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SIMPSON-HADAMARD'S INEQUALITIES FOR HADAMARD K-FRACTIONAL INTEGRALS

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Abstract. In this paper, we present some Hadamard-Simpson type inequalities for Hadamard k-fractional integrals of a function f. These inequalities are based on convexity of $|f'|$, the absolute value of derivative of f. Also, a lower bound for k-fractional integrals is presented in the presence of the convexity of f.

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1. Introduction

The inequalities are very important to know the lower bound and/or upper bound of an important statement such as a mean value of a function. But, these inequalities usually need some conditions such as convexity ([5, 11, 15]). It is well known that the function $f : [a, b] \to \mathbb{R}$ is convex if for each $x_1, x_2 \in [a, b]$

$$f(\mu x_1 + (1-\mu)x_2) \leq \mu f(x_1) + (1-\mu) f(x_2), \qquad \mu \in [0,1].$$

One of the important inequalities presenting an estimation of mean value for a convex function f in a finite interval is Hermite-Hadamard-type inequalities (HHIs) [1]. It states that for a convex function $f : [a, b] \to \mathbb{R}$, where $a < b$ the following holds:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

The proofs for the inequalities of HHIs are found in many literatures ([1, 4, 10, 11]). From view point of numerical analysis, the first and second inequalities in HHIs can be obtained by the error theorems on midpoint rule and trapezoidal

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rule of numerical integration. If $f \in C^2([a,b])$, the midpoint integration rule gives [2]:

$$\int_a^b f(x)\,dx - (b-a)\,f(\frac{a+b}{2}) = \frac{(b-a)^3}{24} f^{''}(\xi), \qquad a < \xi < b.$$

The convexity of f implies that $f^{''} > 0$ and so the first inequality of HHI is concluded.

Moreover, for $f \in C^2([a,b])$, the trapezoidal integration rule gives [2]:

$$\int_a^b f(x)\,dx - \frac{b-a}{2}(f(a)+f(b)) = -\frac{(b-a)^3}{12} f^{''}(\xi), \qquad a < \xi < b.$$

By convexity of f, we have $f^{''} > 0$ and so the second inequality of HHIs is concluded.

Simpson's integration rule also follows an important inequality for mean value of a function f in $[a,b]$. If $f \in C^4([a,b])$, the Simpson's integration rule gives [2]:

$$\int_a^b f(x)\,dx - \frac{1}{6}(b-a)\left(f(a) + 4f(\frac{a+b}{2}) + f(b)\right) = -\frac{(b-a)^5}{2880} f^{(4)}(\xi), \quad a < \xi < b.$$

Let f'' be a convex function on (a,b), then $f^{(4)} > 0$ that implies

$$\frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a) + 4f(\frac{a+b}{2}) + f(b)}{6}.$$

Then, if f and f'' are convex functions on (a,b), then

$$f(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a) + 4f(\frac{a+b}{2}) + f(b)}{6},$$

that are called Hadamard-Simpson type inequalities (HSIs).

Simpson-Hadamard's inequalities are a powerful tool in the study of Hadamard k-fractional integrals. These inequalities provide bounds on the k-fractional integrals of a function, which are useful in many areas of mathematics, including analysis, probability theory, and mathematical physics. The Hadamard k-fractional integral is a generalization of the Riemann-Liouville fractional integral, and it has many applications in the study of differential equations, partial differential equations, and other areas of mathematics. Simpson-Hadamard's inequalities provide a way to estimate the value of the k-fractional integral of a function over a given interval, which is important in many applications. In particular, Simpson-Hadamard's inequalities are useful in the study of functions that are not necessarily continuous or differentiable. They provide a way to estimate the value of the k-fractional integral of such functions, which can be difficult to compute directly. By providing bounds on the k-fractional integral, Simpson-Hadamard's inequalities allow mathematicians to study the behavior of these functions in a rigorous and systematic way.

