J. Appl. Math. & Informatics Vol. 42(2024), No. 5, pp. 1155 - 1170
https://doi.org/10.14317/jami.2024.1155

A STUDY OF LINEAR MAPPING PRESERVING PYTHAGOREAN ORTHOGONALITY IN INNER PRODUCT SPACES†

S. SYLVIANI*, A. TRISKA, L. RATHOUR, H. FULHAMDI, D.A. KUSUMA, K. PARMIKANTI, F.C. PERMANA

Abstract. The concept of orthogonality is widely used in various fields of study, both within and outside the scope of mathematics, especially algebra. The concept of orthogonality gives a picture of the relationship between two vectors that are perpendicular to each other, or the inner product in both of them is zero. However, the concept of orthogonality has undergone significant development. One of the developments is Pythagorean orthogonality. In this paper, it is explored topics related to Pythagorean orthogonality and linear mappings in inner product spaces. It is also examined how linear mappings preserve Pythagorean orthogonality and provides insights into how mathematical transformations affect geometric relationships. The results reveal several properties that apply to linear mappings preserving Pythagorean orthogonality.

AMS Mathematics Subject Classification : 15A86, 47B49, 47B50.
Key words and phrases : Linear mapping, Pythagorean orthogonality, orthogonality preserving mapping.

1. Introduction

The concept of orthogonality emerges as a fundamental cornerstone in the rich landscape of mathematical structures, influencing the understanding of vector spaces and their geometric relationships. The use of the vector concept has not only developed within pure mathematics but also finds extensive application in applied research, see for example [25, 26]. The intriguing interplay between vectors and angles lies at the heart of this concept, giving rise to the notion of orthogonality. At the core of our exploration rests the concept of an inner

Received December 13, 2023. Revised June 19, 2024. Accepted July 9, 2024. *Corresponding author.
†This research was funded by Hibah Riset Unpad Riset Data Pustaka Daring (RDPD) Universitas Padjadjaran grant number 1549/UN6.3.1/PT.00/2023.

product space, a space where vectors possess not only magnitude and direction but also equipped with a mathematical operation that quantifies the angle between them. This operation, known as the inner product, encapsulates fundamental ideas such as length, projection, and orthogonality within a unified framework.

The concept of orthogonality boasts multiple applications in diverse fields, including its relevance in cipher theory and cryptography [12]. However, it is important to note that the concept of orthogonality has been extensively explored by researchers, leading to the emergence of various types of orthogonality. Notable among these developments are Pythagorean orthogonality [8, 22, 38], isosceles orthogonality [22, 8, 40], Birkhoff-James orthogonality [6], Roberts' orthogonality [6, 7], and Bisectrix orthogonality [39].

The Pythagorean concept, known through the Pythagorean Theorem, is a fundamental principle in mathematics, particularly in geometry and linear algebra. The theorem states that in a right-angled triangle, the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides. Beyond its crucial role in Euclidean geometry, the Pythagorean concept has extensive applications in various fields of mathematical and physical research. For instance, in the [33] that produces a significant advancement in the study of fuzzy set theory and group theory by redefining key concepts and providing a new framework for understanding Pythagorean Fuzzy Subgroups and Fermatean Fuzzy Subgroups in the context of t-norm and t-conorm functions. [28] proposes a mediative Pythagorean fuzzy technique with a novel mediative correlation coefficient for multi-criteria in decision-making and its implementation in diagnostic process. There are many other studies that have developed the Pythagorean concept in various fields ranging from mathematics itself to applications [**?**, 18, 19, 30, 31, 32, 34, 35, 36, 14].

Earlier research by [11] explored orthogonality in the context of Hilbert spaces, which are one of the best known inner product spaces. They emphasized the importance of orthogonal preserving structures in the development of operator theory. Furthermore, a study by [20] examined isometric mappings that preserve distance and, thus, orthogonality in Banach spaces. Although important, this research focuses more on isometric mapping without examining in more depth the general linear mapping that preserves Pythagorean orthogonality specifically. Research by [15] and [16] also made important contributions by exploring automorphism in inner product-preserving Hilbert spaces. However, there are still a few studies that specifically examine the conditions under which linear mappings preserving Pythagorean orthogonality.

While those studies provide a solid foundation for our understanding of linear mapping and orthogonality, there are several gaps that need to be filled, including most research focuses on distance-preserving isometry mapping. This research will examine a more general linear mapping, not limited to isometry, but still preserving Pythagorean orthogonality. Furthermore, there are no studies that discuss the characterization of linear mappings that preserve Pythagorean orthogonality in inner product spaces.

Linear mappings, with their ability to transform vectors while preserving linear relationships, assume a pivotal role in our exploration. The notion of Pythagorean orthogonality-preserving linear mapping unveils a captivating dimension: the assurance that if two vectors are orthogonal in their original space, their transformed images under a linear transformation will retain this orthogonal property in the new space. This transformative power upholds the integrity of orthogonality amidst mathematical metamorphosis, offering profound insight into the underlying structures and symmetries that govern vector spaces.

This research was carried out to study the concept and properties of Pythagorean orthogonality, a development of the standard orthogonality concept often encountered. In this article, the concept and properties of orthogonality-preserving mappings previously studied by [13], are developed for Pythagorean orthogonality in the scope of inner product spaces. The characterization of linear mappings that preserve Pythagorean orthogonality is also discussed. The contribution of this research includes an analysis of mapping that preserves Pythagorean orthogonality and its characteristics. Another contribution is that this article provides insight into the development of the standard orthogonality concept and its relationship to the concept of mapping that preserves this property.

On the other hand, in [37], it is states that any quantum mechanical invariance transformation (symmetry transformation) can be represented by a unitary or antiunitary operator on a complex Hilbert space and that, conversely, any operator of that kind represents an invariance transformation. In [29], it is generalized this result by requiring only that T preserves the orthogonality between the one-dimensional subspaces of H. By highlighting the characterization of linear mapping which preserves Pythagorean orthogonality, it provides a bridge for studies in fields of application, one of which is quantum mechanics. By highlighting the properties that apply to mapping that preserves Pythagorean orthogonality, this article also becomes a bridge for studies in fields of application, one of which is quantum mechanics.

2. Methodologies

2.1. Research Methods. To achieve the research objectives and make a significant contribution to understanding linear mappings that preserve Pythagorean orthogonality in inner product spaces, we employ a systematic and structured methodological approach. This research method comprises several key stages designed to identify, characterize, and develop theories and applications of such mappings.

