J. Appl. Math. & Informatics Vol. 42(2024), No. 5, pp. 1025 - 1037
https://doi.org/10.14317/jami.2024.1025

A STUDY OF THE C-GRAPH AND PROPER C-GRAPH OF THE SYMMETRIC GROUP S_n, $n \in \mathbb{N}^{\dagger}$

SINU N. VIJAYAN*, ANJALY KISHORE

Abstract. Defined the proper C-graph $G^*(\Gamma)$ of a group Γ of order n as a subgraph of the C-graph $G(\Gamma)$ induced by the vertex set $\Gamma \setminus \{i\}$, where i is the identity element of the group Γ. Observed that, the number of undirected edges of these two graphs are same, but the number of directed edges of $G^*(\Gamma)$ is $n-1$ less than that of $G(\Gamma)$. Established a characterization of the groups for which the proper C-graph is a complete undirected graph. Obtained a method to identify permutations in S_n that are either a transposition or a product of disjoint transpositions, directly from the corresponding proper C-graph $G^*(S_n)$.

AMS Mathematics Subject Classification : 05C90.
Key words and phrases : C-graph, induced subgraph, proper C-graph, symmetric group, transposition.

1. Introduction

Representing algebraic structures as graphs is an easy way to study its properties, and is a largely focused area for research. Many graphical representations of algebraic strucutres such as, groups, semigroups, commutative and non commutative rings are available in the literature. In fact, most of these are on groups, like Cayley graph[11][4], Commuting graph[19], Directed power graph[16], Divisibility graph, Annhilator graph[15], Relative non nil-n graph[3] Enhanced power graph[7], Non exterior square graph[20], Co-prime order graph[24], Non commuting graph [2]. The Undirected power graph[13] and Identity graph[10] are constructed for semigroups. The Zero divisor graph is defined for both commutative[6], and non commutative [21] rings. The Total graph[9] is also a graphical representation of commutative rings. Most of the

Received April 3, 2023. Revised June 5, 2024. Accepted September 13, 2024. *Corresponding author.
†This research work is funded by University Grants Commision, India and was carried out at the Department of Mathematics, St. Mary's College, Thrissur, India, affiliated to University of Calicut.

above mentioned graphs are undirected except probably the Cayley graph and Directed power graph.

C-graph is another representation of the algebraic structure group, with vertex set as the group itself. The construction of C-graph is based on the question, 'What is the key role of an element in the group that distinguishes it from other elements?'[22]. In a group theoretical view, it can be the order of each element, or more clearly, it can be the set of all elements generated by an element (cyclic subgroup) in the group. Thus, in this graph, the adjacency relation between vertices is based on the cyclic subgroups generated by corresponding elements of the group. A study of the structural properties of C-graphs on the basis of corresponding group structure and vice versa can be carry out in this area of research.

Main objective of this paper is to deduce the concept of proper C-graph of a group from its C-graph and discuss some of the structural properties of the proper C-graph of the symmtric group S_n, $n \in \mathbb{N}$. The paper comprises of two sections - the first one is on the induced subgraph of C-graph and the other is on the proper C-graph of groups.

Throughout this paper, Γ is a group and G is a graph. For an element a in Γ, $o(a)$ is the order of a. The identity element of the group is denoted by i.

2. Preliminaries and Notations

To start this discussion of interconnection between two branches of mathematics, namely Abstract Algebra and Graph Theory, some basic understanding of group theory is required. Let us begin with few elementary definitions as follows;

Definition 2.1. [11] *A permutation of a set A is a function $\phi : A \longrightarrow A$ that is both one to one and onto. In other words, a permutation of A is a one-to-one function from A onto A.*

Definition 2.2. [11] *Let A be the finite set $A = \{1, 2, \ldots, n\}$. The group of all permutations of A is the symmetric group on n letters, and is denoted by S_n.*

Definition 2.3. [11] *A permutation $\sigma \in S_n$ is a cycle, if it has at most one orbit containing more than one element. The length of a cycle is the number of elements in its largest orbit.*

Definition 2.4. [11] *A cycle of length 2 is a transposition*

Before entering into the definition of C-graph of a group, let us discuss the notion of comparability of elements of a group.

Observations[22] For a group Γ and two distinct elements x and y of Γ,

(1) If either $\langle x \rangle \subseteq \langle y \rangle$ or $\langle y \rangle \subseteq \langle x \rangle$, then we can say that $\langle x \rangle$ and $\langle y \rangle$ are comparable in Γ and so are the elements x and y.

