complexity in estimating load-sharing parameters due to the interdependent failure rates in multi-component systems. Their reliance on numerical methods, such as the Gauss-Seidel method, underscored the challenges in deriving closed-form solutions. Park's subsequent advancements provided a significant leap forward, offering closed-form solutions that enhance computational efficiency and robustness in parameter estimation.

Through sensitivity analysis, this research demonstrated the critical impact of load-share parameters on system reliability. The analysis revealed that while the system's failure rate is a major determinant of reliability, the load-sharing parameters post-failure also play a significant role. This underscores the importance of a comprehensive maintenance strategy that not only aims to reduce failure rates but also optimizes load distribution among surviving components after failures.

The findings of this study have several practical implications. Firstly, they suggest that while numerical optimization techniques remain valuable, closed-form solutions offer a more efficient and sometimes more accurate alternative for certain reliability systems. Secondly, the sensitivity analysis emphasizes the need for effective load management and preventive maintenance strategies to enhance system reliability and reduce operational costs.

In conclusion, this research contributes to a deeper understanding of load-sharing parameter estimation in reliability engineering. By comparing numerical and closed-form methods, and highlighting the conditions under which closed-form solutions are advantageous, this study provides valuable insights for both researchers and practitioners in the field. Future research could further explore the applicability of these solutions in more complex and diverse reliability systems, potentially integrating other advanced estimation techniques to enhance robustness and accuracy.

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