(1) The initial step of this research is to conduct a comprehensive literature review to understand the context and recent developments in the field of linear mappings and orthogonality.

(2) Establishing clear and consistent definitions and notations to be used throughout this study. This includes definitions of inner product spaces and Pythagorean orthogonality and formal definitions of linear mappings that preserve Pythagorean orthogonality
(3) This step involves the development of theories and mathematical proofs necessary to identify and characterize linear mappings that preserve Pythagorean orthogonality.

2.2. Terms and Definition. To further understand the concepts and findings presented in this research, it is essential to establish the terms and definitions that will be used throughout the article. This section will detail the key terminology and notation related to inner product spaces, Pythagorean orthogonality, and relevant linear mappings. A clear understanding of these terms and definitions will help readers follow the arguments and results of the research more effectively.

Definition 2.1 ([23]). A norm on a vector space D is any function $\|\cdot\| : D \to R$ with the properties of vector length:

(1) $\|\cdot\| \geq 0$ and $\|c\| = 0$ if and only if $c = 0$.
(2) $\|c + d\| \leq \|c\| + \|d\|$
(3) $\|\phi c\| = |\phi| \, \|c\|$
(4) $\|c + d\| \leq \|c\| + \|c\|$

where $c, d \in D$ and $\phi \in R$ then. A vector space with a norm is called a normed space

Definition 2.2 ([5]). An inner product on a real vector space A is a function that associates a real number $\langle c, d\rangle$ with each pair of vectors in A in such a way that the following axioms are satisfied for all vectors u, v, and e in V and all scalars k:

$$\begin{array}{lcc} (i) & \langle c, d\rangle = \langle d, c\rangle & (Symmetry) \\ (ii) & \langle c+d, e\rangle = \langle c, e\rangle + \langle d, e\rangle & (Homogeneity) \\ (iii) & \langle kc, d\rangle = k\,\langle c, d\rangle & (Homogeneity) \\ (iv) & \langle c, c\rangle \geq 0 & and \\ & \langle c, c\rangle = 0 \Leftrightarrow c = 0 & (Positivity) \end{array}.$$

A real vector space with an inner product is called a real inner product space

Definition 2.3 ([5]). If D is an inner product space, the norm or length of $c \in D$ is defined by $\|c\| = \sqrt{\langle c, c\rangle}$

Theorem 2.4 ([24]). *The parallelogram law states that the norm induced by a scalar product satisfies*

$$\|c + d\|^2 + \|c - d\|^2 = 2\left(\|c\|^2 + \|d\|\right)^2.$$

In this context, c and d represent elements of D as explained in Definition 2.3

Theorem 2.5 ([24]). *The induced norm can also be obtained from the inner product using a formula known as the polarization identity,*

$$\langle c, d \rangle = \frac{1}{4} \|c + d\|^2 - \frac{1}{4} \|c - d\|^2 ,$$

where c,d belong to an inner product space.

Corollary 2.6. $\|c - d\|^2 = \|c + d\|^2 - 4 \langle c, d \rangle$.

Definition 2.7 ([5]). Two nonzero vectors a and b in R^n are said to be orthogonal $(a \perp b)$ if $\langle a, b \rangle = 0$.

Definition 2.8 ([5]). If $L : D \to E$ is a mapping from a vector space D to a vector space E, then L is called a linear transformation from D to E if the following two properties hold for all vectors u and v in D and for all scalars k:

(1) $L(c + d) = L(c) + L(d)$.

(2) $L(kc) = kL(c)$.

In the special case where $D = E$, the linear transformation L is called a linear operator on the vector space D.

Definition 2.9 ([4]). Two nonzero vectors a and b in R^n are said to be Pythagorean orthogonal $(a \perp_p b)$ if $\|a + b\|^2 = \|a\|^2 + \|b\|^2$

Definition 2.10 ([27]). Let A and B be two inner product spaces. A mapping $L : A \to B$ is called orthogonality preserving if:

$$\forall a, b \in A, a \perp b \Rightarrow L(a) \perp L(b).$$

Definition 2.11 ([27]). Let A and B be two inner product spaces. A mapping $L : A \to B$ is called strongly orthogonality preserving if the above conditions apply both ways:

$$\forall a, b \in A, a \perp b \Leftrightarrow L(a) \perp L(b).$$

3. Main results

3.1. Properties of Pythagorean Orthogonality. Pythagorean orthogonality can be seen as an extension of standard orthogonality, illustrated in the theorem below. This concept introduces a new perspective on traditional notions of orthogonality.

Theorem 3.1. *If A is an inner product space and $a, b \in A$, then $a \perp b \Leftrightarrow a \perp_p b$.*

Proof. [$\Rightarrow$]Suppose $a, b \in A$ where A is an inner product space, and we have $a \perp b$ means $\langle a, b \rangle = 0$, it will be shown $a \perp_p b$, it must be shown $\|a + b\|^2 = \|a\|^2 + \|b\|^2$. Take arbitrary $a, b \in A$, then $\|a + b\|^2 = (\sqrt{\langle (a + b), (a + b) \rangle})^2 = \langle (a + b), (a + b) \rangle = \|a\|^2 + 2(\langle a, b \rangle) + \|b\|^2 = \|a\|^2 + 2(0) + \|b\|^2 = \|a\|^2 + \|b\|^2$. It follows that $a \perp b \Rightarrow a \perp_p b$.

[$\Leftarrow$]Take arbitrary $a, b \in A$, where A is an inner product space. Assume that the condition $a \perp_p b$ holds, it means $\|a + b\|^2 = \|a\|^2 + \|b\|^2$, it will be shown that

$a \perp b$, in other words it must be shown $\langle a, b \rangle = 0$. Take arbitrary $a, b \in A$, then

$$\begin{aligned} \|a+b\|^2 + \|a-b\|^2 &= 2(\|a\|^2 + \|b\|^2), \\ \|a+b\|^2 + \|a+b\|^2 - 4\langle a, b \rangle &= 2(\|a\|^2 + \|b\|^2), \\ 2\|a+b\|^2 - 4\langle a, b \rangle &= 2(\|a\|^2 + \|b\|^2). \end{aligned}$$

substitute $\|a+b\|^2 = \|a\|^2 + \|b\|^2$, then we have

$$\begin{aligned} 2(\|a\|^2 + \|b\|^2) - 4\langle a, b \rangle &= 2(\|a\|^2 + \|b\|^2) \\ 0 &= -4\langle a, b \rangle \end{aligned}$$