An extension of HHI [8] and HSIs [3] is on the Hadamard fractional integrals in which the kernel is a log function. If f is a function in $L_1([a,b])$ in which $0<a<b$, then the left and right Hadamard fractional integrals of f from order α denoted respectively by $J^{\alpha}_{a^+}f$ and $J^{\alpha}_{b^-}f$ are defined by [6, 8, 14]

$$J^{\alpha}_{a^+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x\left(\log\frac{x}{t}\right)^{\alpha-1}f(t)dt,\qquad x\geq a,$$

and

$$J^{\alpha}_{b^-}f(x)=\frac{1}{\Gamma(\alpha)}\int_x^b\left(\log\frac{x}{t}\right)^{\alpha-1}f(t)dt,\qquad x\leq b,$$

where

$$\Gamma(\alpha)=\int_0^{\infty}x^{\alpha-1}e^{-x}dx,\qquad \alpha>0,$$

is the Gamma function.
Note that when $\alpha=1$, we have $\Gamma(1)=1$ and from there $J^{\alpha}_{a^+}f(x)$ and $J^{\alpha}_{b^-}f(x)$ will be classic integral.
A generalization of Hadamard fractional integrals is Hadamard k-fractional integrals that are defined by [7, 10]

$$J^{\alpha,k}_{a^+}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_a^x\left(\log\frac{x}{t}\right)^{\frac{\alpha}{k}-1}f(t)dt,\qquad x\geq a,$$

and

$$J^{\alpha,k}_{b^-}f(x)=\frac{1}{k\Gamma_k(\alpha)}\int_x^b\left(\log\frac{t}{x}\right)^{\frac{\alpha}{k}-1}f(t)dt,\qquad x\leq b,$$

where

$$\Gamma_k(\alpha)=\int_0^{\infty}t^{\alpha-1}e^{-\frac{t^k}{k}}dt,\qquad \alpha>0,$$

is the k-Gamma function. It is notable that the particular case $k=1$ gives the usual Hadamard fractional integrals.

(1) Some equalities on the Hadamard fractional integrals

In this section, we present some equalities on the Hadamard k-fractional integrals.

Theorem 1. Let $a<b$ and $f:[a,b]\to\mathbb{R}$ be a differentiable function such that $f,f^{'}\in L_1([a,b])$. Then, for each positive integer k and real positive α, the following holds:

$$\begin{aligned}
k\Gamma_k(\alpha) \quad & \Big[\tfrac{1}{6a\mu\left(0,\log\frac{a+b}{2a}\right)} J^{\alpha,k}_{\left(\frac{a+b}{2}\right)^-} f(a) + \tfrac{2}{3(a+b)\eta\left(0,\log\frac{a+b}{2a}\right)} J^{\alpha,k}_{a^+} f\left(\tfrac{a+b}{2}\right) \\
& + \tfrac{2}{3(a+b)\mu\left(0,\log\frac{2b}{a+b}\right)} J^{\alpha,k}_{b^-} f\left(\tfrac{a+b}{2}\right) + \tfrac{1}{6b\eta\left(0,\log\frac{2b}{a+b}\right)} J^{\alpha,k}_{\left(\frac{a+b}{2}\right)^+} f(b)\Big] \\
& - \tfrac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6} = \tfrac{1}{6\mu(0,\log\frac{a+b}{2a})} \int_a^{\frac{a+b}{2}} \mu\left(\log\tfrac{t}{a}, \log\tfrac{a+b}{2a}\right) f^{'}(t)dt \\
& - \tfrac{1}{3\eta\left(0,\log\frac{a+b}{2a}\right)} \int_a^{\frac{a+b}{2}} \eta\left(\log\tfrac{a+b}{2t}, \log\tfrac{a+b}{2a}\right) f^{'}(t)dt \\
& + \tfrac{1}{3\mu\left(0,\log\frac{2b}{a+b}\right)} \int_{\frac{a+b}{2}}^{b} \mu\left(\log\tfrac{2t}{a+b}, \log\tfrac{2b}{a+b}\right) f^{'}(t)dt \\
& - \tfrac{6}{6\eta\left(0,\log\frac{2b}{a+b}\right)} \int_{\frac{a+b}{2}}^{b} \eta\left(\log\tfrac{b}{t}, \log\tfrac{2b}{a+b}\right) f^{'}(t)dt\},
\end{aligned}$$

where

$$\eta(p,q) = \int_p^q x^{\frac{\alpha}{k}-1} e^{-x} dx,$$

and

$$\mu(p,q) = \int_p^q x^{\frac{\alpha}{k}-1} e^{x} dx \ .$$

Proof.
Using integration by parts and taking $u(t) = f(t)$ and $dV = \left(\log\frac{a+b}{2\tau}\right)^{\frac{\alpha}{k}-1} d\tau$ we have

$$V(t) = \int_a^t \left(\log\frac{a+b}{2\tau}\right)^{\frac{\alpha}{k}-1} d\tau \ .$$

