J. Appl. Math. & Informatics Vol. 42(2024), No. 5, pp. 1183 - 1193
https://doi.org/10.14317/jami.2024.1183

FINITE ELEMENT METHOD FOR SOLVING BOUNDARY CONTROL PROBLEM GOVERNED BY PARABOLIC VARIATIONL INEQUALITIES OF INFINITE ORDER

GHADA. E. MOSTAFA

ABSTRACT. Finite element method is used here in this article to solve boundary control problem governed by parabolic variational inequalities, where the operator is of infinite order. In the handled problem the cost function is quadratic w. r. to the state of the system.The error estimation between the contionuous problem(P) and the discritisation problem is obtained.

AMS Mathematics Subject Classification : 46E35, 46N40, 49J50.
Key words and phrases : Variational inequalities, operator of infinite order, finite element method.

1. Introduction

To apply Ritz and Galerkin methods on Hilbert spaces of finite dimension such as Soboleve spaces we use the finite element method . In F.E.M. we follow the following steps:

• Discritization of the domain Ω in a collection of subdomains which are triangles mostly. This subdomains are called elements.

• A space H of functions defined on Ω is then approximated by appropriate functions defined on each subdomain with suitable matching conditions at interface [18].

Distributed control of a system governed by Dirichlet and Neumann problems for elliptic equations of infinite order have been discussed by I.M.Ghali , H.A.El-Saify and S.A.ElZahaby in (1983). Optimal control of system governed by elliptic operator of infinite order is obtained by I.M.Ghali in (1984). Optimal control of variational inequalities for infinite order are established by El-Zahaby and Gh.E.Mostafa in (2005).

Received February 12, 2024. Revised May 12, 2024. Accepted June 13, 2024. *Corresponding author.

In this paper we shall use the theory of Barbu (1981, 1982, 1984, 1993) to introduce boundary control problem governed by parabolic equation with non-linear boundary value condition in the case of infinite order and will apply finite element method.

This paper is organized as follows:

In section 2, some functional spaces of infinite order are introduced. In section 3 , we introduce the main results and the error estimation.

2- Preliminaries

A Sobolev space of infinite order of periodic functions is defined as follows:

$$W^{\infty}\{a_\alpha, p\} = \left\{ u(x) \in C^{\infty}(\mathrm{R^n}) : \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha(u)\|_p^p < \infty \right\}$$

We recall that $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a multi-index for differentiation,

$$|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \quad , \quad a_\alpha \geq 0$$

is a numerical sequence, $p \geq 1$ and $\|.\ \|_p$ is the canonical norm in the space $L_p(\mathrm{R^n})$.

The structure of $W^{-\infty}\{a_\alpha, p\}$ is based on the fact that for any function $h(x) \in W^{-\infty}\{a_\alpha, p\}$ the equation

$$L(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} D^\alpha \left(a_\alpha |D^\alpha(u)|^{p-2} D^\alpha(u) \right) = h(x)$$

$$D^w u|_\Gamma = 0, \quad |w| = 0, 1, \dots$$

has a unique solution $u(x) \in W_0^{\infty}\{a_\alpha, p\}$.

We have the representation

$$W^{-\infty}\{a_\alpha, q\} = \Big\{ h(x) : h(x) = \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha h_\alpha(x); h_\alpha(x) \in L_q(\mathrm{R^n}) \text{ and } \sum_{|\alpha|=0}^{\infty} a_\alpha \|h_\alpha\|_{L_q}^q < \infty \Big\}$$

and $q = \frac{p}{p-1}$.

The imbedding problem for non-trivial Sobolev spaces of infinite order are investigated in [12, 13, 14].

The imbedding $W^{\infty}\{a_\alpha, p\} \subset W^{\infty}\{b_\alpha, p'\}$ is compact if and only if

$$\lim_{m \to \infty} \sum_{|\alpha|=m+1}^{\infty} b_\alpha \|D^\alpha(u)\|_{p'}^{p'} = 0$$

uniformly on the unit ball of the space $W^{\infty}\{a_\alpha, p\}$ [13].
Now consider the space $W^{\infty}\{a_\alpha, 2\}$ of functions $u(x)$ defined on the Euclidean space $R^n, n \geq 1$,

$$W^{\infty}\{a_\alpha, 2\} = \left\{ u(x) \in C^{\infty}(R^n) : \sum_{|\alpha|=0}^{\infty} a_\alpha \|D^\alpha\|_2^2 < \infty \right\}$$

An imbedding criterion established in terms of the characteristic function of these spaces.