Hence, we have $\langle a, b \rangle = 0$. It has been proved that if $a \perp_p b$ then $a \perp b$. Thus the biimplication is hold. $\square$

In standard orthogonality, several properties apply, such as non-degeneracy, symmetry, homogeneity, simplification, right and left additivity. These properties also apply to Pythagorean orthogonality in inner product spaces. These properties are briefly described as follows.Let a, b are vectors in a product space in A such that:

(1) Non-degeneracy, that is $a \perp_p a \Leftrightarrow a = 0$.
(2) Symmetry, that is $a \perp_p b \Leftrightarrow b \perp_p a$.
(3) Homogeneity, that is $a \perp_p b \Leftrightarrow \phi a \perp_p \delta b$, for all $\phi, \delta \in R$.
(4) Simplification, that is $a \perp_p b \Leftrightarrow \phi a \perp_p \phi b$, for all $\phi \in R$.
(5) Right additivity, that is $a \perp_p b, a \perp_p c \Rightarrow a \perp_p (b+c)$.
(6) Left additivity, that is if $b \perp_p a, c \perp_p a$, then $(b+c) \perp_p a$.

To provide a clearer understanding of these properties, the detailed proofs for all six properties will be presented below.

Lemma 3.2 (Non-degeneracy). *If A is an inner product space and $a \in A$, then $a \perp_p a \Leftrightarrow a = 0$.*

Proof. [$\Rightarrow$]For any inner product space A, suppose $a \in A$, and we have $a \perp_p a$. It means that $\|a+a\|^2 = \|a\|^2 + \|a\|^2$. It will be shown $a = 0$. Take an arbitrary $a \in A$, then

$$\begin{aligned} \|a+a\|^2 &= \|a\|^2 + \|a\|^2 \\ \|2a\|^2 &= (2\|a\|)^2 \\ \left(|2|^2\|a\|\right)^2 &= (2\|a\|)^2 \\ (4\|a\|)^2 &= (2\|a\|)^2 . \end{aligned}$$

That conditon will hold when $a = 0$. It follows that $a \perp_p a \Rightarrow a = 0$.
[$\Leftarrow$]For any inner product space A, suppose $a \in A$, and we have $a = 0$, it will be shown $a \perp_p a$. It must be shown $\|a+a\|^2 = \|a\|^2 + \|a\|^2$. Take arbitrary $a \in A$, then $\|a+a\|^2 = \|2a\|^2 = \|a\|^2 + \|a\|^2 + 2\|0\|^2 = \|a\|^2 + \|a\|^2$. It means $a \perp_p a$, therefore $a = 0 \Rightarrow a \perp_p a$. Thus, $a \perp_p a \Leftrightarrow a = 0$. $\square$

Lemma 3.3 (Symmetry). *If A is an inner product space and a, b are any two elements in A, then $a \perp_p b \Leftrightarrow b \perp_p a$.*

Proof. [$\Rightarrow$] Suppose A is an inner product space $a, b \in A$, and we have $a \perp_p b$. It means that $\|a+b\|^2 = \|a\|^2 + \|b\|^2$. It will be shown that $b \perp_p a$, or in other

words It must be shown that $\|b+a\|^2 = \|b\|^2 + \|a\|^2$. Take arbitrary $a, b \in A$, then $\|b+a\|^2 = \|a+b\|^2 = \|a\|^2 + \|b\|^2 = (\|b\|)^2 + \|a\|^2$. It follows that $b \perp_p a$. Hence $a \perp_p b \Rightarrow b \perp_p a$.
[$\Leftarrow$] Suppose A is an inner product space $a, b \in A$, and we have $b \perp_p a$. It means that $\|b+a\|^2 = \|b\|^2 + \|a\|^2$, it will be shown $a \perp_p b$, or in other words, It must be shown $\|a+b\|^2 = \|a\|^2 + \|b\|^2$. Take arbitrary $a, b \in A$, then $\|a+b\|^2 = \|b+a\|^2 = \|b\|^2 + \|a\|^2 = \|a\|^2 + \|b\|^2$. It follows that $a \perp_p b$, therefore $b \perp_p a \Rightarrow a \perp_p b$. Hence $a \perp_p b \Leftrightarrow b \perp_p a$. ☐

Lemma 3.4 (Homogeneity)**.** *If A is an inner product space $a, b \in A$ and $\phi, \delta \in R$, then we have $a \perp_p b \Leftrightarrow \phi a \perp_p \delta b$.*

Proof. [$\Rightarrow$] Suppose A is an inner product space $a, b \in A$, and we have $a \perp_p b$. Based on previous result, it is obtained that $a \perp_p b$ is equivalent to $a \perp b$, means that $\langle a, b\rangle = 0$. it will be shown $\phi a \perp_p \delta b$ for all $\phi, \delta \in R$. It must be shown $\|\phi a + \delta b\|^2 = \|\phi a\|^2 + \delta b^2$. Take $a, b \in A$ and $\phi, \delta \in R$ arbitrary, then

$$\begin{aligned}
\|\phi a + \delta b\|^2 &= (\sqrt{(\langle(\phi a + \delta b), (\phi a + \delta b)\rangle)})^2 \\
&= (\sqrt{\langle(\phi a), (\phi a)\rangle})^2 + 2\phi\delta \langle a, b\rangle + (\sqrt{\langle(\delta b), (\delta b)\rangle})^2 \\
&= \|\phi a\|^2 + 2\phi\delta \langle a, b\rangle + \|\delta b\|^2 \\
&= \|\phi a\|^2 + 2\phi\delta \cdot 0 + \|\delta b\|^2 \\
&= \|\phi a\|^2 + \|\delta b\|^2 .
\end{aligned}$$

It follows that $\phi a \perp_p \delta b$, therefore it follows $a \perp_p b \Rightarrow \phi a \perp_p \delta b$.
[$\Leftarrow$] Suppose A is an inner product space, $a, b \in A$ and $\phi, \delta \in R$, and we have $\phi a \perp_p \delta b$. It means that $\|\phi a + \delta b\|^2 = \|\phi a\|^2 + \|\delta b\|^2$, it will be shown $a \perp_p b$. on the other words, It must be shown $\|a+b\|^2 = \|a\|^2 + \|b\|^2$. Consider

$$\begin{aligned}
\|\phi a + \delta b\|^2 &= (\sqrt{\langle(\phi a + \delta b), (\phi a + \delta b)\rangle})^2 \\
&= (\sqrt{\langle(\phi a), (\phi a)\rangle})^2 + 2\phi\delta \langle a, b\rangle + (\sqrt{\langle(\delta b), (\delta b)\rangle})^2 \\
&= \|\phi a\|^2 + 2\phi\delta \langle a, b\rangle + \|\delta b\|^2 \, \|\phi a + \delta b\|^2 \\
&= \|\phi a\|^2 + 2\phi\delta \langle a, b\rangle + \|\delta b\|^2 \, \|\phi a\|^2 + \|\delta a\|^2 \\
&= \|\phi a\|^2 + 2\phi\delta \langle a, b\rangle + \|\delta b\|^2 \\
2\phi\delta \langle a, b\rangle &= 0 \\
2\phi\delta \left(\frac{\phi^{-1}\delta^{-1}}{2}\right) \langle a, b\rangle &= 0 \cdot \left(\frac{\phi^{-1}\delta^{-1}}{2}\right) \\
\langle a, b\rangle &= 0.
\end{aligned}$$