(2) For any two comparable elements x and y of Γ,

(a) If $\langle x\rangle \subset \langle y\rangle$, then $\langle x\rangle$ is weaker and $\langle y\rangle$ is stronger cyclic subgroups and also x is weaker and y is stronger element in the group Γ.
(b) If $\langle x\rangle = \langle y\rangle$, then the two cyclic subgroups are equal in strength and so are the elements x and y.

Thus, for any two comparable elements x and y of Γ, the element which generates weaker cyclic subgroup is the weaker element and the other is stronger. So, the power of generating more elements of the group is considered as a measure of strength for comparable elements.

C-graph is the graphical representation based on the concept of comparability of elements of a group.
Given below is the definition of C-graph of a group.

Definition 2.5. [22] *For a group Γ, the C-graph $G = (\Gamma, E, E')$ is the graph whose vertex set $V(G) = \Gamma$ and for $x, y \in \Gamma$, $x \neq y$,*
if $\langle x\rangle \subset \langle y\rangle$, then there is a directed edge from x to y.
if $\langle x\rangle = \langle y\rangle$, then there is an undirected edge between x and y.
where E is the set of all undirected edges and E' is the set of all directed edges of G.

For two elements x and y of the group Γ, an undirected edge between them in the corresponding C-graph $G = (\Gamma, E, E')$, is denoted as xy and a directed edge from x to y is denoted as $x \longrightarrow y$.
Given below observations explain how the adjacencies in C-graph reveals the comparability of elements of the group.

Observations[22] For a group Γ and C-graph $G = (\Gamma, E, E')$,

(1) Any two elements x and y are comparable in the group Γ if and only if there is an edge between x and y in the C-graph G.
(2) For $x, y \in \Gamma$,
$x \longrightarrow y$ in G if and only if x is weaker and y is stronger in Γ.
xy in G if and only if x and y are equal in strength in Γ.

That is, the C-graph is defined as a mixed graph(a graph with both directed and undirected edges), in order to analyse the pattern of communication between nodes. Here, an undirected edge represents the two way communication between the nodes and a directed edge denotes the one way communication(in which the direction imply the flow of communication) betweeen the nodes. Thus, if we consider it as a communication network, the flow of communication is more explicit here.

For a vertex x of the C-graph G of Γ, there are three types of degrees namely, **degree**, the number of undirected edges incident with x, **in-degree**, the number of directed edges with x as their terminal vertex and **out-degree**, the number of directed edges with x as their initial vertex. These three degrees are denoted by $deg(x)$, $deg^-(x)$ and $deg^+(x)$ respectively, for a vertex x of the C-graph $G(\Gamma)$.

3. Induced subgraph

In this section, the nature of the subgraph of the C-graph $G(\Gamma)$ induced by a subgroup H of the group Γ is studied. For this, let us recall the definition of induced subgraph of a group.

Definition 3.1. *Let $G = (V, E)$ be any graph, and let $S \subset V$ be any subset of vertices of G. Then, the induced subgraph G_S is the graph whose vertex set is S and whose edge set consists of all of the edges in E that have endpoints in S. That is, for any two vertices $u, v \in S$, u and v are adjacent in G_S if and only if they are adjacent in G.*

Note 3.1. *Let Γ be a group and $G(\Gamma)$ be its C-graph, then corresponding to any subgroup H of Γ, $G(\Gamma)_H$ denotes the subgraph of $G(\Gamma)$ induced by the subset H of the vertex set Γ.*

Theorem 3.1. *For a group Γ and subgroup H of Γ, $G(\Gamma)_H = G(H)$, where $G(H)$ is the C-graph of H (This is possible, since being a subgroup of Γ, H itself is a group).*

Proof. From the definition of C-graph of a group and induced subgraph, it is clear that, $V(G(\Gamma)_H) = H = V(G(H))$.
Now, let $x, y \in H$
Case 1 : $x \longrightarrow y$ in $G(\Gamma)_H \iff x \longrightarrow y$ in $G(\Gamma)$.
$\iff \langle x\rangle \subset \langle y\rangle$ in Γ.
$\iff \langle x\rangle \subset \langle y\rangle$ in H.
$\iff x \longrightarrow y$ in $G(H)$.
Case 2 : xy in $G(\Gamma)_H \iff xy$ in $G(H)$, by an argument similar to case 1.
Thus, $G(\Gamma)_H$ and $G(H)$ are two graphs with the same vertex set and adjacencies.
Hence, $G(\Gamma)_H = G(H)$. □

That is, the above theorem says that, for any subgroup H of a group Γ, the induced subgraph of the C-graph $G(\Gamma)$ induced by H is same as the C-graph of the subgroup H.

