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STUDY ON S-PRIME IDEAL AS NILPOTENT IDEAL

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Abstract. Let S be a multiplicative subset of a commutative ring $\mathcal{R}$ with unity and I_s be an S-prime ideal of $\mathcal{R}$ which is disjoint from the multiplicative subset S. In this paper, some properties of the S-prime ideal, namely sum, union and intersection of two S-prime ideals are studied in a commutative ring $\mathcal{R}$ with unity. It is proved that a nilradical of $\mathcal{R}$ is the S-prime ideal of $\mathcal{R}$. Zorn's lemma is used to state that an S-prime ideal is unique in a local ring $\mathcal{R}$. Finally, the S-prime ideals in the semilocal ring are classified. The generalized S-prime ideal and its multiplicative subsets of a finite commutative ring with unity are presented.

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1. Introduction

In the present paper, the ring $\mathcal{R}$ is a commutative with unity. Let $N(\mathcal{R})$ be the nilradical ideal containing all the nilpotent elements of $\mathcal{R}$ and $Z(\mathcal{R})$ be the set containing all the zero-divisors of $\mathcal{R}$. The prime ideals have a fundamental role in commutative ring theory. This ideal is the foundation for many researches. Some kinds of prime ideals and related ideals are defined and developed by many researchers.

In 2007, Ayman Badawi [5] introduced a 2-absorbing ideal of a commutative ring with $1 \neq 0$ which is a generalization of prime ideal. A non-zero proper ideal I of R is called a 2-absorbing ideal of $\mathcal{R}$ if $a, b, c \in \mathcal{R}$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. In 2017, Driss Bennis [10] and Brahim Fahid showed interest with the rings in which every 2-absorbing ideals are prime ideal and this class of rings is called 2-AB rings. The authors [10] have shown that in an integral valuation domain $\mathcal{R}$ is a 2-AB domain if and only if $P^2 = P$ for every prime ideal P of $\mathcal{R}$.

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In 2020, Yassine, Nikmehr and Nikandish [22] introduced the concept of 1-absorbing prime ideals of commutative rings which look like both the prime and the 2-absorbing ideals of a ring $\mathcal{R}$, i.e., if for all nonunit elements $a, b, c \in \mathcal{R}$ such that $abc \in I$, then either $ab \in I$ or $c \in I$. Further, the properties of that 2-absorbing ideal are studied. In 2016, Beddani and Messirdi [7] also introduced a proper ideal I of a ring $\mathcal{R}$ called 2-prime ideal; if for all $x, y \in \mathcal{R}$ such that $xy \in I$, then either x^2 or y^2 lies in I. Likewise, in 2021, Suat [22] developed and proposed weakly 2-prime ideals in commutative rings based on the idea of weakly prime ideals by Anderson and Smith [3]. In 2012, Manish Kant Dubey [18] discussed prime and weakly prime ideals in semirings.

Later on, many ideals are introduced and studied by many authors; 2-absorbing and weakly 2-absorbing primary ideals of a commutative semiring [11] by Fatemeh Soheilnia, I-prime ideals [2] by Akray, (m, n)-absorbing ideals of commutative rings [6] by Batool Zarei J. Abadi and Hosen F. Moghimi, weakly S-2-absorbing submodules [13] by Govindarajulu, and so on.

The idea of S-prime ideal was introduced by Ahmed Hamed and Achraf Malek [1] in 2019. The S-prime ideal is the generalization of the prime ideal of $\mathcal{R}$. An ideal I of a commutative $\mathcal{R}$ is called S-prime ideal of $\mathcal{R}$, if there exists an $s \in S$ such that for all $a, b \in \mathcal{R}$ with $ab \in I$, then $sa \in I$ or $sb \in I$, where S is the multiplicative subset of $\mathcal{R}$ which is disjoint from the ideal I. It is also shown that every prime ideal is S-prime. In general, the converse is not true. An S-prime ideal is prime and its disjoint multiplicative subset S contains all the units of $\mathcal{R}$. The authors have shown that an ideal I is an S-prime ideal of a ring $\mathcal{R}$ if and only if $I : s$ is a prime ideal of $\mathcal{R}$ for some $s \in S$, where S is the multiplicative subset of $\mathcal{R}$ which is disjoint from an S-prime ideal I. The notion $I : s$ is an ideal of $\mathcal{R}$ and it is defined by $I : s = \{x \in \mathcal{R} \mid sx \in I\}$. Furthermore, is it proved that P be an S-prime ideal of $\mathcal{R}$ if and only if there exists s in S such that for all $I_1, I_2, I_3, ..., I_n$ ideals of $\mathcal{R}$, and if $I_1.I_2.I_3, ..., I_n \subseteq P$, then $sI_j \subseteq P$ for some $j \in \{1, 2, 3, ..., n\}$. Consequently, the S-prime ideals over S-Noetherian rings are studied. A ring $\mathcal{R}$ is called S-Noetherian if every ideal of $\mathcal{R}$ is finite and an ideal I of $\mathcal{R}$ is S-finite if $sI \subseteq J \subseteq I$ for some finitely generated ideal J of $\mathcal{R}$ and some $s \in S$.