The duality of the spaces $W^{\infty}\{a_\alpha, 2\}$ and $W^{-\infty}\{a_\alpha, 2\}$ is given by the relation

$$\langle h, v \rangle = \sum_{|\alpha|=0}^{\infty} a_\alpha \int_G h_\alpha(x) D^\alpha v(x) dx$$

so that $W^{\infty}\{a_\alpha, 2\}$ is every where dense in $L_2(R^n)$ with topological inclusion, and $W^{-\infty}\{a_\alpha, 2\}$ denotes the topological dual space with respect to $L_2(R^n)$ and then we have the following chain

$$W^{\infty}\{a_\alpha, 2\} \subseteq L_2(R^n) \subseteq W^{-\infty}\{a_\alpha, 2\}$$

similar to the above chain we have

$$W_0^{\infty}\{a_\alpha, 2\} \subseteq L_2(R^n) \subseteq W_0^{-\infty}\{a_\alpha, 2\}$$

where $W_0^{\infty}\{a_\alpha, 2\}$ is the set of all functions of $W^{\infty}\{a_\alpha, 2\}$ which vanish on the boundary Γ of R^n.

The space $L_2(0,\ \mathrm{T}; L_2(\mathrm{R^n}))$ will be denoted by $L_2(Q)$, where $Q = \mathrm{R^n} \times]0, T[$, and $L_2(0,\ \mathrm{T}; L_2(\mathrm{R^n}))$ is the space of all measurable functions $\mathrm{t} \to \phi(\mathrm{t})$, the variable t denotes the time ; $t \in]0, T[, T < \infty$ with the Lebesgue measure dt on $]0,\ \mathrm{T}[$ such that

$$\|\phi\|_{L_2(Q)} = \left(\int_0^T \|\phi(\mathrm{t})\|_2^2 dt \right)^{\frac{1}{2}} < \infty$$

is endowed with the scalar product

$$(f, g)_{L_2(0,\ \mathrm{T}; L_2(\mathrm{R^n}))} = \int_0^T (f(t), g(t))_{L_2(\mathrm{R^n})} dt$$

which is a Hilbert space[15].

Similarly we define the spaces $L_2(0,\ \mathrm{T}; W^{\infty}\{a_\alpha, 2\}), L_2(0,\ \mathrm{T}; W_0^{\infty}\{a_\alpha, 2\})$ and $L_2(0,\ \mathrm{T}; W^{-\infty}\{a_\alpha, 2\}), L_2\left(0,\ \mathrm{T}; W_0^{-\infty}\{a_\alpha, 2\}\right)$ which are its conjugates respectively.

We have the following chains:

$$L_2(0,\ \mathrm{T}; W^{\infty}\{a_\alpha, 2\}) \subseteq L_2(Q) \subseteq L_2\left(0,\ \mathrm{T}; W^{-\infty}\{a_\alpha, 2\}\right)$$
$$L_2(0,\ \mathrm{T}; W_0^{\infty}\{a_\alpha, 2\}) \subseteq L_2(Q) \subseteq L_2\left(0,\ \mathrm{T}; W_0^{-\infty}\{a_\alpha, 2\}\right)$$

Finally we shall denote by $W(Q)$ the space of the functions $y \in L_2(0,\ \mathrm{T}; W^{\infty}\{a_\alpha, 2\})$ such that $\frac{dy}{dt} \in L_2(0,\ \mathrm{T}; W^{-\infty}\{a_\alpha, 2\})$ where $\frac{dy}{dt}$ is the derivative of y in the sense of $W^{-\infty}\{a_\alpha, 2\}$ valued distribution on $]0, T[$.$W(Q)$ is a Banach space endowed with the norm

$$\|y\|^2_{W(Q)} = \|y\|^2_{L_2(0,\ \mathrm{T}; W^{\infty}\{a_\alpha,2\})} + \left\|\frac{dy}{dt}\right\|^2_{L_2(0,\ \mathrm{T}; W^{-\infty}\{a_\alpha,2\})}$$

3-Main results

We introduce a convex control problem governed by boundary-value problem of the form

$$\begin{aligned} y_t + Ay = 0 \quad & \text{in } Q = \Omega \times]\,0, T[\\ \frac{\partial y}{\partial v} + \beta_i(y) \ni u_i + f_i \quad & \text{in} \quad \Sigma_i = \Gamma \times \Big]\, 0, T[, i = 1, 2 \\ y(x, 0) = y_0(x) \quad & \text{in } \Omega \end{aligned} \tag{3.1}$$

Here, Ω is a bounded and open set in R^n with boundary Γ consists of two parts Γ_1 and Γ_2, i.e., $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \Phi$, and $\Sigma = \Gamma \times]\,0, T[$ is the lateral boundary of Q.

