J. Appl. Math. & Informatics Vol. 42(2024), No. 5, pp. 1007 - 1023
https://doi.org/10.14317/jami.2024.1007

STANDARD FRACTIONAL VECTOR CROSS PRODUCT IN EUCLIDEAN 3-SPACE

MANISHA M. KANKAREJ* AND JAI P. SINGH

Abstract. In this paper we are able to define a standard fractional vector cross product(SFVCP) of two vectors in a Euclidean 3-space where it satisfies all the conditions of geometrical reality. For $\gamma = 1$ this definition satisfies the conditions of standard vector cross product(SVCP). The formulae for euclidean norm and fractional triple vector cross product of two vectors with standard fractional vector cross product are presented. Fractional curl and divergence of an electromagnetic vector field are presented using the new definition. All the properties are further supported with particular cases at $\gamma = 0$, $\gamma = 1$ and examples on standard orthogonal basis in R^3. This concept has application in electrodynamics, elastodynamics, fluid flow etc.

AMS Mathematics Subject Classification : 15A30, 15A63, 26A99, 26B12.
Key words and phrases : Vector cross product, standard fractional vector cross product(SFVCP), fractional vector triple product, fractional curl, divergence.

1. Introduction

Crowe [1] laid the foundation of vector analysis. Das [5] modified the existing definition of vector cross product and gave another definition of fractional cross product. He also studied fractional curl with application in vector field of electromagnetic theory. Later Tripathi and Kim [3] presented α - fractional cross product operation of two vectors in Euclidean 3-space which resembled the definition given by Das [5]. Recently Kankarej and Singh [2] extended this work to define β - fractional cross product operation of two vectors. In this paper we present new definition of fractional vector cross product called standard fractional vector cross product (SFVCP) in Euclidean 3-space. This new fractional product satisfies all the conditions of geometrical reality which makes it more

Received March 27, 2023. Revised January 4, 2024. Accepted January 22, 2024.
*Corresponding author.

useful in the real setup.

To be in 3-dimensional space a vector product must be represented in 3-dimension, which is satisfied in new definition.
For $\gamma = 0$, the vector cross product operation is represented in 2-dimensional plane.
For $0 \leq \gamma \leq 1$, the vector cross product operation is an arbitrary vector in 3-dimension.
For $\gamma = 1$, the vector cross product behaves like a standard vector cross product(SVCP).

All mentioned concepts make it geometrically real and visually more presentable. Further this new definition can be applied in various fields of electrodynamics, elastodynamics, fluid flow etc. In later sections of this paper we have presented properties of norm of fractional cross product of two vectors and fractional triple vector cross product. In the last section fractional curl and divergence are discussed.

2. Main results

With the new definition of SFVCP presented in this research, we have given the
1. Graphical representation of new definition.
2. Matrix representation of new definition.
3. Some properties with respect to new definition.
4. Divergence, Curl, Divergence of curl and Curl of divergence of an electromagnetic vector field.

It is noticed that all the results follow the properties of standard vector cross product at $\gamma = 1$.

3. Standard fractional vector cross product (SFVCP)

Definition 3.1. Let R^3 be the euclidean 3-space endowed with standard inner product $\langle .,. \rangle$. Let (e_1, e_2, e_3) is a standard orthonormal basis of R^3 and $\gamma \in [0,1]$ a real number. Then, for vectors $a = a_1e_1 + a_2e_2 + a_3e_3,\ b = b_1e_1 + b_2e_2 + b_3e_3$ in R^3, the *standard fractional vector cross product* is represented as