4. Proper C-graph of a group

In this section the subgraph of $G(\Gamma)$ induced by the set of all non identity elements of the group is studied. The resultant graph is named as proper C-graph $G^*(\Gamma)$ of a group Γ. The purpose of considering $G^*(\Gamma)$ is to reduce the complexity of C-graph of larger groups, so that some characteristics of non identity elements of the group can be more easily identified. Because of the presence of directed edges from the identity to all the non identity elements, the C-graph G of any group Γ of order n has the following two properties;

(1) No isolated vertices are in G.
(2) Number of directed edges in $G \geq$ n-1.

Thus, the identity vertex performs like a center of the network, which can pass information or data to all the other vertices, but no data is sent back to this center vertex. So, in order to study the transmission of data between the non identity vertices, which received data from the identity, the proper C-graph can be used. Also, this leads to the identification of the stability of communication between the non identity elements of the group.

According to the concept of comparability of elements of a group (in which the ordering of elements of a group is done on the basis of the cyclic subgroups generated by each element), the identity element has the property that, it is weaker than every other element of the group, and it is the only element with this property in any group. This leads to the uniqueness of the identity element in the group. So, here onwards, the identity element is named as the unique **Totally weaker element** T_{we} of the group. In other words we can say that, every element of the group is atleast comparable to the identity. More clearly, every element of the group is atleast stronger than the identity. Similarly, in the corresponding C-graph, presence of directed edges from the identity to all the other vertices, make the identity **Totally weaker vertex** T_{wv} in the C-graph. Also, this T_{wv} ensures the fact that there is no isolated vertex in the C-graph G of a group Γ.
Now, we consider the Totally weaker vertex of a subgroup of the C-graph.

Definition 4.1. *A vertex x of a subgraph H of the C-graph $G(\Gamma)$ of a group Γ is said to be a **Totally weaker vertex**, if it is weaker than every other vertex of H. In that case, x is denoted as $T_{wv}(H)$.*

Next, we are going to consider stronger elements and stronger vertices of a group and C-graph respectively.

Definition 4.2. *An element y in a group Γ is said to be a **Totally stronger element**, if for any $x \neq y$ in Γ, either y is stronger than or equal in strength that of x.*

Definition 4.3. *A vertex y in a C-graph $G(\Gamma)$ of a group Γ is said to be a **Totally stronger vertex**, if for any other vertex x in $G(\Gamma)$, either there is an undirected edge connecting x and y or a directed edge from x to y in $G(\Gamma)$.*

Note 4.1.
(1) *A group can have more than one Totally stronger elements. The set of all Totally stronger elements of a group Γ is denoted by $T_{se}(\Gamma)$.*
(2) *Similarly, a C-graph $G(\Gamma)$ can have more than one Totally stronger vertices. The set of all Totally stronger vertices in a C-graph $G(\Gamma)$ is denoted as $T_{sv}(G(\Gamma))$.*

Definition 4.4. *Let Γ be a group and $G(\Gamma)$ be its C-graph, the proper C-graph of Γ, denoted as $G^*(\Gamma)$ is the subgraph induced by the set $\Gamma \setminus \{i\}$, where i is the identity element in Γ. That is, $G^*(\Gamma) = G(\Gamma)_{\Gamma \setminus \{i\}}$.*

Example 4.1. *Given below are the C-graph and proper C-graph of the group* $\Gamma_1 = S_3$

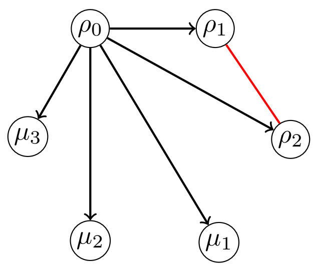

Fig.1 C-graph $G(S_3)$

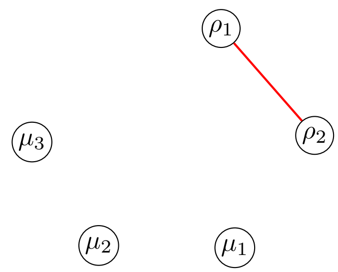

Fig.2 Proper C-graph $G^*(S_3)$

Example 4.2. *C-graph and proper C-graph of the group* $\Gamma_2 = \mathbb{Z}_8$

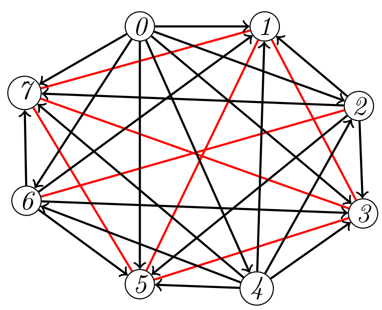

Fig.3 C-graph $G_2(\mathbb{Z}_8)$

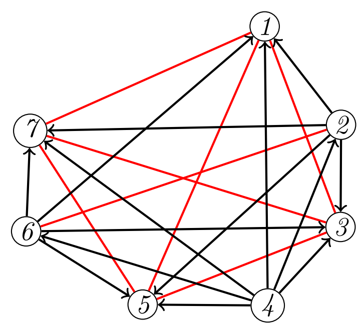

Fig.4 C-graph $G_2^*(\mathbb{Z}_8)$

Remark 4.1. *Let* Γ *be a finite group of order* n*,* $G(\Gamma)$ *and* $G^*(\Gamma)$ *be its C-graph and proper C-graph respectively. then,* G *and* G^* *have the same number of undirected edges. Also, the number of directed edges of* G^* *is* $n-1$ *less than the number of directed edges of* G*.*