$\frac{\partial}{\partial v}$ is the outward normal derivative corresponding to A, and β_i are maximal monotone graphs in $R \times R$, which satisfy the conditions

$$\beta_i(0) \ni 0, \quad i = 1, 2 \tag{3.2}$$

the controls u_i are taken from the Hilbert spaces $L_2(\Sigma_i), i = 1, 2$. The functions y_0, f_i are fixed in $L_2(\mathrm{R^n})$ and $L_2(\Sigma_i), i = 1, 2$ respectively.

A is elliptic, bounded and self-adjoint operator of infinite order with finite dimension, that maps $W_0^{\infty}\{a_\alpha, 2\}$ onto $W_0^{-\infty}\{a_\alpha, 2\}$ and $A(u)$ is given by:

$$\begin{array}{c} A(u) = \sum_{|\alpha|=0}^{\infty} (-1)^{|\alpha|} a_\alpha D^{2\alpha}(u) = h(x) \quad , a_\alpha \geq 0 \\ D^w u|_\Gamma = 0, \quad |w| = 0, 1, \ldots \quad , |w| \leq \alpha \end{array} \tag{3.3}$$

The operator $\frac{\partial}{\partial t} + A$ is an infinite order parabolic operator which maps

$$L_2(0,\ \mathrm{T}; W_0^{\infty}\{a_\alpha, 2\}) \text{ onto } L_2\left(0,\ \mathrm{T}; W_0^{-\infty}\{a_\alpha, 2\}\right)$$

We introduce the following continuous bilinear form on $W_0^{\infty}(R^n)$:

$$\pi(t; \phi, \psi) = (A(t)\phi, \psi)_{L_2(R^n)} \quad \forall \phi, \psi \in W_0^{\infty}(R^n) \tag{3.4}$$

Lemma 3.1 Consider (3.4) the continuous bilinear form on $W_0^{\infty}\{a_\alpha, 2\}$, then $\pi(t; \phi, \psi)$ is coercive this means $\pi(t; \phi, \phi) \geq \lambda \|\phi\|^2_{W_0^{\infty}\{a_\alpha,2\}} \quad \lambda > 0$,

Proof

$$\pi(t;\phi,\phi)=(A(t)\phi,\phi)_{L^2(R^n)}=\left(\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{2\alpha}\phi(x),\phi(x)\right)_{L^2(R^n)}$$

$$=\int_{R^n}\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha D^{\alpha}\phi(x)D^{\alpha}\phi(x)dx$$

$$=\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha\left(D^{\alpha}\phi(x),D^{\alpha}\phi(x)\right)_{L^2(R^n)}=\sum_{|\alpha|=0}^{\infty}(-1)^{|\alpha|}a_\alpha\left\|D^{\alpha}\phi(x)\right\|^2_{L^2(R^n)}$$

$$\geq\lambda\|\phi\|^2_{W_0^\infty\{a_\alpha,2\}}$$

then $\pi(t;\phi,\phi)\geq\lambda\|\phi\|^2_{W_0^\infty\{a_{\alpha,2},2\}}$

Definition 3.1. A function $y\in W(Q)$ is a solution to (3.1) if there exist functions $\omega_i\in L_2\left(\Sigma_i\right),i=1,2$, such that $\omega_i(\sigma,t)\in\beta_i(y(\sigma,t))$ a.e. $(\sigma,t)\in\Sigma_i,i=1,2$,

and

$$\int_Q yk_t dp(x)dt+\int_0^T\pi(y,k)dt+\sum_{i=1}^{2}\int_{\Sigma_i}\left(\omega_i-v_i\right)kd\Gamma dt=\int_{R^n}y_0(x)k(x,0)dp(x) \tag{3.7}$$

for all $k\in W(Q)$ such that $k(x,T)=0$. Here $\pi(y,k)$ is bilinear functional has the form (3.4) , condition (3.7) can be equivalently defined as

$$\frac{d}{dt}(y(t),\psi)+\pi(y(t),\psi)+\sum_{i=1}^{2}\int_{\Gamma_i}\left(\omega_i-v_i\right)\psi d\Gamma=0\text{ a.e. }t\in]0,T[$$
$$y(0)=y_0,\qquad\text{for all }\psi\in W^\infty\left\{a_\alpha,2\right\} \tag{3.7'}$$