$$\begin{aligned} a \times^\gamma b = &\Big\{(a_2b_3 - a_3b_2)\sin\big(\frac{\gamma\pi}{2}\big) + (a_2 + a_3)b_1\cos\big(\frac{\gamma\pi}{2}\big)\Big\} - (b_2 + b_3)a_1\cos\big(\frac{\gamma\pi}{2}\big)\Big\}e_1 \\ &+\Big\{(a_3b_1 - a_1b_3)\sin\big(\frac{\gamma\pi}{2}\big) + (a_3 + a_1)b_2\cos\big(\frac{\gamma\pi}{2}\big)\Big\} - (b_3 + b_1)a_2\cos\big(\frac{\{\gamma\pi}{2}\big)\Big\}e_2 \\ &+\Big\{(a_1b_2 - a_2b_1)\sin\big(\frac{\gamma\pi}{2}\big) + (a_1 + a_2)b_3\cos\big(\frac{\gamma\pi}{2}\big)\Big\} - (b_1 + b_2)a_3\cos\big(\frac{\gamma\pi}{2}\big)\Big\}e_3 \end{aligned} \tag{1}$$

From (1) we have,

$$e_i \times^\gamma e_j = \cos\left(\frac{\gamma\pi}{2}\right)e_j + \sin\left(\frac{\gamma\pi}{2}\right)e_k - \cos\left(\frac{\gamma\pi}{2}\right)e_i \tag{2}$$

$$e_j \times^\gamma e_i = \cos\left(\frac{\gamma\pi}{2}\right)e_i - \sin\left(\frac{\gamma\pi}{2}\right)e_k - \cos\left(\frac{\gamma\pi}{2}\right)e_j \tag{3}$$

$$e_l \times^\gamma e_l = 0 \; for \; l = \{1, 2, 3\} \tag{4}$$

where (i, j, k) remains a cyclic permutation of class (1, 2, 3). The equations (2), (3), (4) are similar to that in [6] and [3].

3.1. Graphical Representation. We can express equation (1) graphically as:

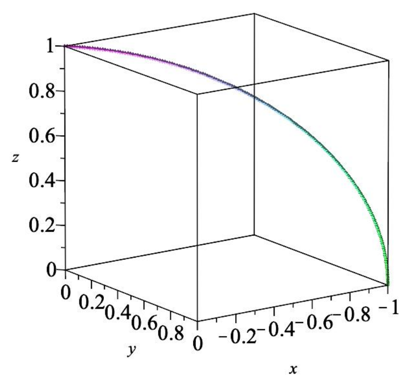

Fig 1: Graphical representation for $e_i \times e_j$

Fig 1 shows that the SFVCP represented by (1) is a curve whose value moves from $x - y$ plane at $\gamma = 0$ to z-axis at $\gamma = 1$. The movement of the vectors for different values of γ is represented in following Fig 2.

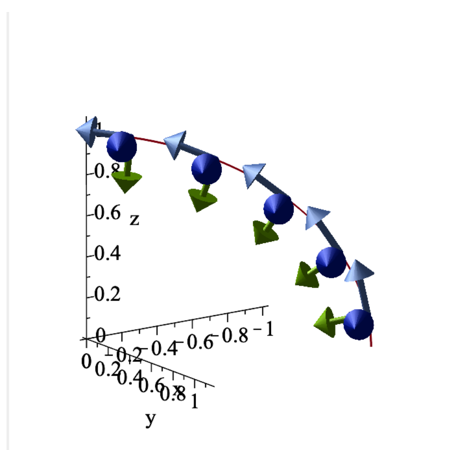

Fig 2: Graphical representation for the movement of the vector in the SFVCP $e_i \times e_j$

Fig 2 verifies the fact that SFVCP represented by (1) is a curve whose value moves from $x - y$ plane at $\gamma = 0$ to z-axis at $\gamma = 1$.