In 2021, Fuad Ali Ahmahdi, El Mehdi Bouba and Mohammed Tamekkante [12] introduced the idea of weakly S-prime ideal which is the generalization of weakly prime ideals of a ring $\mathcal{R}$, an ideal I of $\mathcal{R}$ disjoint with S and if there exists $s \in S$ such that for all $a, b \in \mathcal{R}$, if $0 \neq ab \in I$, then $sa \in I$ or $sb \in I$. In the same year, Wala'a Alkasasbeh and Malik Bataineh [21] introduced a proper ideal P called almost S-prime ideal if there exists $s \in S$ such that for all $x, y \in \mathcal{R}$ and if $xy \in P - P^2$, then $sx \in P$ or $sy \in P$. Moreover, an almost S-prime ideal in a ring $\mathcal{R}$ with unity is generalized.

Recently, Mohamed Aqalmoun [19] determined the S-prime ideal and S-maximal ideal of a principal domain. They have also shown that the intersection of the descending chain of S-prime ideals in a principal domain is an S-prime ideal. In the present paper, the intersection of S-prime ideals in a commutative ring $\mathcal{R}$ with unity is discussed.

In 2023, Kalamani and Mythily [14] introduced a new graph called S-prime ideal graph of a finite commutative ring with unity, which is from the algebraic definition of the S-prime ideal of $\mathcal{R}$. Let a and b be the vertices of an S-prime ideal graph from the elements of the finite commutative ring $\mathcal{R}$ with unity. The S-prime ideal graph is denoted by $G_{S_d}(\mathcal{R})$, where S_d is the S-prime ideal of $\mathcal{R}$. The two vertices a and b of a graph are joined by an edge if sa or sb is in the S-prime ideal for some s in the multiplicative subset S of $\mathcal{R}$ whenever the product ab is in the S-prime ideal. Further, some algebraic and graph theoretic properties of the S-prime ideal of $\mathcal{R}$ are interpreted.

Kalamani and Ramya [15] also introduced a new graph from the algebraic definition called product maximal graph of a finite commutative ring. They [16] have interpreted some basic properties of product maximal graph and explored some topological indices using resistance distance matrix for the product maximal graph. In 2022, Kiruthika and Kalamani [17] introduced a new graph called the vertex order graph and studied their complements. It is a simple graph and there is an edge between any two vertices if and only if its orders are equal.

The S-prime ideal of a commutative ring $\mathcal{R}$ is denoted by I_s in this present paper. In section 2, some basic definitions derived from Abstract Algebra by Dummit and Foote [9] and Introduction to Lattices and Order by Davey and Priestley [8] are given. In section 3, some basic properties of the S-prime ideals are discussed and in sections 4 and 5, the S-prime ideals on local and semilocal rings are studied.

2. Preliminaries

Some fundamental definitions of ideal, nilpotent and idempotent element, and so on are given in this section, and they have been studied in Abstract Algebra by Dummit and Foote [9]. A partially ordered set and Zorn's lemma are also given here and studied in Introduction to Lattices and Order by Davey and Priestley [8].

Definition 2.1. A subring of the ring $\mathcal{R}$ is a subgroup of $\mathcal{R}$ that is closed under multiplication.

Definition 2.2. A nonzero element a of $\mathcal{R}$ is called a zero-divisor if there is a non-zero element b in $\mathcal{R}$ such that either $ab = 0$ or $ba = 0$. The set of all zero-divisors is denoted by $Z(\mathcal{R})$.