Theorem 4.1. *(Cauchy's Theorem)*[11] *Let* Γ *be a finite group and let* p *divides* $|\Gamma|$*. Then* Γ *has an element of order* p *and, consequently, a subgroup of order* p*.*

Theorem 4.2. [22] *Let* Γ *be a finite group and* $G = (\Gamma, E, E')$ *be the C-graph of* Γ*. For any element* a *of the group* Γ*, order of* a *can be deduced from the C-graph* G*. In particular,* $o(a) = deg(a) + deg^-(a) + 1$*.*

Proposition 4.1. [22] *For any prime* p*, the C-graph* $G = (\Gamma, E, E')$ *of the finite group* Z_p *has the following properties.*

(1) G *contains a complete subgraph with* $p-1$ *vertices all of its edges are undirected.*
(2) G *consists of only* $p-1$ *directed edges, all of it are of the form* $(0, t)$ *where* $t = 1, 2, 3, \ldots, p-1$*.*
(3) $deg(0) = deg^-(0) = 0$*,* $deg^+(0) = p-1$*.*

(4) $deg(x) = p - 1$, *if* $x \neq 0$.
(5) $deg^-(x) = 1$, $deg^+(x) = 0$, *if* $x \neq 0$.

Note 4.2. *For a finite group* Γ*, C-graph* G *and proper C-graph* G^**, the following are true.*

(1) $deg(a)$ *in* G $= deg(a)$ *in* G^*, $\forall a \in \Gamma$, $a \neq i$.
(2) $deg^+(a)$ *in* G $= deg^+(a)$ *in* G^*, $\forall a \in \Gamma$, $a \neq i$.
(3) $deg^-(a)$ *in* G $= deg^-(a)$ *in* G^* $+1$, $\forall a \in \Gamma$, $a \neq i$.

That is, every element other than i in Γ, has the same degree in both C-graph and proper C-graph, the same is true in the case of out-degree also. But, the in-degree of each vertex in G is one more than the in-degree of that vertex in G^*

The following theorem gives a necessary and sufficient condition for the proper C-graph G^* of a finite group Γ to be complete in terms of the order of the group.

Theorem 4.3. *The proper C-graph* G^* *of a finite group* Γ *of order* n *is a complete undirected graph if and only if* n *is prime.*

Proof. Suppose that G^* is a complete undirected graph, ie, $G^* \cong K_{n-1}$. If possible, let us assume that n is a composite number, then there exist a prime number p, $1 \neq p \neq n$, such that p divides n. Now, Cauchy's theorem assures, there exist an element a in Γ such that $o(a) = p$ and $o(\langle a \rangle) = p$.
By theorem 4.2

$$o(a) = deg(a) + deg^-(a) + 1, \ \forall a \in \Gamma \tag{1}$$

where $deg(a)$ and $deg^-(a)$ are the degree and in-degree of the vertex a in G.

$$\implies o(a) = deg(a) \text{ in } G^* + deg^-(a) \text{ in } G^* + 1 + 1, \forall a \in \Gamma, a \neq i, \text{ by note 4.1}$$

which is a contradiction. Therefore, the order n of the group Γ is a prime number.
Conversely, suppose that, $n = p$, a prime number, then $\Gamma \cong Z_p$. Now, by (1) and (2) of proposition 4.1, $G^* \cong K_{p-1}$. That is, the proper C-graph G^* of Γ is a complete undirected graph. □

Rest of this section discusses some observations of the proper C-graph of the symmetric group S_n, $n \in \mathbb{N}$. For this, consider the below theorem on permutations of a finite set.