3.2. Matrix representation. By (2), (3), (4) and linearity we can write equation (1) as

$$\begin{aligned} a \times^{\gamma} b = \sin\big(\frac{\gamma\pi}{2}\big)\big\{(a_2b_3 - a_3b_2)e_1 + (a_3b_1 - a_1b_3)e_2 + (a_1b_2 - a_2b_1)e_3\big\} \\ + \cos\big(\frac{\gamma\pi}{2}\big)\big\{(a_2 + a_3)b_1e_1 + (a_3 + a_1)b_2e_2 + (a_1 + a_2)b_3e_3\big\} \\ - \cos\big(\frac{\gamma\pi}{2}\big)\big\{(b_2 + b_3)a_1e_1 + (b_3 + b_1)a_2e_2 + (b_1 + b_2)a_3e_3\big\} \end{aligned} \tag{5}$$

or

$$\begin{aligned} a \times^{\gamma} b = \sin\big(\frac{\gamma\pi}{2}\big)\big\{(a_2b_3 - a_3b_2)e_1 + (a_3b_1 - a_1b_3)e_2 + (a_1b_2 - a_2b_1)e_3\big\} \\ + \cos\big(\frac{\gamma\pi}{2}\big)(a_1 + a_2 + a_3)b - \cos\big(\frac{\gamma\pi}{2}\big)(b_1 + b_2 + b_3)a \end{aligned} \tag{6}$$

From [4] we have

$$a \times^{\gamma} b = \begin{pmatrix} (a_2 + a_3)\cos\big(\frac{\gamma\pi}{2}\big) & -a_3\sin\big(\frac{\gamma\pi}{2}\big) - a_1\cos\big(\frac{\gamma\pi}{2}\big) & a_2\sin\big(\frac{\gamma\pi}{2}\big) - a_1\cos\big(\frac{\gamma\pi}{2}\big) \\ a_3\sin\big(\frac{\gamma\pi}{2}\big) - a_2\cos\big(\frac{\gamma\pi}{2}\big) & (a_3 + a_1)\cos\big(\frac{\gamma\pi}{2}\big) & -a_1\sin\big(\frac{\gamma\pi}{2}\big) - a_2\cos\big(\frac{\gamma\pi}{2}\big) \\ -a_2\sin\big(\frac{\gamma\pi}{2}\big) - a_3\cos\big(\frac{\gamma\pi}{2}\big) & a_1\sin\big(\frac{\gamma\pi}{2}\big) - a_3\cos\big(\frac{\gamma\pi}{2}\big) & (a_1 + a_2)\cos\big(\frac{\gamma\pi}{2}\big) \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \tag{7}$$

or

$$a \times^{\gamma} b = \begin{pmatrix} -(b_2 + b_3)\cos\big(\frac{\gamma\pi}{2}\big) & b_3\sin\big(\frac{\gamma\pi}{2}\big) + b_1\cos\big(\frac{\gamma\pi}{2}\big) & -b_2\sin\big(\frac{\gamma\pi}{2}\big) + b_1\cos\big(\frac{\gamma\pi}{2}\big) \\ -b_3\sin\big(\frac{\gamma\pi}{2}\big) + b_2\cos\big(\frac{\gamma\pi}{2}\big) & -(b_3 + b_1)\cos\big(\frac{\gamma\pi}{2}\big) & b_1\sin\big(\frac{\gamma\pi}{2}\big) + b_2\cos\big(\frac{\gamma\pi}{2}\big) \\ b_2\sin\big(\frac{\gamma\pi}{2}\big) + b_3\cos\big(\frac{\gamma\pi}{2}\big) & -b_1\sin\big(\frac{\gamma\pi}{2}\big) + b_3\cos\big(\frac{\gamma\pi}{2}\big) & -(b_1 + b_2)\cos\big(\frac{\gamma\pi}{2}\big) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \tag{8}$$

Thus considering $a \in R^3$ as a fixed column vector in R^3, we see that $a \times^{\gamma} b : R^3 \to R^3$ is a linear function as given in eqn (7) and (8).