Definition 2.3. An element $u \in \mathcal{R}$ is called a unit if and only if there exists an element $v \in \mathcal{R}$ such that $uv = 1$, where 1 is the unity in $\mathcal{R}$. The set of all units is denoted by $U(\mathcal{R})$.

Definition 2.4. A subring I of a ring $\mathcal{R}$ is called an ideal of $\mathcal{R}$ if for every $r \in \mathcal{R}$ and every $i \in I$ both ri and ir are in I.

Definition 2.5. Let $\mathcal{R}$ be a commutative ring and an element $a \in \mathcal{R}$ is called nilpotent if $a^n = 0$ for some $n \in \mathbf{Z}^+$. The set of all nilpotent elements of $\mathcal{R}$ form an ideal called nilradical and it is denoted by $N(\mathcal{R})$.

Definition 2.6. An ideal I is called nilpotent ideal if I^n is the zero ideal for some $n \geq 1$.

Definition 2.7. A proper ideal M in an arbitrary ring $\mathcal{R}$ is called a maximal ideal if $M \neq \mathcal{R}$ and the only ideals containing M are M and $\mathcal{R}$.
If a ring $\mathcal{R}$ has a unique maximal ideal, then the ring is called a local ring, otherwise it is a semilocal ring.

Definition 2.8. An ideal P in a ring $\mathcal{R}$ is called a prime ideal if $P \neq \mathcal{R}$ and whenever the product ab of two elements $a, b \in \mathcal{R}$ is an element of P, then atleast one of a and b is an element of P.

Definition 2.9. A nonempty subset S of a ring $\mathcal{R}$ is said to be multiplicatively closed and if it contains a multiplicative identity $1 \in S$ and if $a, b \in S$, then $ab \in S$.

Definition 2.10. An element e in a commutative ring $\mathcal{R}$ with unity is called an idempotent if $e^2 = e$.

Definition 2.11. Let $\mathcal{R}$ be a commutative ring , $S \subseteq \mathcal{R}$ a multiplicative set and I an ideal of $\mathcal{R}$ disjoint with S. It is said that I is S-prime if there exists an $s \in S$ such that for all $a, b \in \mathcal{R}$ with $ab \in I$, there is either $sa \in I$ or $sb \in I$.

Definition 2.12. A commutative ring $\mathcal{R}$ with 1 is called Noetherian if every ideal of $\mathcal{R}$ is finitely generated.

Definition 2.13. Let X be a set and the relation $\leqslant$ on a set X is said to be partially ordered set if the following properties are hold,

(1) Reflexivity: $x \leqslant x$.

(2) Antisymmetry: If $x \leqslant y$ and $y \leqslant x$ then $x = y$.

(3) Transitivity: If $x \leqslant y$ and $y \leqslant z$ then $x \leqslant z$.

for all x, y, z in the set X.

Zorn's lemma:

If every chain in a subset T of a partially ordered set X has an upper bound in X, then X has at least one maximal element.

3. Properties of S-Prime Ideal

In this section, some of the properties of the S-prime ideal of $\mathcal{R}$ with unity are presented which are not discussed in [1]. Ahmed Hamed and Archaf Maliek [1] explained some of the properties of the S-prime ideals in Noetherian ring. In the following theorem, it is proved that the nilradical of $\mathcal{R}$ is the S-prime ideal of $\mathcal{R}$. Further, the properties of the ideals of $\mathcal{R}$ are applied to the S-prime ideals of $\mathcal{R}$.

Theorem 3.1. *Every S-prime ideal I_s is proper.*

Proof. Let $\mathcal{R}$ be a commutative ring with unity and I_s be the S-prime ideal of $\mathcal{R}$. Let S be a multiplicative subset of $\mathcal{R}$ which is disjoint from I_s.

By the definition of multiplicative subset of $\mathcal{R}$, the multiplicative identity $1 \in S$. It means that $1 \notin I$. If I_s has no multiplicative identity, then, $I_s \neq \mathcal{R}$.

Thus, every S-prime ideal is proper. □

In general, the union of an ideal of $\mathcal{R}$ need not be an ideal, which is true when an ideal is contained in another ideal. This condition is used on S-prime and the following theorem is proven.

Theorem 3.2. *Let I_s and J_s be the S-prime ideals of $\mathcal{R}$ and if one is contained in the other, then their union is also an S-prime ideal of $\mathcal{R}$.*

Proof. Let $a, b \in \mathcal{R}$ such that $ab \in I_s \cup J_s$ which implies that $ab \in I_s$ or $ab \in J_s$.