Let ρ be a C_0^∞-mollifie function on R, satisfying $\rho(r)>0$ for $r\in]-1,1[,\rho(r)=0$ for $|r|>1,\rho(r)=\rho(-r)$ for all $\boldsymbol{r}\in\boldsymbol{R}$ and $\int_{-\infty}^{\infty}\rho(r)dr=1$. We define, for $\varepsilon>0$

$$\boldsymbol{\beta_i^\varepsilon(r)}=\int_{-\infty}^{\infty}\beta_{i\varepsilon}(r-\boldsymbol{\varepsilon\theta})\boldsymbol{\rho(\theta)}d\boldsymbol{\theta},\quad i=1,2,r\in R \tag{3.8}$$

where

$$\beta_{i\varepsilon}(r)=\varepsilon^{-1}\left(r-\left(1+\varepsilon\beta_i\right)^{-1}r\right) \tag{3.9}$$

It should be recalled that β_i^ε are monotonically increasing infinitely differentiable functions. Moreover, β_i^ε are Lipschitzian with Lipschitz constant $\mathcal{E}^{-1}$, and in a certain sense which will be explained below they approximate β_i, for $\varepsilon \to 0$. For each $\varepsilon > 0$, consider the approximating system,

$$\begin{array}{cc} y_t + Ay = 0 & \text{in } Q \\ \frac{\partial y}{\partial v} + \beta_i^{\varepsilon}(y) = u_i + f_i & \text{in } \Sigma_i, i = 1,2 \\ y(x,0) = y_0(x) & \text{in } \Omega. \end{array} \tag{3.10}$$

According to a standard existence result due to Lions[15] the system (3.10) has a unique solution $y_\varepsilon \in W(Q)$.

Let $\mathcal{A}_\varepsilon : W^\infty \{a_\alpha, 2\} \to W^{-\infty} \{a_\alpha, 2\}$ be the operator defined by

$$(\mathcal{A}_\epsilon y, \psi) = \pi(y, \psi) + \sum_{i=1}^{2} \int_{\Gamma_i} \beta_i^{\varepsilon}(y) \psi d\sigma \quad y, \psi \in W^\infty \{a_\alpha, 2\} \tag{3.11}$$

And let $f \in L^2 (0, T; W^{-\infty} \{a_\alpha, 2\}$ be given by:

$$(f(t), \psi) = \sum_{i=1}^{2} \int_{\Gamma_i} u_i \psi d\sigma \quad , \quad \psi \in W^\infty \{a_\alpha, 2\} \tag{3.12}$$

Then in the sense of definition 3.1, (3.10) can be written as

$\frac{dy}{dt} + \mathcal{A}_\epsilon y = f, \quad t \in [0,T]$

$y(0) = y_0$

Let $j_i : R \to \bar{R}, i = 1,2$, be tow lower semi continuous convex functions such that $\partial j_i = \beta_i$ (it is well known that such functions always exist).

Under the assumptions and the coerciveness condition (3.5), we have

Theorem 3.2. Let $y_0 \in L_2(R^n)$ and $u_i \in L_2(\Sigma_i), f_i \in L_2(\Sigma_i), i =$ 1,2. Then the system (3.1) has a unique solution $y \in W(Q)$. Furthermore, for $\varepsilon \to 0$ we have

$$y_\varepsilon \to y \text{ strongly in } C([0,T]; L_2(R^n)) \cap L_2(0,T; W^\infty \{a_\alpha, 2\}) \text{ and weakly in } W(Q) \tag{3.14}$$

There exists $c > 0$ independent of u_i such that

$$\|y\|_{W(Q)} + \sum_{i=1}^{2} \|\beta_i(y)\|_{L_2(\Sigma_i)} \leq C \left(\sum_{i=1}^{2} \|u_i\|_{L_2(\Sigma_i)} + 1 \right) \tag{3.15}$$

Proof: We take the inner product of (3.13) with y_ε and integrate over $[0,t]$. By (3.11) and (3.12) it follows that

$$\begin{aligned} &\|y_\varepsilon(t)\|_{L_2(R^n)} + \int_0^t \|y_\varepsilon(s)\|^2_{W^\infty \{a_\alpha, 2\}} ds \\ &\leq C \left(\|u_1\|^2_{L_2(\Sigma_1)} + \|u_2\|^2_{L_2(\Sigma_2)} + 1 \right) \quad t \in [0,T] \end{aligned} \tag{3.16}$$

where t is independent of ε.