Particular case:

Case 1: The SFVCP for $\gamma = 0$:

By eqn (7) we have, $a \times^0 b = \begin{pmatrix} a_2+a_3 & -a_1 & -a_1 \\ -a_2 & a_3+a_1 & -a_2 \\ -a_3 & -a_3 & a_1+a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Case 2: The SFVCP for $\gamma = 1$:
By eqn (5) and (7) we have
$a \times b = \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$

Example 1: Using eqn (1) we have

For $\gamma = 0,\ e_i \times^0 e_j = e_j - e_i,\ e_j \times^0 e_i = e_i - e_j,\ e_l \times^0 e_l = 0$

For $\gamma = 1, e_i \times e_j = e_k,\ e_j \times e_i = -e_k,\ e_l \times e_l = 0$

Thus, the SFVCP for $\gamma = 1$ is simply the SVCP.

4. Properties of standard fractional vector cross product

Theorem 4.1. *The SFVCP satisfies*

$$a \times^\gamma (b + c) = a \times^\gamma b + a \times^\gamma c \tag{9}$$

$$(a + b) \times^\gamma c = a \times^\gamma c + b \times^\gamma c \tag{10}$$

$$(\lambda a + \mu b) \times^\gamma c = \lambda a \times^\gamma c + \mu b \times^\gamma c \tag{11}$$

Proof. From eqns (7), equations (9), (10) and (11) can be proved. □

Particular case:

Case 1: The SFVCP for $\gamma = 0$ satisfies:

$a \times^0 (b + c) = a \times^0 b + a \times^0 c$

$(a + b) \times^0 c = a \times^0 c + b \times^0 c$

$(\lambda a + \mu b) \times^0 c = \lambda a \times^0 c + \mu b \times^0 c$

Case 2: The SFVCP for $\gamma = 1$ behaves as SVCP.

$a \times (b + c) = a \times b + a \times c$

$$(a+b)\times c = a\times c + b\times c$$

$$(\lambda a+\mu b)\times c = \lambda a\times c + \mu b\times c$$

Theorem 4.2. *The SFVCP satisfies*

$$\begin{aligned}\langle a, a\times^{\gamma} b\rangle = \cos\Big(\frac{\gamma\pi}{2}\Big)\big\{a_1b_1(a_2+a_3)+a_2b_2(a_1+a_3)+a_3b_3(a_1+a_2)\big\}\\ -\cos\Big(\frac{\gamma\pi}{2}\Big)\big\{a_1^2(b_2+b_3)+a_2^2(b_1+b_3)+a_3^2(b_1+b_2)\big\}\end{aligned} \tag{12}$$

$$\langle a, a\times^{\gamma} b\rangle = \cos\Big(\frac{\gamma\pi}{2}\Big)(a_1+a_2+a_3)\langle a,b\rangle - \cos\Big(\frac{\gamma\pi}{2}\Big)(b_1+b_2+b_3)\|a\|^2 \tag{13}$$

From eqns (5) and (6), equations (12) and (13) can be proved.

Particular case:

Case 1: The SFVCP for $\gamma = 0$ satisfies:

$$\langle a, a\times^{0} b\rangle = \big\{a_1b_1(a_2+a_3)+a_2b_2(a_1+a_3)+a_3b_3(a_1+a_2)\big\}$$
$$-\big\{a_1^2(b_2+b_3)+a_2^2(b_1+b_3)+a_3^2(b_1+b_2)\big\}$$

$$\langle a, a\times^{0} b\rangle = (a_1+a_2+a_3)\langle a,b\rangle - (b_1+b_2+b_3)\|a\|^2$$

Case 2: The SFVCP for $\gamma = 1$ satisfies:

$$\langle a, a\times b\rangle = 0$$

Example 2: Using eqn (1) we have

$$\langle e_i, e_i\times^{\gamma} e_j\rangle = -\cos\big(\tfrac{\gamma\pi}{2}\big)$$