Then,

$$\begin{aligned}
J^{\alpha,k}_{a^+} f\left(\tfrac{a+b}{2}\right) &= \tfrac{1}{k\Gamma_k(\alpha)} \int_a^{\frac{a+b}{2}} \left(\log\tfrac{a+b}{2t}\right)^{\frac{\alpha}{k}-1} f(t)dt \\
&= \tfrac{1}{k\Gamma_k(\alpha)} \Bigg(f\left(\tfrac{a+b}{2}\right) \int_a^{\frac{a+b}{2}} \left(\log\tfrac{a+b}{2\tau}\right)^{\frac{\alpha}{k}-1} d\tau \\
&- \int_a^{\frac{a+b}{2}} \left(\int_a^t \left(\log\tfrac{a+b}{2\tau}\right)^{\frac{\alpha}{k}-1} d\tau\right) f'(t)dt \Bigg).
\end{aligned} \tag{1}$$

On the other hand, by a change of variable $x = \log\frac{a+b}{2\tau}$, we have

$$\int_a^t \left(\log\frac{a+b}{2\tau}\right)^{\frac{\alpha}{k}-1} d\tau = \frac{a+b}{2} \int_{\log\frac{a+b}{2\tau}}^{\log\frac{a+b}{2a}} x^{\frac{\alpha}{k}-1} e^{-x} dx.$$

This equality and (1) imply

$$\begin{aligned} J_{a^+}^{\alpha,k} f(\tfrac{a+b}{2}) &= \tfrac{a+b}{2\mathrm{k}\Gamma_k(\alpha)}\Bigg(f(\tfrac{a+b}{2}) \int_0^{\log\frac{a+b}{2a}} x^{\frac{\alpha}{k}-1} e^{-x} dx \\ &- \int_a^{\frac{a+b}{2}} \left(\int_{\log\frac{a+b}{2\tau}}^{\log\frac{a+b}{2a}} x^{\frac{\alpha}{k}-1} e^{-x} dx \right) f'(t) dt \Bigg). \end{aligned} \tag{2}$$

the equality (2) becomes

$$J_{a^+}^{\alpha,k} f(\frac{a+b}{2})$$
$$= \frac{a+b}{2\mathrm{k}\Gamma_k(\alpha)} \left(f(\frac{a+b}{2}) \eta\left(0, \log\frac{a+b}{2a}\right) - \int_a^{\frac{a+b}{2}} \eta\left(\log\frac{a+b}{2t}, \log\frac{a+b}{2a}\right) f'(t) dt \right).$$

Then,

$$\begin{aligned} \tfrac{2\mathrm{k}\Gamma_k(\alpha)}{(a+b)\eta\left(0,\log\frac{a+b}{2a}\right)} \quad & J_{a^+}^{\alpha,k} f(\tfrac{a+b}{2}) \\ &= f(\tfrac{a+b}{2}) - \tfrac{1}{\eta\left(0,\log\frac{a+b}{2a}\right)} \int_a^{\frac{a+b}{2}} \eta\left(\log\tfrac{a+b}{2t}, \log\tfrac{a+b}{2a}\right) f'(t) dt. \end{aligned} \tag{3}$$

Similarly, by taking $U(t) = f(t)$ and $dV = \left(\log\frac{t}{a}\right)^{\frac{\alpha}{k}-1} dt$, we have

$$V(t) = -\int_t^{\frac{a+b}{2}} \left(\log\frac{\tau}{a}\right)^{\frac{\alpha}{k}-1} d\tau.$$

Then,

$$\begin{aligned} J_{(\frac{a+b}{2})^-}^{\alpha,k} f(a) &= \tfrac{1}{k\Gamma_k(\alpha)} \int_a^{\frac{a+b}{2}} \left(\log\tfrac{t}{a}\right)^{\frac{\alpha}{k}-1} f(t) dt \\ &= \tfrac{1}{k\Gamma_k(\alpha)} \Bigg(f(a) \int_a^{\frac{a+b}{2}} \left(\log\tfrac{\tau}{a}\right)^{\frac{\alpha}{k}-1} d\tau \\ &+ \int_a^{\frac{a+b}{2}} \left(\int_t^{\frac{a+b}{2}} \left(\log\tfrac{\tau}{a}\right)^{\frac{\alpha}{k}-1} d\tau \right) f'(t) dt \Bigg). \end{aligned} \tag{4}$$