The S-prime ideal I_s is contained in an S-prime ideal J_s.

Therefore $I_s \cup J_s$ is an ideal of $\mathcal{R}$ and the multiplicative subset S of $\mathcal{R}$ which is disjoint from an ideal $I_s \cup J_s$ is considered.

If $ab \in J_s$, then sa or sb in J_s for some s in S. Therefore sa or $sb \in I_s \cup J_s$ for some s in S.

Thus, an ideal $I_s \cup J_s$ is also an S-prime ideal of $\mathcal{R}$. □

The following theorem shows that an ideal $N(\mathcal{R})$ is an S-prime ideal of a ring $\mathcal{R}$ with unity.

Theorem 3.3. *Let $\mathcal{R}$ be a ring. If an ideal $N(\mathcal{R})$ is nilradical of $\mathcal{R}$, then it is an S-prime ideal of $\mathcal{R}$.*

Proof. Let $N(\mathcal{R})$ be an ideal containing all the elements that are nilpotent of $\mathcal{R}$ and S be a multiplicative subset of $\mathcal{R}$ which is disjoint from $N(\mathcal{R})$. Since $N(\mathcal{R})$ is a proper ideal of $\mathcal{R}$ such that the multiplicative identity $1 \notin N(\mathcal{R})$.

Choose $a, b \in \mathcal{R}$ such that $ab \in N(\mathcal{R})$. If either a or $b \in N(\mathcal{R})$, then $sa \in N(\mathcal{R})$ or $sb \in N(\mathcal{R})$ for all $s \in S$.

Let it be assumed that both a and b are not in $N(\mathcal{R})$. Since ab is a nilpotent element, ab is a zero-divisor. Then either a or b is a zero-divisor of a ring $\mathcal{R}$.

Let it be assumed that the element a is a zero-divisor which is not in $N(\mathcal{R})$. Choose the multiplicative subset S as the set generated by the element a.

In this case, $s = a$ can be taken. Since $ab \in N(\mathcal{R}), (ab)^n = 0$. This implies that $sb \in N(\mathcal{R})$ for some $s \in S$.

Thus, an ideal $N(\mathcal{R})$ is an S-prime ideal of a commutative ring $\mathcal{R}$ with unity. □

Corollary 3.4. *In a field, trivial ideal is the only S-prime ideal.*

Example 3.5. Consider the ring of integers $\mathbb{Z}$. The S-prime ideals I_s of $\mathbb{Z}$ are $\{0\}$ and all the prime ideals of $\mathbb{Z}$. The nilradical of $\mathbb{Z}$ is the zero-ideal.

In [9], an ideal I of a ring $\mathcal{R}$ intersection with the subring A of a ring $\mathcal{R}$ is also an ideal of a subring A. The following theorem shows that an intersection of the S-prime ideal with the subring of a commutative ring $\mathcal{R}$ is the S-prime ideal of a subring of $\mathcal{R}$.

Theorem 3.6. *Let I_s be an S-prime ideal of $\mathcal{R}$ and S be a multiplicative subset of $\mathcal{R}$. If A is a subring of $\mathcal{R}$, then $I_s \cap A$ is an S-prime ideal of A.*

Proof. Let $S \subseteq A$ be a multiplicative subset of $\mathcal{R}$ which is disjoint from the S-prime ideal I_s. Let A be a subring of $\mathcal{R}$.

Since I_s is an ideal of $\mathcal{R}$ and A is a subring of $\mathcal{R}$, $I_s \cap A$ is an ideal of A. The set S can be considered as the multiplicative subset of A which is disjoint from $I_s \cap A$.

Choose $a, b \in A$ such that $ab \in I_s \cap A$ which implies that $ab \in I_s$ and $ab \in A$. Since I_s is an S-prime ideal of $\mathcal{R}$, $sa \in I_s$ or $sb \in I_s$ for some $s \in S$.

A is a subring of R such that $sa \in A$ and $sb \in A$ for any $s \in S$. Therefore $sa \in I_s \cap A$ or $sb \in I_s \cap A$ for some $s \in S$.