For $\gamma = 0$, $\langle e_i, e_i\times^{0} e_j\rangle = -1$

For $\gamma = 1$, $\langle e_i, e_i\times e_j\rangle = 0$

Theorem 4.3. *The SFVCP satisfies*

$$\begin{aligned}\langle b, a\times^{\gamma} b\rangle = \cos\Big(\frac{\gamma\pi}{2}\Big)\big\{b_1^2(a_2+a_3)+b_2^2(a_1+a_3)+b_3^2(a_1+a_2)\big\}\\ -\cos\Big(\frac{\gamma\pi}{2}\Big)\big\{a_1b_1(b_2+b_3)+a_2b_2(b_1+b_3)+a_3b_3(b_1+b_2)\big\}\end{aligned} \tag{14}$$

$$\langle b, a\times^{\gamma} b\rangle = \cos\Big(\frac{\gamma\pi}{2}\Big)(a_1+a_2+a_3)\|b\|^2 - \cos\Big(\frac{\gamma\pi}{2}\Big)(b_1+b_2+b_3)\langle a,b\rangle \tag{15}$$

From eqns (5) and (6), equations (14) and (15) can be proved.

Particular case:

Case 1: The SFVCP for $\gamma = 0$ satisfies:

$$\langle b, a \times^0 b\rangle = \big\{b_1^2(a_2+a_3) + b_2^2(a_1+a_3) + b_3^2(a_1+a_2)\big\}$$
$$-\big\{a_1b_1(b_2+b_3) + a_2b_2(b_1+b_3) + a_3b_3(b_1+b_2)\big\}$$

$$\langle b, a \times^0 b\rangle = (a_1+a_2+a_3)\|b\|^2 - (b_1+b_2+b_3)\langle a, b\rangle$$

Case 2: The SFVCP for $\gamma = 1$ satisfies:

$$\langle a, a \times b\rangle = 0$$

Example 3: Using eqn (1) we have

$$\langle e_j, e_i \times^\gamma e_j\rangle = \cos\left(\tfrac{\gamma\pi}{2}\right)$$

For $\gamma = 0$, $\langle e_j, e_i \times^0 e_j\rangle = 1$

For $\gamma = 1$, $\langle e_j, e_i \times e_j\rangle = 0$

Theorem 4.4. *The SFVCP satisfies*

$$\langle a+b, a \times^\gamma b\rangle =$$
$$\cos\left(\frac{\gamma\pi}{2}\right)\big\{(a_1+b_1)(a_2+a_3)b_1 + (a_2+b_2)(a_1+a_3)b_2 + (a_3+b_3)(a_1+a_2)b_3\big\}$$
$$-\cos\left(\frac{\gamma\pi}{2}\right)\big\{a_1(a_1+b_1)(b_2+b_3) + a_2(a_2+b_2)(b_1+b_3) + a_3(a_3+b_3)(b_1+b_2)\big\} \tag{16}$$

$$\langle a+b, b \times^\gamma a\rangle =$$
$$\cos\left(\frac{\gamma\pi}{2}\right)\big\{a_1(a_1+b_1)(b_2+b_3) + a_2(a_2+b_2)(b_1+b_3) + a_3(a_3+b_3)(b_1+b_2)\big\}$$
$$-\cos\left(\frac{\gamma\pi}{2}\right)\big\{(a_1+b_1)(a_2+a_3)b_1 + (a_2+b_2)(a_1+a_3)b_2 + (a_3+b_3)(a_1+a_2)b_3\big\} \tag{17}$$

$$\langle a+b, a \times^\gamma b\rangle - \langle a+b, b \times^\gamma a\rangle =$$
$$2\cos\left(\tfrac{\gamma\pi}{2}\right)\left\{\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_1+a_2 & a_2+a_3 & a_3+a_1 \\ b_1+b_2 & b_2+b_3 & b_3+b_1 \end{vmatrix} - \begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}\right\}$$

Using eqn (12) we can prove the theorem.