Theorem 4.4. [11] *Every permutation* σ *of a finite set is a product of disjoint cycles.*

Theorem 4.5. *Consider the symmetric group* S_n *of order* n*,* $n \in \mathbb{N}$*. A vertex* x *in the proper C-graph* $G^*(S_n)$ *with* $deg(x) = t$ *and* $deg^-(x) = 0$*, where* t *is an odd number such that* $0 < t \leq n - 2$*, is a* $t + 2$ *cycle in* S_n*.*

Proof. For the vertex x in the proper C-graph $G^*(S_n)$,

$$\begin{aligned} o(x) &= deg(x) + deg^-(x) + 1 \text{ in } G(S_n) \text{ by theorem 4.2} \\ &= deg(x) + deg^-(x) \text{ in } G^*(S_n) + 2 \text{ by note 4.1} \\ &= t + 0 + 2 \\ &= t + 2 \end{aligned}$$

That is, x is a permutation of order $t+2$. Now, it is only need to prove that x is a cycle. For this, consider $\langle x \rangle$.

$$\begin{aligned} deg(x) = t &\implies \text{There exist } t \text{ disjoint vertices, say } y_j, \\ & \quad\ j = 1, 2, \dots, t \text{ different from } x \text{ in } G^*(S_n), \\ & \quad\ \text{such that } xy_j \text{is an undirected edge.} \\ &\implies \langle x \rangle = \langle y_j \rangle \ \ j = 1, 2, \dots t \\ &\implies y_j \in \langle x \rangle \ \ j = 1, 2, \dots t \end{aligned}$$

Thus, $\langle x \rangle = \{i, x, y_1, y_2, \dots, y_t\}$.
Suppose that x is not a cycle. Then, by the theorem 4.4, x is a product of cycles. Let $x = x_1 x_2 \cdots x_k$, where x_m's are disjoint cycles with length t_m. Then, order of the permutation x is the least common multiple of the lengths of the cycles x_m, $m = 1, 2, \dots, k$.
That is, $t = o(x) = lcm(t_1, t_2, \dots, t_k)$
Claim : $\forall m = 1, 2, \dots, k$, t_m is odd.
Suppose that, for an $l \in \{1, 2, \dots, k\}$, $t_l = 2d$, $d \in \mathbb{N}$.
Then,

$$\begin{aligned} t &= lcm(t_1, t_2, \dots, t_l, \dots, t_k) \\ &= \text{multiple of } t_l \\ &= \text{multiple of } 2 \end{aligned}$$

$\implies t$ is even, which is a contradiction.
Now, choose a prime number p satisfying one of the following conditions.

(1) $p = t_j$, for some $j = 1, 2, \dots, k$
(2) p divides t_j, for some $j = 1, 2, \dots, k$

Then, clearly p divides t.
Applying Cauchy's theorem to $\langle x \rangle$ gives a subgroup H of $\langle x \rangle$, so that $o(H) = p$, since $\langle x \rangle$ is cyclic, H is also cyclic.
That is, $\exists\, y \in S_n$, $y \neq i$, $y \neq x$ such that, $H = \langle y \rangle \subset \langle x \rangle$ and $|\langle y \rangle| = p$

$$\begin{aligned} &\implies \exists \text{ a directed edge } y \longrightarrow x \text{ in } G(S_n),\ y \neq i. \\ &\implies \exists \text{ a directed edge } y \longrightarrow x \text{ in } G^*(S_n). \\ &\implies deg^-(x) > 1 \text{ in } G^*(S_n), \text{ which is a contradiction.} \end{aligned}$$

$\therefore$ The permutation x in S_n is a cycle of length $t+2$. □

Thus, a vertex in the proper C-graph G^* of the symmetric group S_n with an odd degree and nil in-degree corresponds to an odd cycle in S_n.

Note 4.3. *The number t such that $deg(x) = t$ and $deg^-(x) = 0$ as mentioned in theorem 4.5 can not be even.*
If possible, Suppose that t is even.
Consider $\{y \in \Gamma : y \neq x$ and $\langle y \rangle = \langle x \rangle\} = \{x_1, x_2, \dots x_t\}$ (say), since $deg(x) = t$. Let $A = \{x, x_1, x_2, \dots x_t\}$, then $o(A) = t + 1$ is odd. Here, A is a set of

permutations in S_n all of which generates the same cyclic subgroup. Since, $|A|$ is odd, by pairing of elements of A with its inverse results in isolation of a single element say x_k, and then, $x_k^{-1} = x_k \Longrightarrow x_k^2 = i$, the identity permutation in S_n $\Longrightarrow \langle x_k \rangle = \{i, x_k\}$, which is a contradiction.
$\therefore$ There does not exist a vertex x in $G^*(S_n), n \in \mathbb{N}$, with $deg(x) = t$, $deg^-(x) = 0$, $0 < t \leq n-2$, t is even.