By change of variable $x = \log\frac{\tau}{a}$, the relation (4) is rewritten by

$$\begin{aligned} J_{(\frac{a+b}{2})^-}^{\alpha,k} f(a) &= \tfrac{a}{k\Gamma_k(\alpha)} \Bigg(f(a) \int_0^{\log\frac{a+b}{2a}} x^{\frac{\alpha}{k}-1} e^x dx \\ &+ \int_a^{\frac{a+b}{2}} \left(\int_{\log\frac{t}{a}}^{\log\frac{a+b}{2a}} x^{\frac{\alpha}{k}-1} e^x dx \right) f'(t) dt \Bigg). \end{aligned} \tag{5}$$

The relation (5) is rewritten by

$$J^{\alpha,k}_{(\frac{a+b}{2})^-}f(a)$$
$$= \frac{a}{k\Gamma_k(\alpha)}\left(f(a)\mu\left(0,\log\frac{a+b}{2a}\right)+\int_a^{\frac{a+b}{2}}\mu\left(\log\frac{t}{a},\log\frac{a+b}{2a}\right)f^{'}(t)dt\right).$$

Then,

$$\frac{k\Gamma_k(\alpha)}{a\mu\left(0,\log\frac{a+b}{2a}\right)}J^{\alpha,k}_{(\frac{a+b}{2})^-}f(a)$$
$$= f(a)+\frac{1}{\mu\left(0,\log\frac{a+b}{2a}\right)}\int_a^{\frac{a+b}{2}}\mu\left(\log\frac{t}{a},\log\frac{a+b}{2a}\right)f^{'}(t)dt. \tag{6}$$

In a similar manner, we obtain

$$\frac{k\Gamma_k(\alpha)}{b\eta\left(0,\log\frac{2b}{a+b}\right)}J^{\alpha,k}_{(\frac{a+b}{2})^+}f(b)$$
$$= f(b)-\frac{1}{\eta\left(0,\log\frac{2b}{a+b}\right)}\int_{\frac{a+b}{2}}^{b}\eta\left(\log\frac{b}{t},\log\frac{2b}{a+b}\right)f^{'}(t)dt\ . \tag{7}$$

and

$$\frac{2\mathrm{k}\Gamma_k(\alpha)}{(a+b)\mu\left(0,\log\frac{2b}{a+b}\right)}J^{\alpha,k}_{b^-}f(\tfrac{a+b}{2})$$
$$= f(\tfrac{a+b}{2})+\frac{1}{\mu\left(0,\log\frac{2b}{a+b}\right)}\int_{\frac{a+b}{2}}^{b}\mu\left(\log\frac{2t}{a+b},\log\frac{2b}{a+b}\right)f^{'}(t)dt\ . \tag{8}$$

Thus, in view of (3) and (6-8), the theorem is proved. ■
By a change of variable $x \to -x$, we have

$$\mu(p,q) = (-1)^{\frac{\alpha}{k}-1}\eta(-q,-p).$$

This equality and noting that $-\log x = \log 1/x$, we can deduce the following corollary:

Corollary 2. Under the conditions given in Theorem 1, the following holds:

$$k\Gamma_k(\alpha)\left[\frac{(-1)^{\frac{\alpha}{k}-1}}{6a\eta\left(\log\frac{2a}{a+b},0\right)}J^{\alpha,k}_{(\frac{a+b}{2})^-}f(a)+\frac{2}{3(a+b)\eta\left(0,\log\frac{a+b}{2a}\right)}J^{\alpha,k}_{a^+}f(\tfrac{a+b}{2})\right.$$
$$\left.+\frac{2(-1)^{\frac{\alpha}{k}-1}}{3(a+b)\eta\left(\log\frac{2b}{a+b},0\right)}J^{\alpha,k}_{b^-}f(\tfrac{a+b}{2})+\frac{2}{3b\eta\left(0,\log\frac{2b}{a+b}\right)}J^{\alpha,k}_{(\frac{a+b}{2})^+}f(b)\right]-\frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}$$

$$= \frac{(-1)^{\frac{\alpha}{k}-1}}{6\eta\left(\log\frac{2a}{a+b},0\right)} \int_a^{\frac{a+b}{2}} \mu\left(\log\frac{t}{a}\ ,\log\frac{a+b}{2a}\ \right) f^{'}(t)dt$$