Thus, $I_s \cap A$ is an S-prime ideal of A. □

The converse of the Theorem 3.6 is not true. Let $A = \mathbb{Z}$ and $\mathcal{R} = \mathbb{Q}$ be taken. Since $\mathbb{Q}$ is a field, $I_s =< 0 >$ is the only S-prime ideal of $\mathbb{Q}$.
Any S-prime ideal $< p >$ of A is not in the form of $I_s \cap A$, where p is a prime.

Theorem 3.7. *Let S be a multiplicative subset of $\mathcal{R}$ which is disjoint from the S-prime ideal I_s of $\mathcal{R}$ and then $SI_s = I_s$.*

Proof. Let I_s be the S-prime ideal of a ring $\mathcal{R}$ which is disjoint from the multiplicative set $S \subseteq \mathcal{R}$ so that the S-prime ideal I_s contains the product SI_s. That is, $SI_s \subseteq I_s$.

Let a be an element of I_s and it is not in the multiplicative subset S of $\mathcal{R}$. The element a in I_s can be written as $a = 1.\, a$ where $1 \in S$. Therefore $a \in SI_s$.

Thus, $SI_s = I_s$. □

Corollary 3.8. *An S-prime ideal of $\mathcal{R}$ is the zero-ideal of $\mathcal{R}$ and then $SI_s = \{0\}$.*

4. S-Prime Ideal in Local Ring

In this section, the ring $\mathcal{R}$ is considered as local which contains a unique maximal ideal of $\mathcal{R}$ and in [8] every maximal ideal is the prime ideal of a ring $\mathcal{R}$. By [1], every prime ideal is the S-prime ideal of a ring $\mathcal{R}$ with unity. If e is any idempotent element of $\mathcal{R}$, then $e\mathcal{R}$ is an ideal of $\mathcal{R}$.

In this section, the result is based on idempotent element of $\mathcal{R}$ as some idempotent element e of $\mathcal{R}$ generates the S-prime ideal of $\mathcal{R}$. The authors [4] have shown that the alternative definition of nilradical of a ring is the intersection of all the prime ideals of a ring by using Zorn's lemma. In the present study also, Zorn's lemma is used to prove the following equivalent conditions.

Theorem 4.1. *If $\mathcal{R}$ is a commutative ring with unity, the following are equivalent:*

1) *$\mathcal{R}$ is a local ring*
2) *$\mathcal{R}$ has a unique S-prime ideal*
3) *Every zero-divisor of $\mathcal{R}$ is a nilpotent element of $\mathcal{R}$.*

Proof. Let it be assumed that $\mathcal{R}$ is a local ring. It has a unique maximal ideal I_s which is the S-prime ideal I_s of $\mathcal{R}$. Let S be the multiplicative subset of $\mathcal{R}$.

As I_s is maximal, the multiplicative subset S contains only units of a ring $\mathcal{R}$.

If I'_s is an another S-prime ideal of $\mathcal{R}$, then $I'_s : s$ is the prime ideal of $\mathcal{R}$ for some s in S which is disjoint from I'_s. Since $s \in S$ is a unit, $I'_s : s = I'_s$.

It implies that I'_s is prime, but I_s is the unique prime ideal and hence, $I_s = I'_s$.

Let it be assumed that the ring $\mathcal{R}$ has a unique S-prime ideal and to prove that every zero-divisor of a ring is nilpotent.

A non-zero element $a \in Z(\mathcal{R})$ not in $N(\mathcal{R})$ can be considered, where $N(\mathcal{R})$ is the set of all nilpotent elements of $\mathcal{R}$ and let X_a is the collection of ideals I_s with the following characteristic

$$X_a = \{I_s \subset \mathcal{R} \mid a^n \notin I_s, for\ all\ n > 0\}.$$

As the ideal $\{0\}$ is in a set X_a, X_a is non-empty and any non-zero ideals in a set X_a contain zero-ideal $\{0\}$ of $\mathcal{R}$. This implies the set X_a satisfies the properties of partial order via inclusion.

Then applying the Zorn's lemma on the set X_a, it has the maximal ideal M in a set X_a.

It is needed to show that M is an S-prime ideal of $\mathcal{R}$, where S is the multiplicative subset of $\mathcal{R}$ which is disjoint from M.