Particular case:

Case 1: The SFVCP for $\gamma = 0$ satisfies:

$$\langle a+b, a \times^0 b\rangle = \\ \big\{(a_1+b_1)(a_2+a_3)b_1 + (a_2+b_2)(a_1+a_3)b_2 + (a_3+b_3)(a_1+a_2)b_3\big\} \\ -\big\{a_1(a_1+b_1)(b_2+b_3) + a_2(a_2+b_2)(b_1+b_3) + a_3(a_3+b_3)(b_1+b_2)\big\}$$

$$\langle a+b, b \times^0 a\rangle = \\ \big\{a_1(a_1+b_1)(b_2+b_3) + a_2(a_2+b_2)(b_1+b_3) + a_3(a_3+b_3)(b_1+b_2)\big\} \\ -\big\{(a_1+b_1)(a_2+a_3)b_1 + (a_2+b_2)(a_1+a_3)b_2 + (a_3+b_3)(a_1+a_2)b_3\big\}$$

$$\langle a+b, a \times^0 b\rangle - \langle a+b, b \times^0 a\rangle = \\ 2\Big\{ \begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_1+a_2 & a_2+a_3 & a_3+a_1 \\ b_1+b_2 & b_2+b_3 & b_3+b_1 \end{vmatrix} - \begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \Big\}$$

Case 2: The SFVCP for $\gamma = 1$ satisfies:

$$\langle a+b, a \times b\rangle = 0, \langle a+b, b \times a\rangle = 0$$

$$\langle a+b, a \times b\rangle - \langle a+b, b \times a\rangle = 0$$

Example 4: Using eqn (1) we have

$$\langle e_i+e_j, e_i \times^\gamma e_j\rangle - \langle e_i+e_j, e_j \times^\gamma e_i\rangle = 0$$

For $\gamma = 0, \langle e_i+e_j, e_i \times^0 e_j\rangle - \langle e_i+e_j, e_j \times^0 e_i\rangle = 0$

For $\gamma = 1, \langle e_i+e_j, e_i \times e_j\rangle - \langle e_i+e_j, e_j \times e_i\rangle = 0$

Theorem 4.5. *The SFVCP satisfies*

$$(a \times^\gamma b) + (b \times^\gamma) = 0 \tag{18}$$

Using eqn (5) we can prove the theorem.

Particular case:

Case 1: The SFVCP for $\gamma = 0$ satisfies:

$$(a \times^0 b) + (b \times^0 a) = 0 \tag{19}$$

Case 2: The SFVCP for $\gamma = 1$ satisfies:

$$(a \times b) + (b \times a) = 0 \tag{20}$$

Example 5: Using eqn (1) we have

$(e_i \times^\gamma e_j) + (e_j \times^\gamma e_i) = 0$

For $\gamma = 0$, $(e_i \times^0 e_j) + (e_j \times^0 e_i) = 0$

For $\gamma = 1$, $(e_i \times e_j) + (e_j \times e_i) = 0$

Theorem 4.6. *The SFVCP satisfies*

$$\|a \times^\gamma b\|^2 = \left\{\|a\|^2\|b\|^2 - \langle a, b\rangle^2\right\} \sin^2\left(\frac{\gamma\pi}{2}\right) + 2\begin{vmatrix} b_1(a_2+a_3) & b_2(a_3+a_1) & b_3(a_1+a_2) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \sin(\gamma\pi) - 2\begin{vmatrix} a_1(b_2+b_3) & a_2(b_1+b_3) & a_3(b_1+b_2) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \sin(\gamma\pi) + \left\{[b_1(a_2+a_3) - a_1(b_2+b_3)]^2 + [b_2(a_1+a_3) - a_2(b_1+b_3)]^2 + [b_3(a_1+a_2) - a_3(b_1+b_2)]^2\right\} \cos^2\left(\frac{\gamma\pi}{2}\right) \tag{21}$$