$$-\frac{1}{3\eta\left(0,\log\frac{a+b}{2a}\ \right)} \int_a^{\frac{a+b}{2}} \eta\left(\log\frac{a+b}{2t}\ ,\log\frac{a+b}{2a}\ \right) f^{'}(t)dt$$

$$+\frac{(-1)^{\frac{\alpha}{k}-1}}{3\eta\left(\log\frac{2b}{a+b},0\ \right)} \int_{\frac{a+b}{2}}^{b} \mu\left(\log\frac{2t}{a+b}\ ,\log\frac{2b}{a+b}\ \right) f^{'}(t)dt$$

$$-\frac{1}{6\eta\left(0,\log\frac{2b}{a+b}\ \right)} \int_{\frac{a+b}{2}}^{b} \eta\left(\log\frac{b}{t}\ ,\log\frac{2b}{a+b}\ \right) f^{'}(t)dt.$$

(1) Some inequalities for Hadamard k-fractional integrals

In this section, we present, a lower bound and an upper bound for Hadamrd k-fractional integral.

Theorem 3. Let $a < b$ and $f : [a, b] \to \mathbb{R}$ be a differentiable function such that $f, f^{'} \in L_1([a, b])$. Moreover, assume that f' is a convex function. Then, for each positive integer k and real positive α, the following inequality holds:

$$\left|\frac{k\Gamma_k(\alpha)}{b-a}\left[\frac{(-1)^{\frac{\alpha}{k}-1}}{6a\eta\left(\log\frac{2a}{a+b},0\ \right)} J^{\alpha,k}_{\left(\frac{a+b}{2}\right)^-} f(a) + \frac{2}{3(a+b)\eta\left(0,\log\frac{a+b}{2a}\ \right)} J^{\alpha,k}_{a^+} f\left(\frac{a+b}{2}\right)\right.\right.$$
$$\left.+\frac{2(-1)^{\frac{\alpha}{k}-1}}{3(a+b)\eta\left(\log\frac{2b}{a+b},0\ \right)} J^{\alpha,k}_{b^-} f\left(\frac{a+b}{2}\right) + \frac{2}{3b\eta\left(0,\log\frac{2b}{a+b}\ \right)} J^{\alpha,k}_{\left(\frac{a+b}{2}\right)^+} f(b)\right]$$
$$\left.-\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}\right| \le \frac{1}{3}\left(|f^{'}(a)| + |f^{'}(b)|\right).$$

Proof. By Corollary 2 and triangular inequality, we have

$$\left|k\Gamma_k(\alpha)\left[\frac{(-1)^{\frac{\alpha}{k}-1}}{6a\eta\left(\log\frac{2a}{a+b},0\ \right)} J^{\alpha,k}_{\left(\frac{a+b}{2}\right)^-} f(a) + \frac{2}{3(a+b)\eta\left(0,\log\frac{a+b}{2a}\ \right)} J^{\alpha,k}_{a^+} f\left(\frac{a+b}{2}\right)\right.\right.$$
$$\left.+\frac{2(-1)^{\frac{\alpha}{k}-1}}{3(a+b)\eta\left(\log\frac{2b}{a+b},0\ \right)} J^{\alpha,k}_{b^-} f\left(\frac{a+b}{2}\right) + \frac{2}{3b\eta\left(0,\log\frac{2b}{a+b}\ \right)} J^{\alpha,k}_{\left(\frac{a+b}{2}\right)^+} f(b)\right]$$
$$\left.-\frac{f(a)+4f\left(\frac{a+b}{2}\right)+f(b)}{6}\right|$$
$$\le \left|\int_a^{\frac{a+b}{2}} \left[\frac{\eta\left(\log\frac{2a}{a+b}\ ,\log\frac{a}{t}\ \right)}{6\eta\left(\log\frac{2a}{a+b},0\ \right)} - \frac{\eta\left(\log\frac{a+b}{2t}\ ,\log\frac{a+b}{2a}\ \right)}{3\eta\left(0,\log\frac{a+b}{2a}\ \right)}\right] f^{'}(t)dt\right|$$
$$+\left|\int_{\frac{a+b}{2}}^{b} \left[\frac{\eta\left(\log\frac{a+b}{2t},\log\frac{a+b}{2b}\ \right)}{3\eta\left(\log\frac{a+b}{2b},0\ \right)} - \frac{\eta\left(\log\frac{b}{t}\ ,\log\frac{2b}{a+b}\ \right)}{6\eta\left(0,\log\frac{2b}{a+b}\ \right)}\right] f^{'}(t)dt\right|.$$