Theorem 4.6. *Any vertex x in the proper C-graph $G^*(S_n)$ of the symmetric group S_n, $n \in \mathbb{N}$ with $deg(x) = deg^-(x) = deg^+(x) = 0$ corresponds to a transposition in S_n.*

Proof. Consider a vertex x in $G^*(S_n)$ with $deg(x) = deg^-(x) = deg^+(x) = 0$.

$$\begin{aligned} o(x) &= deg(x) + deg^-(x) + 1 \text{ in } G \\ &= deg(x) + deg^-(x) \text{ in } G^* + 2 \\ &= 2 \end{aligned}$$

Then, the permutation x is either a transposition or a product of disjoint transpositions.
If possible, suppose that, x is not a transpostion, then x is of the form

$$\begin{aligned} x &= x_1 x_2 \dots x_t, \text{ where } x_j\text{'s are } t \text{ disjoint transpostions.} \\ &= (p_1q_1)(p_2q_2)\dots(p_{t-1}q_{t-1})(p_tq_t), \qquad t \leq \tfrac{n}{2}, p_j, q_j \in \{1,2,\dots n\}, \\ &\qquad p_j \text{ and } q_j \text{ are all distinct } j = 1,2,\dots t. \end{aligned}$$

Clearly $x^2 = i$, the identity permutation in S_n and so $o(x) = 2$. Now, consider a permutation $y = (p_1 q_t q_{t-1} \dots q_2 q_1 p_t p_{t-1} \dots p_2)$ in S_n. (Obviously $y \in S_n$, since $p_j, q_j \in \{1, 2, \dots, n\}$ and p_j, q_j are distinct). Here y is a cycle in S_n of length $2t$.
Claim : $y^t = x$

$$\begin{aligned} y(p_1) &= q_t \\ y^2(p_1) &= y(y(p_1)) = y(q_t) = q_{t-1} \\ y^3(p_1) &= y(y^2(p_1)) = y(q_{t-1}) = q_{t-2} \\ \dots & \quad \dots \\ y^t(p_1) &= q_1 \end{aligned}$$

$$\begin{aligned} y(q_t) &= q_{t-1} \\ y^2(q_t) &= y(y(q_t)) = y(q_{t-1}) = q_{t-2} \\ \dots & \quad \dots \\ y^t(q_t) &= p_t \end{aligned}$$

That is, in y^t, the k^{th} element of y maps to $(k+t)^{th}$ element of the cycle y, $k = 1, 2, \dots, t$.
So, we have

$$\begin{aligned} y^t(q_{t-1}) &= p_{t-1} \\ \dots & \quad \dots \\ y^t(q_2) &= p_2 \\ y^t(q_1) &= p_1 \\ y^t(p_t) &= q_t \end{aligned}$$

$$\begin{array}{lll} y^t(p_{t-1}) & = & q_{t-1} \\ \dots & & \dots \\ y^t(p_2) & = & q_2 \end{array}$$

$\implies y^t = (p_1q_1)(p_2q_2)\dots(p_{t-1}q_{t-1})(p_tq_t) = x.$
$\implies x \in \langle y \rangle.$
$\implies \langle x \rangle \subset \langle y \rangle$ in S_n.
$\implies x \longrightarrow y$ in $G(S_n)$.
$\implies x \longrightarrow y$ in $G^*(S_n)$.
which is a contradiction, since $deg^+(x) = 0$ in $G^*(S_n)$.
$\therefore$ The permutation x in S_n is a transposition. □

Thus, the theorem says that every isolated vertex in $G^*(S_n)$, $n \in \mathbb{N}$ corresponds to a transpostion.

Corollary 4.1. *Any vertex x in the proper C-graph $G^*(S_n)$ of the symmetric group S_n, $n \in \mathbb{N}$ with $deg(x) = deg^-(x) = 0$ corresponds to a permutation that is the product of disjoint transpositions in S_n.*

Proof. Directly follows from the proof of theorem 4.6 □

Example 4.3. *Consider the symmetric group $S_4 = \{x_0, x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{15}, x_{16}, x_{17}, x_{18}, x_{19}, x_{20}, x_{21}, x_{22}, x_{23}\}$, where x_0 is the identity permutation, $x_1 = (12)$, $x_2 = (23)$, $x_3 = (132)$, $x_4 = (123)$, $x_5 = (13)$, $x_6 = (34)$, $x_7 = (12)(34)$, $x_8 = (243)$, $x_9 = (1432)$, $x_{10} = (1243)$, $x_{11} = (143)$, $x_{12} = (234)$, $x_{13} = (1342)$, $x_{14} = (24)$, $x_{15} = (142)$, $x_{16} = (13)(24)$, $x_{17} = (1423)$, $x_{18} = (1234)$, $x_{19} = (134)$, $x_{20} = (124)$, $x_{21} = (14)$, $x_{22} = (1324)$, $x_{23} = (14)(23)$. The C-graph and proper C-graph of S_4 are given below.*