Hence, it is equivalent to $\|a+b\|^2 = \|a\|^2 + \|b\|^2$. It follows that $\|a+b\|^2 = \|a\|^2 + \|b\|^2$ means that $a \perp_p b$, therefore it follows that $\phi a \perp_p \delta b \Rightarrow a \perp_p b$. Thus, it follows that $a \perp_p b \Leftrightarrow \phi a \perp_p \delta b$. ☐

Lemma 3.5 (Simplification)**.** *If A is an inner product space, $a, b \in A$ and $\phi \in R$, then $a \perp_p b \Leftrightarrow \phi a \perp_p \phi b$.*

Proof. [$\Rightarrow$] Suppose A is an inner product space $a, b \in A$, and $a \perp_p b$ means that $\|a+b\|^2 = \|a\|^2 + \|b\|^2$. It will be shown $\phi a \perp_p \phi b$ for all $\phi \in R$. Take arbitrary $a \in R$, then $\|\phi a + \phi b\|^2 = ((\|\phi(a+b)\|))^2 = (|\phi|\,(\|a\|))^2 + (|\phi|\,(\|b\|))^2 = \|\phi a\|^2 +$

$\|\phi b\|^2$. It follows that $\phi a \perp_p \phi b$, therefore $a \perp_p b \Rightarrow \phi a \perp_p \phi b$.
[$\Leftarrow$] Suppose A is an inner product space, $a, b \in A$, and $\phi \in R$, and we have $\phi a \perp_p \phi b$. It means that $\|\phi a + \phi b\|^2 = \|\phi a\|^2 + \|\phi b\|^2$. It must be shown $\|a + b\|^2 = \|a\|^2 + \|b\|^2$.

$$\begin{aligned}
\|\phi a + \phi b\|^2 &= \|\phi a\|^2 + \|\phi b\|^2 \, \|\phi(a+b)\|^2 \\
&= \|\phi a\|^2 + \|\phi b\|^2 \, (|\phi| \, (\|a+b\|))^2 \\
&= (|\phi| \, (\|a\|))^2 + (|\phi| \, (\|b\|))^2 \, |\phi|^2 \, \|a+b\|^2 \\
&= |\phi|^2 \, \|a\|^2 + |\phi|^2 \, \|b\|^2 \, |\phi|^2 \, \|a+b\|^2 \\
&= |\phi|^2 (\|a\|^2 + \|b\|^2) \, \|a+b\|^2 \\
&= \|a\|^2 + \|b\|^2 .
\end{aligned}$$

It follows that $a \perp_p b$, therefore $\phi a \perp_p \phi b \Rightarrow a \perp_p b$. Thus $a \perp_p b \Leftrightarrow \phi a \perp_p \phi b$. □

Lemma 3.6 (Right additivity). *If A is an inner product space and $a, b, c \in A$, then we have $a \perp_p b, a \perp_p c \Rightarrow a \perp_p (b + c)$.*

Proof. Suppose A is an inner product space $a, b, c \in A$, and we have $a \perp_p b$ is equivalent to $\langle a, b \rangle = 0$ and $a \perp_p c$ is equivalent to $\langle a, c \rangle = 0$. It will be shown $a \perp_p (b + c)$. Consider

$$\begin{aligned}
\|a + (b+c)\|^2 &= (\sqrt{\langle (a + (b+c)), (a + (b+c)) \rangle})^2 \\
&= \langle (a + (b+c)), (a + (b+c)) \rangle \\
&= \langle a, (a + (b+c)) \rangle + \langle (b+c), (a + (b+c)) \rangle \\
&= \langle (a + (b+c)), a \rangle + \langle (a + (b+c)), (b+c) \rangle \\
&= \langle a, a \rangle + \langle (b+c), a \rangle + \langle a, (b+c) \rangle + \langle (b+c), (b+c) \rangle \\
&= \langle a, a \rangle + \langle (b+c), a \rangle + \langle (b+c), a \rangle + \langle (b+c), (b+c) \rangle \\
&= \langle a, a \rangle + \langle (b+c), (b+c) \rangle + 2 \, \langle (b+c), a \rangle \\
&= \langle a, a \rangle + \langle (b+c), (b+c) \rangle + 2(\langle b, a \rangle + \langle c, a \rangle) \\
&= \langle a, a \rangle + \langle (b+c), (b+c) \rangle + 2(\langle a, b \rangle + \langle a, c \rangle) \\
&= \langle a, a \rangle + \langle (b+c), (b+c) \rangle + 2(0 + 0) \\
&= (\sqrt{\langle a, a \rangle})^2 + (\sqrt{\langle (b+c), (b+c) \rangle})^2 \\
&= \|a\|^2 + \|b + c\|^2 .
\end{aligned}$$

Thus $a \perp_p b, a \perp_p c \Rightarrow a \perp_p (b + c)$. □

Lemma 3.7 (Left additivity). *If A is an inner product space and $a, b, c \in A$, then $b \perp_p a, c \perp_p a \Rightarrow (b + c) \perp_p a$.*

Proof. Suppose A is an inner product space $a, b, c \in A$. We have that $b \perp_p a$ is equivalent to $\langle b, a \rangle = 0$ and $c \perp_p a$ is equivalent to $\langle c, a \rangle = 0$. Next it will be shown $(b + c) \perp_p a$,or in other words it must be shown $\|(b + c) + a\|^2 = \|b + c\|^2 + \|a\|^2$. Consider

$$\begin{aligned}\|(b+c)+a\|^2 &= (\sqrt{\langle((b+c)+a),((b+c)+a)\rangle})^2\\ &= \langle((b+c)+a),((b+c)+a)\rangle\\ &= (\sqrt{\langle(b+c),(b+c)\rangle})^2+(\sqrt{\langle a,a\rangle})^2+2(\langle a,b\rangle+\langle a,c\rangle)\\ &= \left(\|b+c\|^2+\|a\|\right)^2+2(\langle b,a\rangle+\langle c,a\rangle)\\ &= \left(\|b+c\|^2+\|a\|\right)^2+2(0+0)\\ &= \left(\|b+c\|^2+\|a\|\right)^2.\end{aligned}$$

Thus it follows that $b \perp_p a, c \perp_p a \Rightarrow (b+c) \perp_p a$

□

Several characteristics of Pythagorean orthogonality have been outlined. The following section will examine the identification of linear mappings that preserve Pythagorean orthogonality.