Now,

$$M_1 := \max_{a\le t\le \frac{a+b}{2}} \left| \frac{\eta\left(\log\frac{2a}{a+b}\ ,\log\frac{a}{t}\ \right)}{6\eta\left(\log\frac{2a}{a+b},0\ \right)} - \frac{\eta\left(\log\frac{a+b}{2t}\ ,\log\frac{a+b}{2a}\ \right)}{3\eta\left(0,\log\frac{a+b}{2a}\ \right)} \right| = \frac{1}{3},$$

and

$$M_2 := \max_{\frac{a+b}{2}\le t\le b} \left| \frac{\eta\left(\log\frac{a+b}{2t},\log\frac{a+b}{2b}\ \right)}{3\eta\left(\log\frac{a+b}{2b},0\ \right)} - \frac{\eta\left(\log\frac{b}{t}\ ,\log\frac{2b}{a+b}\ \right)}{6\eta\left(0,\log\frac{2b}{a+b}\ \right)} \right| = \frac{1}{3}.$$

This implies that

$$\Bigg| k\Gamma_k(\alpha) \quad \Bigg[\tfrac{(-1)^{\frac{\alpha}{k}-1}}{6a\eta\left(\log\frac{2a}{a+b},0\ \right)} J^{\alpha,k}_{(\frac{a+b}{2})^-} f(a) + \tfrac{2}{3(a+b)\eta\left(0,\log\frac{a+b}{2a}\ \right)} J^{\alpha,k}_{a^+} f(\tfrac{a+b}{2})$$

$$+\tfrac{2(-1)^{\frac{\alpha}{k}-1}}{3(a+b)\eta\left(\log\frac{2b}{a+b},0\ \right)} J^{\alpha,k}_{b^-} f(\tfrac{a+b}{2}) + \tfrac{2}{3b\eta\left(0,\log\frac{2b}{a+b}\ \right)} J^{\alpha,k}_{(\frac{a+b}{2})^+} f(b) \Bigg]$$

$$-\tfrac{f(a)+4f(\frac{a+b}{2})+f(b)}{6} \Bigg| \le (M_1+M_2)\int_a^b |f^{'}(t)|dt = \tfrac{2}{3}\int_a^b |f^{'}(t)|dt.$$

Now, taking $t = a\left(1-s\right)+bs$, for $0\le s\le 1$ and using the convexity of f', we have

$$\Bigg| k\Gamma_k(\alpha) \quad \Bigg[\tfrac{(-1)^{\frac{\alpha}{k}-1}}{6a\eta\left(\log\frac{2a}{a+b},0\ \right)} J^{\alpha,k}_{(\frac{a+b}{2})^-} f(a) + \tfrac{2}{3(a+b)\eta\left(0,\log\frac{a+b}{2a}\ \right)} J^{\alpha,k}_{a^+} f(\tfrac{a+b}{2})$$

$$+\tfrac{2(-1)^{\frac{\alpha}{k}-1}}{3(a+b)\eta\left(\log\frac{2b}{a+b},0\ \right)} J^{\alpha,k}_{b^-} f(\tfrac{a+b}{2}) + \tfrac{2}{3b\eta\left(0,\log\frac{2b}{a+b}\ \right)} J^{\alpha,k}_{(\frac{a+b}{2})^+} f(b) \Bigg]$$

$$-\tfrac{f(a)+4f(\frac{a+b}{2})+f(b)}{6} \Bigg| \le \tfrac{2}{3}\left(b-a\right)\int_0^1 \left|f^{'}\left(a\left(1-s\right)+bs\right)\right|ds$$

$$\le \tfrac{b-a}{3}\left(|f^{'}(a)|+|f^{'}(b)|\right). \qquad \blacksquare$$

NOTE: If $J^{\alpha,k}_{(\frac{a+b}{2})^-} f(a)$, $J^{\alpha,k}_{a^+} f(\frac{a+b}{2})$, $J^{\alpha,k}_{b^-} f(\frac{a+b}{2})$ and $J^{\alpha,k}_{(\frac{a+b}{2})^+} f(b)$ are positive values, then in view of Theorem 3,