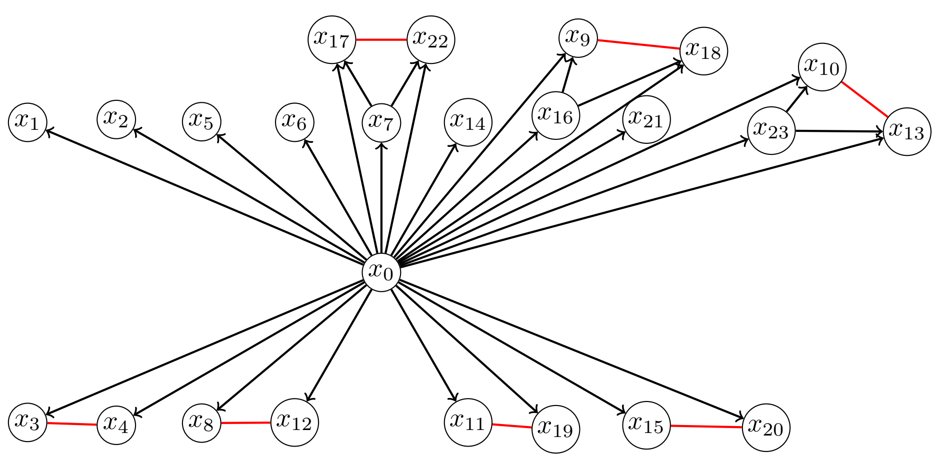

Fig.5 C-graph $G(S_4)$

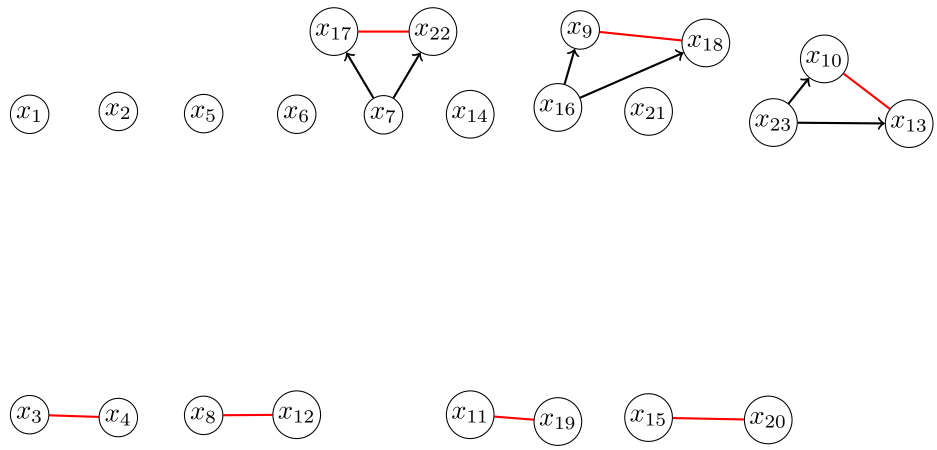

Fig.6 Proper C-graph $G^*(S_4)$

Here, in $G^*(S_4)$,

- *For $x = x_3, x_4, x_8, x_{12}, x_{11}, x_{19}, x_{15}, x_{20}$, $deg(x) = 1$, an odd number and $deg^-(x) = 0$, these 8 vertices corresponds to 8 three cycles in the group S_4.*
- *For $x = x_1, x_2, x_5, x_6, x_{14}, x_{21}$, $deg(x) = deg^-(x) = deg^+(x) = 0$, these 6 vertices corresponds to 6 transpositions in S_4.*
- *For $x = x_7, x_{16}, x_{23}$, $deg(x) = deg^-(x) = 0$, these 3 vertices corresponds to 3 double transpositions (product of two disjoint transpositions) in S_4.*

5. Conclusion

C-graph is a mixed graphical representation of groups which discusses some group theoretical properties from the graph itself. That is, it is a pictorial representation of a group, in which the adjacency between the vertices is an indication of comparability. The identical characteristics of the vertex corresponding to i in every group (ie, irrespective of the group) allow an opportunity to avoid that vertex from C-graph and to form the proper C-graph. This proper C-graph can be studied further to identify some structural properties of groups in a more convenient way.

Conflicts of interest : The authors declare no conflict of interest.

Acknowledgments : The authors thank the Referee for their valuable suggestions towards the improvement of the paper.