3.2. Preservation of Pythagorean Orthogonality in Linear Mapping. As briefly discussed in the introduction, the concept of linear mappings that preserve orthogonality provides a bridge between pure mathematics and quantum mechanics. Chmieliński has characterized linear mappings that preserve standard orthogonality. In this article, we extend Chmieliński's results to the concept of Pythagorean orthogonality. Several characterizations have been established, with the first being described by Theorem 3.8 below.

Theorem 3.8. *Let A and B be two inner product spaces. Suppose $L : A \to B$ is a linear mapping, and there exists a nonzero real number ϕ such that for all $c \in A$, we have $\langle L(c), L(c)\rangle = \phi\langle c, c\rangle$. Then, for all $c, d \in A$, it follows that $\langle L(c), L(d)\rangle = \phi\langle c, d\rangle$.*

Proof. Suppose that A and B are two inner product spaces. Let $L : A \to B$ be a linear mapping and there exists a nonzero real number ϕ such that for any $c \in A$ $\langle L(c), L(c)\rangle = \phi\,\langle c, c\rangle$. It will be shown that for every $c, d \in A$, we have $\langle L(c), L(d)\rangle = \phi\,\langle c, d\rangle$. Take $c, d \in A$ arbitrary. Based on the polarization identity, we obtain

$$\begin{aligned}&\langle L(c), L(d)\rangle\\ &= \tfrac{1}{4}\|L(c)+L(d)\|^2-\tfrac{1}{4}\|L(c)-L(d)\|^2\\ &= \tfrac{1}{4}\|L(c)+L(d)\|^2-\tfrac{1}{4}\|L(c)+L(-d)\|^2\\ &= \tfrac{1}{4}\|L(c+d)\|^2-\tfrac{1}{4}\|L(c-d)\|^2\\ &= \tfrac{1}{4}((\|L(c+d)\|)^2-\|L(c-d)\|^2)\\ &= \tfrac{1}{4}((\sqrt{\langle L(c+d),L(c+d)\rangle})^2-(\sqrt{\langle L(c-d),L(c-d)\rangle})^2)\\ &= \tfrac{1}{4}(\langle L(c+d),L(c+d)\rangle-\langle L(c-d),L(c-d)\rangle)\\ &= \tfrac{1}{4}(\phi\,\langle c+d,c+d\rangle-\phi\,\langle c-d,c-d\rangle)\\ &= \tfrac{1}{4}\phi(\langle c+d,c+d\rangle-\langle c-d,c-d\rangle)\\ &= \tfrac{1}{4}\phi(\langle c,c+d\rangle+\langle d,c+d\rangle-(\langle c,c-d\rangle+\langle -d,c-d\rangle))\\ &= (\langle c,c+d\rangle+\langle d,c+d\rangle-(\langle c,c-d\rangle-\langle d,c-d\rangle))\\ &= \tfrac{1}{4}\phi(\langle c+d,c\rangle+\langle c+d,d\rangle-(\langle c-d,c\rangle-\langle c-d,d\rangle))\end{aligned}$$

$$
\begin{aligned}
&= \tfrac{1}{4}\phi(\langle c+d,c\rangle + \langle c+d,d\rangle - \langle c-d,c\rangle + \langle c-d,d\rangle) \\
&= \tfrac{1}{4}\phi(\langle c,c\rangle + \langle d,c\rangle + \langle c,d\rangle + \langle d,d\rangle - (\langle c,c\rangle + \langle -d,c\rangle) + \langle c,d\rangle + \langle -d,d\rangle) \\
&= \tfrac{1}{4}\phi(\langle c,c\rangle + \langle d,c\rangle + \langle c,d\rangle + \langle d,d\rangle - (\langle c,c\rangle - \langle d,c\rangle) + \langle c,d\rangle - \langle d,d\rangle) \\
&= \tfrac{1}{4}\phi(\langle c,c\rangle + \langle d,c\rangle + \langle c,d\rangle + \langle d,d\rangle - \langle c,c\rangle + \langle d,c\rangle + \langle c,d\rangle - \langle d,d\rangle) \\
&= \tfrac{1}{4}\phi(\langle c,c\rangle + \langle c,d\rangle + \langle c,d\rangle + \langle d,d\rangle - \langle c,c\rangle + \langle c,d\rangle + \langle c,d\rangle - \langle d,d\rangle) \\
&= \tfrac{1}{4}\phi(4\langle c,d\rangle) \\
&= \phi\langle c,d\rangle .
\end{aligned}
$$

Thus, it is proven that $\langle L(c), L(d)\rangle = \phi(\langle c,d\rangle)$. □

Theorem 3.9. *Suppose A and B are two inner product spaces. Let L be a mapping that maps A to B. If there exists a nonzero real number ϕ such that for every $c,d \in A$ we have $\langle L(c), L(d)\rangle = \phi\langle c,d\rangle$, then L is a linear mapping and preserves Pythagorean orthogonality strongly.*

Proof. Suppose L is a mapping that maps the inner product space of A to B. Suppose also that $c,d \in A$, there exists a nonzero real number ϕ such that

$$\langle L(c), L(d)\rangle = \phi\langle c,d\rangle . \tag{1}$$

We will prove:

(1) The mapping L is a linear mapping.
(2) The mapping L strongly preserves Pythagorean orthogonality.

First by definition, it will be shown that for every $a,b,c,d \in A$ and for all scalars δ , we have:

(1) $\langle L(a+b), L(c+d)\rangle = \langle L(a)+L(b), L(c)+L(d)\rangle$.
(2) $\langle L(\delta a), L(\delta b)\rangle = \langle \delta L(a), \delta L(b)\rangle$.