Next we take the inner product of (3.13) with $\beta_i^\varepsilon(y_\varepsilon)$. In as much as $\pi(\psi, \beta_i^\varepsilon(\psi)) \geq 0$, for all $\psi \in W^\infty\{a_\alpha, 2\}$, we find, after some calculations,

$$\int_{R^n} j_i^\varepsilon(y_\varepsilon)\,dx + \sum_{i=1}^{2} \int_{\Sigma_i} \beta_i^\varepsilon(y_\varepsilon - u_i)\,\beta_i^\varepsilon(y_\varepsilon)\,d\sigma dt \leq \int_{R^n} j_i^\varepsilon(y_0)\,dx \quad \text{for } i = 1, 2 \tag{3.17}$$

where

$$j_i^\varepsilon(r) = \int_0^r \beta_i^\varepsilon(s)ds, \quad i = 1, 2$$

Along with assumption (3.2) , (3.17) yields
$\sum_{i=1}^{2} \|\beta_i^\varepsilon(y_\varepsilon)\|_{L_2(\Sigma_i)}^2 \leq C\left(\sum_{i=1}^{2} \|u_i\|_{L_2(\Sigma_i)}^2 + 1\right)$
and by (3.13) , (3.16) we see that
$\|y_\varepsilon\|_{W(Q)}^2 + \sum_{i=1}^{2} \|\beta_i^\varepsilon(y_\varepsilon)\|_{L_2(\Sigma_i)}^2 \leq C\left(\sum_{i=1}^{2} \|u_i\|_{L_2(\Sigma_i)}^2 + 1\right)$
Where C is independent of ε.
Now, using (3.13), for $\varepsilon, \lambda > 0$ we get

$$\|y_\varepsilon(t) - y_\lambda(t)\|_{L_2(R^n)}^2 + \|y_\varepsilon(t) - y_\lambda(t)\|_{L_2(0,T;H^1(R^\infty))}^2 + C\sum_{i=1}^{2} \int_{\Sigma_i} \left(\beta_i^\varepsilon(y_\varepsilon) - \beta_i^\lambda(y_\lambda)\right)(y_\varepsilon - y_\lambda)\,d\sigma dt \leq 0$$

If we take into account (3.8), (3.9) and (3.18) and the monotonicity of β_i, we find

$$\|y_\varepsilon - y_\lambda\|_{C([0,T];W^\infty\{a_\alpha,2\})}^2 + \|y_\varepsilon - y_\lambda\|_{L_2([0,T];W^\infty\{a_\alpha,2\})} \leq C(\varepsilon + \lambda) \tag{3.20}$$

Hence y exits in the strong topology of

$$L_2([0,T]; W^\infty\{a_\alpha, 2\}) \cap C([0,T]; L_2([0,T]; W^\infty\{a_\alpha, 2\}))$$

Which yields

$$y_\varepsilon \to y \text{ strongly in } L_2\left([0,T]; W^{\frac{1}{2}}(\Gamma)\right) \subset L_2(\Sigma)$$

In order to get (3.14) , (3.15) we let ε tend to zero in (3.19) and $\lambda \to 0$ in (3.20) which completes the proof.

We shall study the following control problem (P) Minimize

$$\frac{1}{2}\int_Q h(x,t)\,|y(x,t) - y_d(x,t)|^2\,dp(x)dt + \psi_1(u_1) + \psi_2(u_2) + \varphi(y(T)) = J(y,u) \tag{3.21}$$

on the class of all $u_i \in L_2(\Sigma_i), i = 1, 2$, and $y \in W(Q)$ subject to the state system (3.1).

We shall assume that the following conditions are satisfied

(1) $U_i = L_2(\Sigma_i), i = 1, 2$, are the spaces of controls $u_i, i = 1, 2$.