References

1. G. Aalipur, S. Akbari, P.J. Cameron, R. Nikandish, F. Shaveisi, *On the tructure of the power graph and enhanced power graph of a group*, Electron. J. Combin. **24** (2017), 3-16.
2. A. Abdollahi, S. Akbari, H.R. Maimani, *Non commuting Graph of a Group*, Journal of Algebra **298** (2006), 468-492.
3. Ahmed Erfanian, Behnaz Tolue, *Relative non nil-n graphs of finite groups*, Science Asia **38** (2012), 201-206.
4. Alain Bretto, Alain Faisant, Luc Gilibert, *G-graphs A new representation of groups*, Journal of Symbolic Computation **42** (2007), 549-560.
5. S. Banerjee, *On a new graph defined on the order of elements of a finite group*, Preprint, 2019. Available at https:|| arxiv.org | abs | 1911.02763
6. I. Beck, *Coloring of Commutative Rings*, Journal of Algebra **116** (1988), 208-226.
7. S. Bera, A.K. Buniya, *On enhanced power graph*, J. Algebra Appl. **17** (2018), pp 8.
8. P.J. Cameron, *The Power graph of a finite group ii*, Journal of Group Theory **13** (2010), 779-783.
9. David F.Anderson, Ayman Badawi, *The total graph of a commutative ring*, Journal of Algebra **320** (2008), 2706-2719.
10. Floretin Smarandache, W.B. Vasantha Kandaswamy, *Groups as graphs*, Mathematics and Statistics Faculty and Staff Publications, 2009.
11. J.B. Fraliegh, *A first course in Abstract Algebra*, Pearson Education, 1967.
12. Frank Harary, *Graph Theory*, Narosa Publishing House, 1988.
13. Ivy Chakravarty, Shamik Ghosh and M.K. Sen, *Undirected Power graphs of Semigroups*, Semigroup Forum **78** (2009), 410-426.
14. Kazem Khashyarmanesh, Mahdi Riza Khorsandi, *Projective total graphs of commutative rings*, Rocky Mountain Journal of Mathematics **43** (2013), 1207-1213.
15. A.V. Kelarev and S.J. Quinn, *Directed Graphs and Combinatorial Properties of Semigroups*, Journal of Algebra **251** (2002), 16-26.
16. A.V. Kelarev and S.J. Quinn, *A combinatorial Property and Power graphs of Semigroups*, Commentationes Mathematicae Universitatis Cardinae **45** (2004), 1-7.
17. Maria Vianney, Any Herawati, Priscila Seplinia, Henryanti, Ricky Aditya, *Identity graph of a finite cyclic group*, International Journal of Applied Sciences and Smart technologies **3** (2021), 101-110.
18. A.M. Mohammed, *Mixed graph representation and mixed graph isomorphism*, Gazi University Journal of Science **30** (2017), 303-310.
19. B.H. Neumann, *A problem of Paul Erdös on Groups*, Journal of the Australian Mathematical Society **21** (1976), 467-472.
20. P. Niroomand, A. Erfanian, M. Parvizi, B. Tolue, *Non-Exterior Square Graph of Finite group*, Filomat **31** (2017), 877-883.
21. Shani P. Redmond, *The zero divisor graph of a Non commutaive ring*, International Journal of Commutative Algebra **1** (2002), 203-211.
22. Sinu N. Vijayan, Anjaly Kishore, *C-graphs-A mixed graphical representation of groups*, WSEAS Transactions on Mathematics **20** (2021), 569-580.
23. M.I. Sowaity, B. Sarada and A.M. Naji, *The identity graph of a multigroup*, Far East Journal of Mathematical Sciences **117** (2019), 67-86.
24. Xuanlong Ma, Zhonghua Way, *The Co-prime order graph associated with a finite group*, arxiv:2011,13547v1
25. S. Zahirovic, I. Bosnjak, R. Madarasz, *Enhanced Power graph of Finite groups*, 2018. arXiv:1810.07627v1

Sinu N. Vijayan received MSc, MPhil and PhD from University of Calicut. Awarded UGC JRF from University Grants Commission, India. Currectly working as an Assistant

Professor on contract at St.Mary's College, Thrissur. Her research interest include Graph theory, Abstract algebra, Algebraic graph theory and Hypergraph theory.

Department of Mathematics, St. Mary's College, Thrissur, India, affiliated to University of Calicut.
e-mail: sinu.n.vijayan.res@smctsr.ac.in

Anjaly Kishore received MSc from Mahatma Gandhi University, PhD from National Institute of Technology, Calicut. Awarded UGC JRF from University Grants Commission, India. Currently working as an Assistant Professor at St. Mary's College, Thrissur. She has workrd as an Assistant Professor at Vimala College, Thrissur from 2012 to 2024. Her research interest include Graph theory, Fuzzy graph theory and Algebraic graph theory.

Department of Mathematics, St. Mary's College, Thrissur, India, affiliated to University of Calicut.
e-mail: dr.anjalykishore@gmail.com