Let $a, b \in \mathcal{R}$ and $ab \in M$. If $sa \notin M$ and $sb \notin M$ for all $s \in S$, then $M + < a >$ is not contained in M and $M + < b >$ is not contained in M and hence $M + < a >$ and $M + < b >$ are not in the set X_a. Therefore a^n is in $M + < a >$ and a^m is in $M + < b >$ for some $m \neq n$.

This implies that a^{n+m} is in $M + < ab >$ and hence $M + < ab >$ is not in the set X_a. It contradicts to the present assumption that ab is an element of M. Hence M is the only S-prime ideal of a ring $\mathcal{R}$.

As a^n is not in M, a is also not in M. It means that a is not a zero-divisor of $\mathcal{R}$. It also contradicts the current assumption that a is non-nilpotent and zero-divisor, hence every zero-divisor is a nilpotent.

Let it be considered that every zero-divisor of $\mathcal{R}$ is a nilpotent element of $\mathcal{R}$ and to prove that the commutative ring $\mathcal{R}$ with unity is local.

Since $N(\mathcal{R})$ is an ideal of $\mathcal{R}$, $Z(\mathcal{R})$ is an ideal of $\mathcal{R}$ and it contains all the zero-divisors and it is a maximal ideal of $\mathcal{R}$. There is no other maximal ideal exists in $\mathcal{R}$.

Thus, $Z(\mathcal{R})$ is the unique maximal ideal of $\mathcal{R}$. Hence $\mathcal{R}$ is local. □

Example 4.2. Let a finite ring $\mathcal{R} = \mathbb{Z}_8$ be considered. The ring $\mathbb{Z}_8$ is a local ring having a unique S-prime ideal $< 2 >$. The zero-divisors $Z(\mathbb{Z}_8)$ are $\{0, 2, 4, 6\}$ and the nilradical $N(\mathbb{Z}_8)$ are $\{0, 2, 4, 6\}$. Thus, $Z(\mathcal{R}) = N(\mathcal{R})$.

Theorem 4.3. *In a field, an ideal generated by an element $e = 0$ is the S-prime ideal if and only if it is the prime ideal.*

Proof. Let it be assumed that $< e >$ is the S-prime ideal of a field.

In a field, $\{0\}$ is the only proper ideal of $\mathcal{R}$ by Theorem 3.1 and by above Theorem 4.1 it is the maximal ideal of a ring $\mathcal{R}$ since every maximal ideal is the prime ideal of $\mathcal{R}$.

Thus, $< e >= 0$ is the only S-prime ideal of a field.

Conversely, it can be assumed that the ideal generated by e is the prime of a field. By [1], the prime ideal is S-prime ideal of $\mathcal{R}$. □

Corollary 4.4. *An S-prime ideal of a field is nilpotent.*

5. S-Prime Ideals in Semilocal Ring

In this section, $\mathcal{R}$ is considered as the semilocal ring with unity. The semilocal rings play an important role in algebra which contains more than one maximal ideal.

Theorem 5.1. *In a ring $\mathcal{R}$, a non-prime S-prime ideal is a nilpotent ideal.*

Proof. Let S be the multiplicative subset of $\mathcal{R}$ and I_s be the non-prime S-prime ideal of the ring $\mathcal{R}$ which is disjoint from the set S.

It is needed to prove that I_s is a nilpotent ideal of $\mathcal{R}$.

Let $P_1, P_2, P_3, ..., P_r$ be the prime ideals of a ring $\mathcal{R}$ where $r > 1$.

As a non-prime ideal is contained in a prime ideal of $\mathcal{R}$ and the ideal I_s is non-prime which is properly contained in a prime ideal, say P_1.

Every element of I_s will be of the form $k_1 p_1$ where $k_1 \in \mathcal{R}$ and $p_1 \in P_1$.

As I_s is non-prime, both k_1 and p_1 are not in I_s, but sa or sb is in I_s for some s in the multiplicative subset S.

As s and k_1 are not in I_s, also sk_i is not in I_s for all $i = 1, 2, 3, ..., r$. This implies that sp_1 is an element of I_s. The element s in S is not in the prime ideal P_1 and since s is the zero-divisor, it will be an element of any prime ideal, except P_1.

Let s be in P_2 such that $s = k_2 p_2$ where $k_2 \in R$, $p_2 \in P_2$.

This implies that $k_2 p_1 p_2$ is the element of I_s.