$$\|b \times^\gamma a\|^2 = \left\{\|a\|^2\|b\|^2 - \langle a, b\rangle^2\right\} \sin^2\left(\frac{\gamma\pi}{2}\right) + 2\begin{vmatrix} a_1(b_2+b_3) & a_2(b_1+b_3) & a_3(b_1+b_2) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \sin(\gamma\pi) - 2\begin{vmatrix} b_1(a_2+a_3) & b_2(a_3+a_1) & b_3(a_1+a_2) \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \sin(\gamma\pi) + \left\{[a_1(b_2+b_3) - b_1(a_2+a_3)]^2 + [a_2(b_1+b_3) - b_2(a_1+a_3)]^2 + [a_3(b_1+b_2) - b_3(a_1+a_2)]^2\right\} \cos^2\left(\frac{\gamma\pi}{2}\right) \tag{22}$$

Particular case:

Case 1: The SFVCP for $\gamma = 0$ satisfies:

$\|a \times^0 b\|^2 = \{[b_1(a_2+a_3) - a_1(b_2+b_3)]^2 + [b_2(a_1+a_3) - a_2(b_1+b_3)]^2 + [b_3(a_1+a_2) - a_3(b_1+b_2)]^2$

$\|b \times^0 a\|^2 = [a_1(b_2+b_3) - b_1(a_2+a_3)]^2 + [a_2(b_1+b_3) - b_2(a_1+a_3)]^2 + [a_3(b_1+b_2) - b_3(a_1+a_2)]^2$

Case 2: The SFVCP for $\gamma = 1$ satisfies:

$$\|a \times b\|^2 = \left\{\|a\|^2\|b\|^2 - \langle a, b\rangle^2\right\}, \; \|b \times a\|^2 = \left\{\|a\|^2\|b\|^2 - \langle a, b\rangle^2\right\}$$

Example 6: $\|e_i \times^\gamma e_j\|^2 = \cos\left(\frac{\gamma\pi}{2}\right)^2, \|e_j \times^\gamma e_i\|^2 = cos\left(\frac{\gamma\pi}{2}\right)^2$

For $\gamma = 0$, $\|e_i \times^0 e_j\|^2 = 1$, $\|e_j \times^0 e_i\|^2 = 1$

For $\gamma = 1$, $\|e_i \times e_j\|^2 = 0$, $\|e_j \times e_i\|^2 = 0$

Theorem 4.7. *The SFVCP satisfies*

$$\begin{aligned} \langle a \times^\gamma b, c\rangle &= \sin\left(\frac{\gamma\pi}{2}\right)\begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ &+ \cos\left(\frac{\gamma\pi}{2}\right)(a_1+a_2+a_3)\langle b, c\rangle - \cos\left(\frac{\gamma\pi}{2}\right)(b_1+b_2+b_3)\langle a, c\rangle \end{aligned} \tag{23}$$

Using eqn (12) we can prove the theorem.

Particular case:

Case 1: The SFVCP for $\gamma = 0$ satisfies :

$$\langle a \times^0 b, c\rangle = (a_1+a_2+a_3)\langle b, c\rangle - (b_1+b_2+b_3)\langle a, c\rangle$$

Case 2: The SFVCP for $\gamma = 1$ satisfies:

$$\langle a \times b, c\rangle = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

Example 7: $\langle e_i \times^\gamma e_j, e_k\rangle = \sin\left(\frac{\gamma\pi}{2}\right)$