Take $a,b,c,d \in A$ and an arbitrary scalar δ. Based on equation (1) it is obtained

$$
\begin{aligned}
\langle L(a+b), L(c+d)\rangle &= \phi\langle a+b, c+d\rangle \\
&= \phi(\langle a,c+d\rangle + \langle b,c+d\rangle) \\
&= \phi(\langle c+d,a\rangle + \langle c+d,b\rangle) \\
&= \phi(\langle c,a\rangle + \langle d,a\rangle + \langle c,b\rangle + \langle d,b\rangle) \\
&= \phi\langle c,a\rangle + \phi\langle d,a\rangle + \phi\langle c,b\rangle + \phi\langle d,b\rangle \\
&= \langle L(c),L(a)\rangle + \langle L(d),L(a)\rangle + \langle L(c),L(b)\rangle + \langle L(d),L(b)\rangle \\
&= \langle L(c)+L(d),L(a)\rangle + \langle L(c)+L(d),L(b)\rangle \\
&= \langle L(a),L(c)+L(d)\rangle + \langle L(b),L(c)+L(d)\rangle \\
&= \langle L(a)+L(b),L(c)+L(d)\rangle .
\end{aligned}
$$

It is proven that the first is satisfied. Furthermore, based on equation (1) it is also obtained that

$$\begin{aligned}
\langle L(\delta a), L(\delta b)\rangle &= \phi\,\langle \delta a, \delta b\rangle \\
&= \phi\delta\,\langle a, \delta b\rangle \\
&= \phi\delta\,\langle \delta b, a\rangle \\
&= \phi\delta^2\,\langle b, a\rangle \\
&= \delta^2\phi\,\langle b, a\rangle \\
&= \delta^2\,\langle L(b), L(a)\rangle \\
&= \delta\,\langle \delta L(b), L(a)\rangle \\
&= \delta\,\langle L(a), \delta L(b)\rangle \\
&= \langle \delta L(a), \delta L(b)\rangle\,.
\end{aligned}$$

It is proven that the second condition is satisfied. Thus, it is proven that the mapping L is a linear mapping.

Secondly it will be shown that L preserves Pythagorean orthogonality strongly. It will be shown that for every $a, b \in A$ we have $a \perp_p b$ if and only if $L(a) \perp_p L(b)$. [$\Rightarrow$] Suppose A is an inner product space and L a linear mapping. If for every $a, b \in A$ we have $a \perp_p b$, it means that if $\|a+b\|^2 = \|a\|^2 + \|b\|^2$, then $L(a) \perp_p L(b)$. It will be shown that $\|L(a)+L(b)\|^2 = \|L(a)\|^2 + \|L(b)\|^2$. Take arbitrary $a, b \in A$, then

$$\begin{aligned}
\|L(a)+L(b)\|^2 &= \|L(a+b)\|^2 \\
&= (\sqrt{\langle L(a+b), L(a+b)\rangle})^2 \\
&= \langle L(a+b), L(a+b)\rangle \\
&= \phi\,\langle a+b, a+b\rangle \\
&= \phi(\sqrt{\langle a+b, a+b\rangle})^2 \\
&= \phi\,\|a+b\|^2\,.
\end{aligned}$$

Substitute $\|a+b\|^2 = \|a\|^2 + \|b\|^2$, we obtain

$$\begin{aligned}
\|L(a)+L(b)\|^2 &= \phi(\|a\|^2 + \|b\|^2) \\
&= \phi((\sqrt{\langle a, a\rangle})^2 + (\sqrt{\langle b, b\rangle})^2) \\
&= \phi(\langle a, a\rangle + \langle b, b\rangle) \\
&= \phi\,\langle a, a\rangle + \phi\,\langle b, b\rangle \\
&= \|L(a)\|^2 + \|L(b)\|^2\,.
\end{aligned}$$

Hence it is proven that $a \perp_p b \Rightarrow L(a) \perp_p L(b)$.
[$\Leftarrow$] Suppose A is an inner product space and L a linear mapping, if for every $a, b \in A$ we have $L(a) \perp_p L(b)$, it means that if $\|L(a)+L(b)\|^2 = \|L(a)\|^2 + \|L(b)\|^2$, then $a \perp_p b$. Next, it will be shown that $\|a+b\|^2 = \|a\|^2 + \|b\|^2$. Take arbitrary $a, b \in A$. Consider

$$\begin{aligned}
\|L(a)+L(b)\|^2 &= \|L(a)\|^2 + \|L(b)\|^2 \\
\|L(a+b)\|^2 &= \|L(a)\|^2 + \|L(b)\|^2 \\
(\sqrt{\langle L(a+b), L(a+b)\rangle})^2 &= (\sqrt{\langle L(a), L(a)\rangle})^2 + (\sqrt{\langle L(b), L(b)\rangle})^2 \\
\langle L(a+b), L(a+b)\rangle &= \langle L(a), L(a)\rangle + \langle L(b), L(b)\rangle \\
\phi\,\langle a+b, a+b\rangle &= \phi\,\langle a, a\rangle + \phi\,\langle b, b\rangle \\
\phi\,\langle a+b, a+b\rangle &= \phi(\langle a, a\rangle + \langle b, b\rangle) \\
\langle a+b, a+b\rangle &= \langle a, a\rangle + \langle b, b\rangle
\end{aligned}$$

$$\begin{aligned}(\sqrt{\langle a+b,a+b\rangle})^2 &= (\sqrt{\langle a,a\rangle})^2+(\sqrt{\langle b,b\rangle})^2\\ \|a+b\|^2 &= \|a\|^2+\|b\|^2\,.\end{aligned}$$

Hence $L(a) \perp_p L(b) \Rightarrow a \perp_p b$. Thus it is proven that $a \perp_p b \Leftrightarrow L(a) \perp_p L(b)$which means that L preserves Pythagorean orthogonality strongly. From statements (1) and (2) it can be concluded that for two inner product spaces A and B, L is a mapping that maps A to B. If there exists a nonzero real number ϕ such that for every $c, d \in A$ we have $\langle L(c), L(d)\rangle = \phi \langle c, d\rangle$, then L is a linear mapping and preserves Pythagorean orthogonality strongly.