(2) The functions $\psi_i : L_2(\Sigma_i) \to \bar{R}, u_i, i = 1, 2$ are lower semi continuous convex functions and not identically equal to infinity.

(3) The function $\varphi : L_2(R^n) \to R$ is convex and continuous on $L_2(R^n)$.

(4) $h \in L^\infty(Q)$ and $y_d \in L_2(Q)$ are given; $h \geq 0$ a.e. on Q.

(5) A is the elliptic symmetric operator which is presented by (3.3) and $\beta_i, i = 1, 2$, are two maximal monotone graphs in $R \times R$ which satisfy condition (3.2).

(6) $y_0 \in L_2(R^n)$ and $f_i \in L_2(\Sigma_i), i = 1, 2$, satisfy the assumptions of Theorem 3.2.

Under our assumptions, the coerciveness condition (3.5), we may apply the result of Barbu [3],[6] for every pair $(u_1, u_2) \in L_2(\Sigma_1) \times L_2(\Sigma_2)$. Problem (P), has at least optimal $(y^\bullet, u_1^\cdot, u_2^*)$ where $y^\bullet \in W(Q), u_i \in L_2(\Sigma_i), i = 1, 2$, for which the infimum of the functional (3.10) is attained for $y = y^\bullet$ and $u_i = u_i^\bullet, i = 1, 2$.

The optimality result is given in the case in which β_i are single-valued and satisfy the following condition,

(7) The functions β_i are monotonically increasing and locally Lipschitzian on the real axis R. Moreover, there exists $c > 0$, such that

$$\beta_i'(r) \leq c\,(|\beta_i(r)| + |r| + 1) \quad \text{a.e. } \boldsymbol{r} \in \boldsymbol{R}, \quad i = 1, 2 \tag{3.22}$$

In the following we shall introduce the finite element discretization of the state equation and optimal control problem([1],[14]):

At first let us consider the finite element approximation of the state equation (3.1). For the spatial discretization we consider conforming Lagrange triangle elements. We assume that Ω is a polygonal domain. Let Υ^h be a quasi-uniform partitioning of Ω into disjoint regular triangles τ, so that

$$\bar{\Omega} = \cup_{\tau \in Y^h} \bar{\tau}$$

Associated with Y^h a finite dimensional subspace V^h of $C([0, T]; \bar{\Omega})$, such that for $\chi \in V^h$ and $\tau \in \Upsilon^h$, $\chi|_\tau$ are piecewise linear polynomials. We set

$$V_0^h = V^h \cap W_0^\infty \{a_\alpha, 2\}$$

Let Υ_U^h be a partitioning of Γ into disjoint regular segments s, so that

$$\Gamma = U_{s \in Y_U^h} \bar{s}$$

Associated with Υ_U^h is another finite dimensional subspace U^h of $L_2([0, T]; \Gamma)$, such that for $\chi \in U^h$ and s $\in \Upsilon_U^h$, $\chi|_s$ are piecewise linear polynomials. Here we suppose that Y_U^h is the restriction of Υ^h on the boundary Γ and $U^h = V^h(\Gamma)$, where $V^h(\Gamma)$ is the restriction of V^h on the boundary Γ.

Lagrange interpolation operator $I_h : C([0, T]; \bar{\Omega}) \to V^h$, we have the following error estimate $\|w - I_h w\|_{l, R^n} \leq C h^{m-l} \|v\|_{m, R^n}, 0 \leq l \leq 1 \leq m \leq \infty$

$Q_h : L_2(\Gamma) \to V^h(\Gamma)$ and $\widetilde{Q_h} : L^2(R^n) \to V_0^h$ denote the orthogonal projection operators. Further more, $R_h : W^1([0,T]; R^n) \to V_0^h$ denotes the Ritsz projection operator defined as: $\pi(R_h w, v_h) = \pi(w, v_h) \quad \forall v_h \in V_0^h$

It is well known that the Ritz projection satisfies:

$\|w - R_h w\|_{s,R^n} \le Ch^{l-s}\|w\|_{l,R^n}, w \in W_0^\infty\{a_\alpha, 2\} \cap W^\infty\{a_\alpha, l\}, \forall 0 \le s \le 1 \le l \le \infty$

For the $L_2(\Gamma)$ projection operator Q_h we also have:

$\|w - Q_h w\|_{0,\Gamma} \le Ch^{s-\frac{1}{2}}\|w\|_{s,R^n}, w \in W^\infty\{a_\alpha, s\}, \forall \frac{1}{2} \le s \le \infty$

and

$$\|(I - Q_h)\partial_n w\|_{0,\Gamma} \le Ch^{\frac{1}{2}}\|w\|_{2,R^n}, \text{ for } w \in W^\infty\{a_\alpha, 2\}$$

The semi-discrete finite element approximation of (3.1) reads:

find $y_h \in L_2(V^h)$ such that

$-(y_h, \partial_t v_h)_Q + \pi(y_h, v_h)_Q = (f, v_h)_Q + (y_0^h, v_h(.,0)) \quad \forall v_h \in W^\infty\{a_\alpha, 2\},$

$y_h = Q_h(u)$ on $\Sigma_i = \Gamma\times]0, T[$,

with y_0^h an approximation of y_0 the semi- discrete finite element approximation of (3.10), (3.1) reads as follows

$$\text{Minimize } J(y_h, u_h) \text{ over } u_h \in U_{ad}^h, y_h \in L_2(V^h) \tag{3.28}$$

subject to

$$-(y_h, \partial_t v_h)_Q + \pi(y_h, v_h)_Q = (f, v_h)_Q + (y_0^h, v_h(.,0))\, \forall v_h \in W^\infty\{a_\alpha, 2\}$$
$$y_h = Q_h(u_h) \text{ on } \Sigma_i = \Gamma\times]0, T[\tag{3.29}$$

where U_{ad}^h is an appropriate approximation to $U_i = L_2(\Sigma_i)$.It follows that (3.28) ,(3.29) has a unique solution (y_h, u_h)

We next consider the fully discrete approximation for the above semi-discrete problem by using the dG(0) scheme in time. For simplicity we consider an equi-distant partition of the time interval.

Let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ with $k = \frac{T}{N}$ and $t_i = ik, i = 1, 2, \ldots.N$. We also let $I_i = (t_{i-1}, t_i]$, where $i = 1, 2, \ldots.N$. We construct the finite element spaces $V^h \in W^\infty\{a_\alpha, 2\}$ with the mesh Υ^h. Similarly we construct the finite element spaces U^h of $L_2([0,T]; \Gamma)$, with the mesh Υ_U^h. then we denote by V^h, U^h the finite element spaces defined on Υ^h, Υ_U^h on each time step .

Let V_k dente the space of piecewise constant functions on the time portion. we define the L_2 projection operator $P_k : L_2(0,T) \to V_k$ on I_i through

$(P_k(w)(t) = \frac{1}{k}\int w(s)ds$ for $t \in I_i$

Then we have the following estimate:

$$\|(I - P_k)w\|_{L_2(0,T;H)} \le Ck\|w_t\|_{L_2(0,T;H)}, \forall w \in W^1(0,T;H) \tag{3.30}$$

H denotes a separable Hilbert space.
We consider a dG(0) scheme for the time discretization and let

$$V_{hk} = \left\{\theta; \bar{\Omega} \times [0,T] \to R,\, \theta(.,t)|_{\bar{\Omega}} \in V^h, \quad \theta(x,.)|_{I_n} \in P_0\right\}$$

We introduce for $Y, \Phi \in V_{hk}$

$$A\left(Y_{hk}; \Phi\right) = (\mathrm{f}, \Phi)_Q + \left(y_0, \Phi_+^0\right), \quad \forall \Phi \in V_{hk}^0$$
$$Y_{hk} = \Lambda(u) \quad \text{on } \Gamma \tag{3.31}$$

Where V_{hk}^0 denotes the subspace of V_{hk} with functions vanishing on Γ, and $\Lambda = P_k Q_h$

As a result of the application of finite element method on boundary control problem governed by parabolic Variationl Inequalities with an infinite number of variables. we estimate the error introduced by the discretization of the state equation i.e., The error between the solutions of problem (3.1) and (3.31) by the following theorem :

Theorem 3.3

Suppose that $f \in L_2\left(L_2\left(R^n\right)\right), u \in L_2\left(L_2(\Gamma)\right)$, and $y_0 \in L_2\left(R^n\right)$. let $y \in L_2\left(L_2\left(R^n\right)\right)$ and $Y_{hk} \in V_{hk}$ with $Y_{hk}|_{\Sigma} = \Lambda(\mathrm{u})$ be the solution of problems (3.1), (3.31). respectively. then we have

$$\|y - Y_{hk}\|_{L_2(L_2(R^n))} \leq C\left(h^{\frac{1}{2}} + h^{\frac{1}{4}}\right)\left(\|f\|_{L_2(L_2(R^n))} + \|y_0\|_{0,R^n} + \|u\|_{L_2(L_2(R^n))}\right)$$

Conflicts of interest : The author declares no conflict of interest.