$$\Bigg| \tfrac{\mathrm{mk}\Gamma_k(\alpha)}{(b-a)} \quad \Bigg[J^{\alpha,k}_{(\frac{a+b}{2})^-} f(a) + J^{\alpha,k}_{a^+} f(\tfrac{a+b}{2}) + J^{\alpha,k}_{b^-} f(\tfrac{a+b}{2}) + J^{\alpha,k}_{(\frac{a+b}{2})^+} f(b) \Bigg]$$

$$-\tfrac{f(a)+4f(\frac{a+b}{2})+f(b)}{6} \Bigg| \le \tfrac{1}{3}\left(|f^{'}(a)|+|f^{'}(b)|\right),$$

where

$$m=\min\left\{\frac{(-1)^{\frac{\alpha}{k}-1}}{6a\eta\left(\log\frac{2a}{a+b},0\right)},\frac{2}{3\left(a+b\right)\eta\left(0,\log\frac{a+b}{2a}\right)},\frac{2(-1)^{\frac{\alpha}{k}-1}}{3\left(a+b\right)\eta\left(\log\frac{2b}{a+b},0\right)},\frac{2}{3b\eta\left(0,\log\frac{2b}{a+b}\right)}\right\}.$$

Also, If $J^{\alpha,k}_{(\frac{a+b}{2})^-}f(a)$, $J^{\alpha,k}_{a^+}f(\frac{a+b}{2})$, $J^{\alpha,k}_{b^-}f(\frac{a+b}{2})$ and $J^{\alpha,k}_{(\frac{a+b}{2})^+}f(b)$ are negative values, then in view of Theorem 3,

$$\left|\frac{\mathrm{Mk}\Gamma_k(\alpha)}{(b-a)}\left[J^{\alpha,k}_{(\frac{a+b}{2})^-}f(a)+J^{\alpha,k}_{a^+}f(\tfrac{a+b}{2})+J^{\alpha,k}_{b^-}f(\tfrac{a+b}{2})+J^{\alpha,k}_{(\frac{a+b}{2})^+}f(b)\right]\right.$$
$$\left.-\frac{f(a)+4f(\frac{a+b}{2})+f(b)}{6}\right|\leq\frac{1}{3}\left(|f^{'}(a)|+|f^{'}(b)|\right),$$

where

$$M=\max\left\{\frac{(-1)^{\frac{\alpha}{k}-1}}{6a\eta\left(\log\frac{2a}{a+b},0\right)},\frac{2}{3\left(a+b\right)\eta\left(0,\log\frac{a+b}{2a}\right)},\frac{2(-1)^{\frac{\alpha}{k}-1}}{3\left(a+b\right)\eta\left(\log\frac{2b}{a+b},0\right)},\frac{2}{3b\eta\left(0,\log\frac{2b}{a+b}\right)}\right\}.$$

In the following, we present a lower bound for $J^{\alpha}_{a^+}f(b)+J^{\alpha,k}_{b^-}f(a)$ and $J^{\alpha,k}_{(\frac{a+b}{2})^-}f(a)+J^{\alpha,k}_{a^+}f(\frac{a+b}{2})+J^{\alpha,k}_{b^-}f(\frac{a+b}{2})+J^{\alpha,k}_{(\frac{a+b}{2})^+}f(b)$.

Theorem 4. Let $0<a<b$, and $f:[a,b]\to\mathbb{R}$ be a convex function. Then, the following inequalities hold:

$$(b-a)\,k\Gamma_k(\alpha)f(\frac{a+b}{2})\left(\log\frac{b}{a}\right)^{\frac{\alpha}{k}-1}\leq J^{\alpha}_{a^+}f(b)+J^{\alpha,k}_{b^-}f(a),$$

and

$$(b-a)\,k\Gamma_k(\alpha)\left[f\left(\tfrac{3a+b}{4}\right)\left(\log\tfrac{a+b}{2a}\right)^{\frac{\alpha}{k}-1}+f\left(\tfrac{a+3b}{4}\right)\left(\log\tfrac{2b}{a+b}\right)^{\frac{\alpha}{k}-1}\right]$$
$$\leq J^{\alpha,k}_{(\frac{a+b}{2})^-}f(a)+J^{\alpha,k}_{a^+}f(\tfrac{a+b}{2})+J^{\alpha,k}_{b^-}f(\tfrac{a+b}{2})+J^{\alpha,k}_{(\frac{a+b}{2})^+}f(b).$$