Repeating the same process for all the prime ideals $P_1, P_2, P_3, ..., P_r$, the element of I_s is of the form $k.p_1.p_2...p_r$ where $k \in \mathcal{R}$ and $p_i \in P_i$ for $i = 1, 2, 3, ..., r$. This means that I_s is contained in the intersection of all the prime ideals $P_1, P_2, P_3, ..., P_r$.

As it is known that the intersection of all the prime ideals is the nilradical $N(\mathcal{R})$ of a ring $\mathcal{R}$, the S-prime ideal I_s is contained in $N(\mathcal{R})$ and hence I_s is the nilpotent ideal of a ring $\mathcal{R}$. □

Corollary 5.2. *Let $\mathcal{R}$ be a commutative ring with unity. The S-prime ideals of $\mathcal{R}$ are either prime or nilpotent.*

Example 5.3. Consider the ring $\mathcal{R} = \mathbb{Z}_{36}$ and the S-prime ideals of $\mathcal{R}$ are $< 2 >, < 3 >, < 6 >, < 12 >$ and $< 18 >$. The non-prime S-prime ideals of $\mathcal{R}$ are $< 6 >, < 12 >$ and $< 18 >$. The nilpotent ideals of a ring $\mathbb{Z}_{36}$ are $< 6 >, < 12 >, < 18 >$ and $< 36 >$.

Thus, the non-prime S-prime ideals are nilpotent ideals of a ring $\mathbb{Z}_{36}$.

The converse of the Theorem 5.1 is not true and the nilpotent ideal $< 36 >$ is not an S-prime ideal of the ring $\mathbb{Z}_{36}$.

Theorem 5.4. *Let I_s and J_s be the S-prime ideals of $\mathcal{R}$. If either I_s or J_s is a non-prime S-prime ideal, then $I_s + J_s$ is also an S-prime ideal of $\mathcal{R}$.*

Proof. Let I and J be the ideals of $\mathcal{R}$ and the sum of two ideals is an ideal of $\mathcal{R}$ which contains both I and J.

First, the S-prime ideals in a commutative ring $\mathcal{R}$ are generalized.

By Theorem 3.3, $N(\mathcal{R})$ is an S-prime ideal of $\mathcal{R}$ and by Theorem 5.1, non-prime ideals are nilpotent ideals in a semilocal ring $\mathcal{R}$. By Theorem 4.1, the S-prime ideal is unique in a local ring.

Thus, if the S-prime ideal I_s or J_s is non-prime and the non-prime ideals are nilpotent contained in the prime ideals of $\mathcal{R}$.

$\therefore$ Sum of two S-prime ideals is also an S-prime ideal of $\mathcal{R}$. $\square$

Example 5.5. Let $\mathcal{R} = \mathbb{Z}_{36}$ be a finite commutative ring with unity.

1) Let the ideals $I_s =< 12 >$ and $J_s =< 18 >$ which are the S-prime ideals of $\mathbb{Z}_{36}$. In this, both I_s and J_s are the non-prime S-prime ideals of $\mathbb{Z}_{36}$. Then $I_s + J_s$ is the S-prime ideal generated by the element 6 of $\mathbb{Z}_{36}$.

2) If $I_s =< 2 >$ and $J_s =< 3 >$ are the prime ideals of $\mathbb{Z}_{36}$ which are the S-prime ideals of $\mathbb{Z}_{36}$, then $I_s + J_s = \mathbb{Z}_{36}$.

$\mathbb{Z}_{36}$ is not an S-prime ideal by Theorem 3.1.

Theorem 5.6. *Let J be an ideal of $\mathcal{R}$ such that $J \cap S \neq \phi$. If I_s is a non-prime S-prime ideal of $\mathcal{R}$, then $I_s \cap J$ is an S-prime ideal of $\mathcal{R}$.*

Proof. Let $a, b \in \mathcal{R}$ such that $ab \in I_s \cap J$ and the multiplicative subset S of a commutative ring $\mathcal{R}$ with unity is disjoint from the intersection of I_s and J.

Since I_s is an S-prime ideal of a ring $\mathcal{R}$ and $ab \in I_s$, there exists s_1 in S such that $s_1 a \in I_s$ or $s_1 b \in I_s$.

Let it be assumed that $s_1 a \in I_s$. The intersection of J and S is non-empty such that $s_2 \in J$ implies $s_2 a \in J$ for $a \in \mathcal{R}$. S is the subset of $\mathcal{R}$ and s_1 and s_2 are the elements of $\mathcal{R}$ such that $s_1 s_2 a \in I_s$ and $s_1 s_2 a \in J$.