Data availability : Not applicable

Acknowledgments : In this section you can acknowledge any support given which is not covered by the author contribution or funding sections.

References

1. N.U. Ahmed, *Boundary and point controls for semilinear stochastic partial differential equations*, Nonlinear Funct. Anal. Appl **19** (2014), 639-662.
2. V. Barbu, *Necessary Conditions for Distributed Control Problems Governed by Parabolic Variational Inequalities*, SIAM J. Cont. Optimiz **19** (1981), 64-86.
3. V. Barbu, *Boundary Control Problems With Nonlinear State Equation*, SIAM J. Cont. Optimiz **20** (1982), 125-143.
4. V. Barbu, *Optimal Control of Variational Inequalities*, Research, Notes in Mathematics 100, Boston-London-Melbourne, Pitman, 1984.
5. V. Barbu, *The Time Optimal Control Problem for Parabolic Variational Inequalities*, Applied Mathematics and optimization **11** (1984), 1-22.
6. V. Barbu, *Analysis and Control of Nonlinear Infinite Dimensional Systems*, Mathematics in Science and Engineering, Boston-San Diego-New York, **189**, 1993.

7. E. Browder, *On the Unification of the Calculus of Variational and the Theory of Monotone Nonlinear Operators in Banach Spaces*, Proc. Nat. Acad. Sci. U.S.A. **56** (1966), 419-425.
8. Ju.A. Dubinskii, *Some Imbedding Theorems for Sobolev Spaces of Infinite Order*, Soviet Math. Dokl **19** (1978), 1271-1274.
9. Ju.A. Dubinskii, *About one Method for Solving Partial Differential Equation*, Dokl. Akad. Nauk. USSR **258** (1981), 780-784.
10. S.A. El-Zahaby, and Gh.h. Mostafa, *Optimal Control of Variational Inequalities for Infinite Order*, International Conference on Mathematics: Trends and developments, Cairo, Egypt, 2002, 28-31.
11. D.A. French and J.T. king, *Analysis of a robust finite element approximation for a parabolic equation with rough boundary data*, Math. Comput **60** (1993), 79-104.
12. I.M. Gali, *Optimal Control of System Governed by Elliptic Operator of Infinite Order, Ordinary and Partial Differential Equations*, Proc., Dundee, Scotland, Springer-Verlag Series, Lecture Notes in Mathematics, 1984, 263-272.
13. I.M. Gali, and H.A. El-Saify, and S.A. El-Zahaby, *Distributed Control of a System Governed by Dirichlet and Neumann Problems for Elliptic Equations of Infinite Order*, Proc. of the International Conference of Functional Differential System and Related Topics III, Poland, 1983, 22-29.
14. W. Gong, Micheal Hints, and Zhaojie Zhou, *A finite element method for Dirichlet boundary control problems governed by parabolic*, PDES.[math. OC], 2014.
15. J.L. Lions, *Optimal Control of System Governed by Partial Differential Equations*, Springer-Verlag, New York, Band **170**, 1971.
16. Messaoud Boulbrachene, *Finite element approximation of V.I.: an algorithmic approach*, SQU J. for Sci. **22** (2017), 114-119.
17. Gh.E. Mostfa, *Application of finite element method in solving boundary control problem governed by parabolic Variationl Inequalities with an infinite number of variables*, Nonlinear functional analysis and applications **25** (2020), 605-616.
18. A.H. Siddiqi, *Applied Functional Analysis*, Marcel Dekker, New York, 2004.

Ghada Ebrahim Mostafa Ali Mohamed received M.Sc. from Al-Azhar University, and Ph.D. from Al-Azhar University. He is currently a lecturer at Al-Azhar University since 1999. His research interests are applications of functional analysis, optimal control and applications of distance function.

Department of Mathematics, Faculty of Science, Al-Azhar University-Girls Branch, Nasr City, Cairo, 11651, Egypt.

e-mail: Ghadaali@azhar.edu.eg