□

Theorem 3.10. *Suppose A and B are two inner product spaces. If the mapping $L : A \to B$ is linear and preserves Pythagorean orthogonality strongly then $L : A \to B$ is linear and preserves Pythagorean orthogonality.*

Proof. Suppose A and B are two inner product spaces. Let $L : A \to B$ is a linear mepping and preserves Pythagorean orthogonality strongly. It means that if for every $c, d \in A$ we have $c \perp_p d \Leftrightarrow L(c) \perp_p L(d)$, then L is linear and preserves Pythagorean orthogonality. From the assumtion above, it is clear that L linear mapping that preserves Pythagorean orthogonality. □

4. Conclusions

Investigation into Pythagorean orthogonality-preserving linear mapping within inner product spaces unveils the harmonious interplay of geometry and algebra. This concept bridges the gap between abstract mathematical structures and practical applications, offering insights that redefine the way we perceive vectors, spaces, and their transformations. As we delve into this realm, we find ourselves on a journey that not only enriches our theoretical foundations but also empowers us to create innovative solutions that transcend traditional boundaries. According to the findings, the properties that apply in Pythagorean orthogonality in the inner product space are non-degeneracy, symmetry, homogeneity, simplification, right-additivity, and left-additivity. The characterization of linear mappings that preserve Pythagorean orthogonality in the inner product space includes: Strong preservation of Pythagorean orthogonality, and preservation of orthogonality. What is interesting to study further is to apply these results to a larger space. In [27], it is explained that the inner product space has undergone many developments. Some of them are semi-inner product spaces and indefinite inner product spaces. A paper outlining the results in these spaces is in preparation.

This research also opens up several potential avenues for further exploration. One avenue is to expand the analysis into more complex inner product spaces such as Sobolev spaces or Banach spaces. In addition to theoretical aspects, future research can also explore practical applications. Investigating the practical applications of linear mappings that preserve orthogonality in fields such as information theory, signal processing, and quantum computing can demonstrate the relevance and practical benefits of these findings.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

References

1. A.R. Gairola, A. Singh, L. Rathour, V.N. Mishra, *Improved rate of approximation by modification of Baskakov operator*, Operators and Matrices **16** (2022), 1097-1123. DOI: http://dx.doi.org/10.7153/oam-2022-16-72
2. A.R. Gairola, S. Maindola, L. Rathour, L.N. Mishra, V.N. Mishra, *Better uniform approximation by new Bivariate Bernstein Operators*, International Journal of Analysis and Applications **20** (2022), 1-19. DOI: https://doi.org/10.28924/2291-8639-20-2022-60
3. A.R. Gairola, N. Bisht, L. Rathour, L.N. Mishra, V.N. Mishra, *Order of approximation by a new univariate Kantorovich Type Operator*, International Journal of Analysis and Applications **21** (2023), 1-17. Article No. 106. DOI: https://doi.org/10.28924/2291-8639-21-2023-106
4. J. Alonso, H. Martini, S. Wu, (). *On Birkhoff orthogonality and isosceles orthogonality in normed linear spaces*, Aequationes mathematicae **83** (2012), 153-189.
5. H. Anton, C. Rorres, A. Kaul, *Elementary Linear Algebra*, 12th ed, Wiley, New York, 2019.
6. L. Arambasic, R. Rajic, *On Birkhoff-James and Roberts orthogonality*, Special Matrices **6** (2018), 229–236. https://doi.org/10.1515/spma-2018-0018
7. L.J. Arambasic, R. Rajic, *Roberts orthogonality for 2 * 2 2× 2 complex matrices*, Acta Mathematica Hungarica **157** (2019), 220-228.
8. P.M. Bajracharya, B.P. Ojha, *Birkhoff orthogonality and different particular cases of Carlsson's orthogonality on normed linear Spaces*, Journal of Mathematics and Statistics **16** (2020), 133-141.
9. V. Balakrishnan, *Mathematical physics: applications and problems*, Springer Nature, 2020.
10. D. Bhattacharjee, *An outlined tour of geometry and topology as perceived through physics and mathematics emphasizing geometrization, elliptization, uniformization, and projectivization for Thruston's 8-geometries covering Riemann over Teichmuller spaces*, Authorea Preprints. 2022.
11. G. Birkhoff, & J. Von Neumann, *The logic of quantum mechanics*, The Logico-Algebraic Approach to Quantum Mechanics, Volume I, Historical Evolution, Dordrecht, Springer Netherlands, 1975, 1-26.
12. H. Chen, *New MDS entanglement-assisted quantum codes from MDS Hermitian self-orthogonal codes*, Designs, Codes and Cryptography, **91** (2023), 1-12.
13. J. Chmieli´nski, *Linear mappings approximately preserving orthogonality*, J. Math. Anal. Appl. **304** (2005), 158–169.
14. Deepmala, L.N. Mishra, V.N. Mishra, *Trigonometric Approximation of Signals (Functions) belonging to the* $W(L_r, \xi(t)), (r \geq 1)-$ *class by (E, q) (q>0)-means of the conjugate series of its Fourier series*, Global Journal of Mathematical Sciences (GJMS) **2** (2015), 61-69.
15. J. Dixmier, *Sur la réduction des anneaux d'opérateurs*, In Annales scientifiques de l'École normale supérieure **68** (1951), 185-202.
16. H.A. Dye, *On the geometry of projections in certain operator algebras*, Annals of Mathematics **61** (1955), 73-89.
17. B. Jacob, *Linear Algebra*, University of California, USA, 1990.
18. L.N. Mishra, M. Raiz, L. Rathour, V.N. Mishra, *Tauberian theorems for weighted means of double sequences in intuitionistic fuzzy normed spaces*, Yugoslav Journal of Operations Research **32** (2022), 377-388. DOI: https://doi.org/10.2298/YJOR210915005M