Thus, $s_1 s_2 a \in I_s \cap J$.

Since the set S is multiplicatively closed, there is some $s \in S$ such that $s = s_1 s_2$. It implies that for some s in S, $sa \in I_s \cap J$.

So, $I_s \cap J$ is the S-prime ideal of $\mathcal{R}$. $\square$

Theorem 5.7. *An ideal generated by a non-zero idempotent element e is S-prime if and only if it is the prime ideal of a ring $\mathcal{R}$.*

Proof. Let $I =< e >$ be the prime ideal of $\mathcal{R}$ where e is the non-zero idempotent element of $\mathcal{R}$. In [1], every prime ideal is the S-prime ideal and hence I is the S-prime ideal of $\mathcal{R}$.

Conversely, let it be assumed that the ideal generated by a non-zero idempotent element e of $\mathcal{R}$ is S-prime ideal. As e is a non-zero idempotent, it is not a nilpotent element of R. Hence the ideal I is not nilpotent.

By Theorem 5.1, I is a prime ideal of $\mathcal{R}$.

Thus, the ideal generated by the non-zero idempotent element of $\mathcal{R}$ is the prime ideal of $\mathcal{R}$. □

Example 5.8. Let a finite ring of order 12 be considered and its non-zero idempotent elements are $1, 4$ and 9. The ideal generated by 4 is $\{0, 4, 8\}$ which is not an S-prime ideal of $\mathbb{Z}_{12}$. The element 9 generates an ideal $\{0, 3, 6, 9\}$ which is an S-prime ideal of $\mathbb{Z}_{12}$ which in turn is the prime ideal of $\mathbb{Z}_{12}$.

In [14], the S-prime ideal of a finite commutative ring with unity is explained and denoted by S_d where d is the divisor of the order of the ring $\mathcal{R}$. The following Table:1 gives the generalized S-prime ideals and their multiplicative subset S of a commutative ring $\mathcal{R}$ with unity for some of its order.

S.No	$\mathcal{R}=\mathbb{Z}_n$	**S-prime ideal S_d of $\mathcal{R}$**	**Multiplicative subsets of $\mathcal{R}$**
1	$n = p$	S_p	$\mathcal{R} - \{0\}$
2	$n = pq$	S_p	$\mathcal{R} - S_p$
		S_q	$\mathcal{R} - S_q$
		S_{pq}	Either $\mathcal{R} - S_p$ or $\mathcal{R} - S_q$
3	$n = pqr$	S_p	$\mathcal{R} - S_p$
		S_q	$\mathcal{R} - S_q$
		S_r	$\mathcal{R} - S_r$
		S_{pq}	$\mathcal{R} - S_p$ or $\mathcal{R} - S_q$
		S_{pr}	$\mathcal{R} - S_p$ or $\mathcal{R} - S_r$
		S_{qr}	$\mathcal{R} - S_q$ or $\mathcal{R} - S_r$
		S_{pqr}	$\mathcal{R} - S_p$ or $\mathcal{R} - S_q$ or $\mathcal{R} - S_r$
4	$n = p^t q, t > 1$	S_p	$\mathcal{R} - S_p$
		S_q	$\mathcal{R} - S_q$
		S_{pq}	Either $\mathcal{R} - S_p$ or $\mathcal{R} - S_q$
		$S_{p^2 q}$	Either $\mathcal{R} - S_p$ or $\mathcal{R} - S_q$
		$S_{p^3 q}$	Either $\mathcal{R} - S_p$ or $\mathcal{R} - S_q$
		.	.
		.	.
		.	.
		$S_{p^t q}$	Either $\mathcal{R} - S_p$ or $\mathcal{R} - S_q$

Table:1

6. Conclusion

In this paper, some of the basic properties, namely sum, union and intersection of the S-prime ideals of a ring $\mathcal{R}$ with unity are discussed and it is proved that the nilradical ideals are the S-prime ideal of $\mathcal{R}$. Moreover, the S-prime ideals on a local and semilocal rings $\mathcal{R}$ are characterized. At the end of this paper, it is proved that some idempotent elements generate an S-prime ideal in a commutative ring $\mathcal{R}$ with unity and the S-prime ideals of a finite commutative ring are generalized.

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