Proof. For $0\leq s\leq 1$, let $x=as+(1-s)b$ and $y=bs+(1-s)\,a$. The convexity of f implies that

$$f(\frac{a+b}{2})\leq f\left(\frac{x+y}{2}\right)\leq\frac{1}{2}f\left(x\right)+\frac{1}{2}f\left(y\right).$$

Then,

$$f(\frac{a+b}{2})\leq\frac{1}{2}f\left(as+(1-s)\,b\right)+\frac{1}{2}f\left(bs+(1-s)\,a\right).\tag{9}$$

Now, we define the operator T for each positive real valued p and q,

$$T\left(p-q\right)=\log p-\log q.$$

It is seen that if $p > q$, then $T(p-q)$ is a positive. Multiplying both side of (9) by $T\left[(b-a)s\right]$ and taking integral yield:

$$\begin{aligned}\frac{1}{k\Gamma_k(\alpha)} & \int_0^1 f(\tfrac{a+b}{2})(T\left[(b-a)\,s\right])^{\frac{\alpha}{k}-1}\,ds \\ & \leq \frac{1}{k\Gamma_k(\alpha)}\int_0^1 \tfrac{1}{2}f\left(as+(1-s)\,b\right)(T\left[(b-a)\,s\right])^{\frac{\alpha}{k}-1}ds \\ & +\frac{1}{k\Gamma_k(\alpha)}\int_0^1 \tfrac{1}{2}f\left(bs+(1-s)\,a\right)(T\left[(b-a)\,s\right])^{\frac{\alpha}{k}-1}ds.\end{aligned}$$

We note that

$$\int_0^1 (T\left[(b-a)\,s\right])^{\frac{\alpha}{k}-1}ds = \int_0^1 \left(\log\frac{b}{a}\right)^{\frac{\alpha}{k}-1}ds = \left(\log\frac{b}{a}\right)^{\frac{\alpha}{k}-1}.$$

Then,

$$\begin{aligned}& f(\tfrac{a+b}{2})\left(\log\tfrac{b}{a}\right)^{\frac{\alpha}{k}-1} \\ & \leq \frac{1}{(b-a)k\Gamma_k(\alpha)}\int_a^b \tfrac{1}{2}f\left(x\right)\left(T\left(b-x\right)\right)^{\frac{\alpha}{k}-1}dx \\ & +\frac{1}{(b-a)k\Gamma_k(\alpha)}\int_a^b \tfrac{1}{2}f\left(y\right)\left(T\left[y-a\right]\right)^{\frac{\alpha}{k}-1}dy \\ & = \frac{1}{2(b-a)k\Gamma_k(\alpha)}\left[\int_a^b f\left(x\right)\left(\log\tfrac{b}{x}\right)^{\frac{\alpha}{k}-1}dx + \int_a^b f\left(y\right)\left(\log\tfrac{y}{a}\right)^{\frac{\alpha}{k}-1}dy\right] \\ & = \frac{1}{2(b-a)k\Gamma_k(\alpha)}\left[J^{\alpha}_{a^+}f(b) + J^{\alpha,k}_{b^-}f\left(x\right)\right].\end{aligned}$$

Thus,

$$f(\frac{a+b}{2})\left(\log\frac{b}{a}\right)^{\frac{\alpha}{k}-1} \leq \frac{1}{2\left(b-a\right)k\Gamma_k(\alpha)}\left[J^{\alpha}_{a^+}f(b) + J^{\alpha,k}_{b^-}f(a)\right].$$

This inequality gives:

$$\begin{aligned}& f\left(\tfrac{3a+b}{4}\right)\left(\log\tfrac{a+b}{2a}\right)^{\frac{\alpha}{k}-1} + f\left(\tfrac{a+3b}{4}\right)\left(\log\tfrac{2b}{a+b}\right)^{\frac{\alpha}{k}-1} \\ & \leq \frac{1}{(b-a)k\Gamma_k(\alpha)}\left[J^{\alpha,k}_{(\frac{a+b}{2})^-}f(a) + J^{\alpha,k}_{a^+}f(\tfrac{a+b}{2}) + J^{\alpha,k}_{b^-}f(\tfrac{a+b}{2}) + J^{\alpha,k}_{(\frac{a+b}{2})^+}f(b)\right].\end{aligned}$$

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