19. L.N. Mishra, V.N. Mishra, K. Khatri, Deepmala, *On The Trigonometric approximation of signals belonging to generalized weighted Lipschitz* $W(L^r, \xi(t))(r \geq 1)-$ *class by matrix* $(C^1.N_p)$ *Operator of conjugate series of its Fourier series*, Applied Mathematics and Computation **237** (2014), 252-263.
20. G. Lumer, (). *Semi-inner-product spaces*, Transactions of the American Mathematical Society **100** (1961), 29-43.
21. H.A. Moon, T.J. Asaki, M.A. Snipes, *Inner Product Spaces and Pseudo-Invertibility*, In Application-Inspired Linear Algebra, Cham, Springer International Publishing, 2022, 379-477.
22. B.P. Ojha, P.M. Bajrayacharya, *Relation of Pythagorean and Isosceles Orthogonality with Best approximations in Normed Linear Space*, Mathematics Education Forum Chitwan **4** (2019), 72–78. https://doi.org/10.3126/mefc.v4i4.26360
23. C.C. Pugh, *Real Mathematical Analysis*, Cham, Springer International Publishing, 2015.
24. R. Sen, *A First Course in Functional Analysis Theory and Applications*, Athem Press, London, 2013.
25. S. Sylviani, E. Carnia, A.K. Supriatna, *An algebraic study of the matrix meta-population model*, In Journal of Physics: Conference Series **893** (2017), 012009.
26. S. Sylviani, H. Garminia, P. Astuti, *Behavior for Time Invariant Finite Dimensional Discrete Linear Systems*, Journal of Mathematical Fundamental Sciences **1** (2013).
27. S. Sylviani, H. Garminia, P. Astuti, *Characterization of orthogonality preserving mappings in indefinite inner product spaces*, Journal Mathematics and Computer Science **26** (2021), 10-15.
28. M.K. Sharma, N. Dhiman, Vandana, V.N. Mishra, *Mediative Fuzzy Pythagorean Algorithm to Multi-criteria Decision-Making and Its Application in Medical Diagnostic*, In: Ali, I., Chatterjee, P., Shaikh, A.A., Gupta, N., AlArjani, A. (eds) Computational Modelling in Industry 4.0. Springer, Singapore, 2022. https://doi.org/10.1007/978-981-16-7723-6_14
29. U. Uhlhorn, *Representation of symmetry transformations in quantum mechanics*, Arkiv Fysik **23** (1963), 307-341.
30. V.N. Mishra, L.N. Mishra, *Trigonometric Approximation of Signals (Functions) in Lp-norm*, International Journal of Contemporary Mathematical Sciences **7** (2012), 909-918.
31. V.N. Mishra, K. Khatri, L.N. Mishra, Deepmala; *Inverse result in simultaneous approximation by Baskakov-Durrmeyer-Stancu operators*, Journal of Inequalities and Applications **2013** (2013), 586. doi:10.1186/1029-242X-2013-586
32. V.N. Mishra, K. Khatri, L.N. Mishra; *Statistical approximation by Kantorovich type Discrete* $q-$*Beta operators*, Advances in Difference Equations **2013** (2013), 345. DOI: 10.1186/10.1186/1687-1847-2013-345
33. V.N. Mishra, T. Kumar, M.K. Sharma, L. Rathour, *Pythagorean and fermatean fuzzy sub-group redefined in context of T-norm and S-conorm*, Journal of Fuzzy Extension and Applications **4** (2023), 125-135. https://doi.org/10.22105/jfea.2023.396751.1262
34. V.N. Mishra, K. Khatri, L.N. Mishra, Deepmala, *Trigonometric approximation of periodic Signals belonging to generalized weighted Lipschitz* $W'(L_r, \xi(t)), (r \geq 1)-$ *class by Nörlund-Euler* $(N, p_n)(E, q)$ *operator of conjugate series of its Fourier series*, Journal of Classical Analysis **5** (2014), 91-105. doi:10.7153/jca-05-08
35. V.N. Mishra, *Some Problems on Approximations of Functions in Banach Spaces*, Ph.D. Thesis, Indian Institute of Technology Roorkee 247 667, Uttarakhand, India, 2007.
36. V.N. Mishra, L.N. Mishra, *Trigonometric Approximation of Signals (Functions) in Lp-norm*, International Journal of Contemporary Mathematical Sciences **7** (2012), 909–918.
37. J.S. Lomont, & P. Mendelson, *The Wigner unitarity-antiunitarity theorem*, Annals of Mathematics **78** (1963), 548-559.
38. Z. Yang, Y. Li, *A New Geometric Constant in Banach Spaces Related to the Isosceles Orthogonality*, Kyungpook Mathematical Journal **62** (2022), 271-287.

39. A. Zamani, *Approximately bisectrix-Orthogonality preserving mappings*, Ferdowsi University of Mashhad, 2015.
40. Y. Zhao, M. Yu, S. Wu, C. He. *An orthogonality type based on invariant inner products*, Aequationes mathematicae **97** (2023), 707-724.

Sisilia Sylviani received M.Sc. and Ph.D. from Institut Teknologi Bandung. She is currently a Lecturer at Department of Mathematics Universitas Padjadjaran since 2014. Her research interests are Algebra, applied algebra, and computational algebra.

Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran, Sumedang 45363, Indonesia.
e-mail: sisilia.sylviani@unpad.ac.id

Anita Triska received M.Sc. and Ph.D from Institut Teknologi Bandung. She is currently a Lecturer at Department of Mathematics Universitas Padjadjaran since 2010. Her research interests are Pattern Formation and Biomathematics.

Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran, Sumedang 45363, Indonesia.
e-mail: a.triska@unpad.ac.id

Laxmi Rathour is an active researcher at Dept. of Mathematics, National Institute of Technology, Mizoram, India. She received her Master's from Department of Mathematics, Indira Gandhi National Tribal University, India. Her interests include convex optimization, nonlinear analysis, approximation theory, fixed point theory, and operations research. In the meantime, She has published numerous papers internationally and volunteers as a reviewer for Mathematical Reviews and Zentralblatt Math.

Department of Mathematics, National Institute of Technology, Chaltlang, Aizawl 796 012, Mizoram, India.
e-mail: laxmirathour817@gmail.com, rathourlaxmi562@gmail.com

Hanafi Fulhamdi received bachelor from Universitas Padjadjaran. He was a student at Mathematics Study Program Universitas Padjadjaran and graduated in 2023. His research interests are Algebra.

Mathematics Study Program, Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran, Sumedang 45363, Indonesia.
e-mail: hanafifulhamdi01@gmail.com

Dianne Amor Kusuma received M.Ed and Ph.D from Universitas Pendidikan Indonesia. She is currently a Lecturer at Department of Mathematics Universitas Padjadjaran since 2005. Her research interests are Mathematics and Mathematics Education.

Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran, Sumedang 45363, Indonesia.
e-mail: amor@unpad.ac.id

Kankan Parmikanti received Master Degree from Universitas Padjadjaran. She is currently a Lecturer at Department of Mathematics Universitas Padjadjaran since 1987. Her research interests are Mathematics and Statistics.

Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Padjadjaran, Sumedang 45363, Indonesia.
e-mail: parmikanti@unpad.ac.id

Fahmi Candra Permana received Master Degree from Institut Teknologi Bandung. He is currently a Kampus UPI di Cibiru, Universitas Pendidikan Indonesia since 2017. Her research interests are Computer Science and Multimedia Computing.

Software Engineering Study Program, Kampus UPI di Cibiru, Universitas Pendidikan Indonesia.
`e-mail: fahmi.candrap@upi.edu`