For $\gamma = 0$, $\langle e_i \times^0 e_j, e_k\rangle = 0$

For $\gamma = 1$, $\langle e_i \times e_j, e_k\rangle = 1$

Theorem 4.8. *The SFVCP satisfies*

$$
\begin{aligned}
&(a \times^{\gamma} b) \times^{\gamma} c = \sin\left(\frac{\gamma\pi}{2}\right)^{2}(\langle a, c\rangle b - \langle b, c\rangle a)\\
&+ \cos\left(\frac{\gamma\pi}{2}\right)\Big\{ \sin\left(\frac{\gamma\pi}{2}\right)(a_1 + a_2 + a_3)\begin{vmatrix} e_1 & e_2 & e_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3 \end{vmatrix}\\
&+ \sin\left(\frac{\gamma\pi}{2}\right)(b_1 + b_2 + b_3)\begin{vmatrix} e_1 & e_2 & e_3\\ c_1 & c_2 & c_3\\ a_1 & a_2 & a_3 \end{vmatrix} + \sin\left(\frac{\gamma\pi}{2}\right)(c_1 + c_2 + c_3)\begin{vmatrix} e_1 & e_2 & e_3\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3 \end{vmatrix}\Big\}\\
&+ \cos\left(\frac{\gamma\pi}{2}\right)c\begin{vmatrix} 1 & 1 & 1\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3 \end{vmatrix} + (c_1 + c_2 + c_3)\cos^{2}\left(\frac{\gamma\pi}{2}\right)[(a_1 + a_2 + a_3)b\\
&- (b_1 + b_2 + b_3)a]
\end{aligned}
\tag{24}
$$

Proof. For vectors $c = c_1e_1 + c_2e_2 + c_3e_3$ and $d = d_1e_1 + d_2e_2 + d_3e_3$ in R^3 in view of eqn (6), we have

$$
\begin{aligned}
&d \times^{\gamma} c = \sin\left(\frac{\gamma\pi}{2}\right)\big\{(d_2c_3 - d_3c_2)e_1 + (d_3c_1 - d_1c_3)e_2 + (d_1c_2 - d_2c_1)e_3\big\}\\
&+ \cos\left(\frac{\gamma\pi}{2}\right)\big\{(d_2 + d_3)c_1e_1 + (d_3 + d_1)c_2e_2 + (d_1 + d_2)c_3e_3\big\}\\
&- \cos\left(\frac{\gamma\pi}{2}\right)\big\{(c_2 + c_3)d_1e_1 + (c_3 + c_1)d_2e_2 + (c_1 + c_2)d_3e_3\big\}\\
&= \sin\left(\frac{\gamma\pi}{2}\right)\big\{(d_2c_3 - d_3c_2)e_1 + (d_3c_1 - d_1c_3)e_2 + (d_1c_2 - d_2c_1)e_3\big\}\\
&+ \cos\left(\frac{\gamma\pi}{2}\right)(d_1 + d_2 + d_3)c - \cos\left(\frac{\gamma\pi}{2}\right)(c_1 + c_2 + b_3)d
\end{aligned}
\tag{25}
$$

Using $d = a \times^{\gamma} b$, then from eqn (1) we have,

$$d_1 = \big\{(a_2b_3 - a_3b_2)\sin\left(\tfrac{\gamma\pi}{2}\right) + \{(a_2 + a_3)b_1 - (b_2 + b_3)a_1\}\cos\left(\tfrac{\gamma\pi}{2}\right)\big\}e_1$$

$$d_2 = \big\{(a_3b_1 - a_1b_3)\sin\left(\tfrac{\gamma\pi}{2}\right) + \{(a_3 + a_1)b_2 - (b_3 + b_1)a_2\}\cos\left(\tfrac{\gamma\pi}{2}\right)\big\}e_2$$

$$d_3 = \big\{(a_1b_2 - a_2b_1)\sin\left(\tfrac{\gamma\pi}{2}\right) + \{(a_1 + a_2)b_3 - (b_1 + b_2)a_3\}\cos\left(\tfrac{\gamma\pi}{2}\right)\big\}e_3$$

Using above details with eqn (25), we can prove theorem 3.8.

□

Particular case:

Case 1: The SFVCP for $\gamma = 0$ satisfies:

$$(a \times^0 b) \times^0 c = c \begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$
$$+ (c_1 + c_2 + c_3)[(a_1 + a_2 + a_3)b - (b_1 + b_2 + b_3)a]$$

Case 2: The SFVCP for $\gamma = 1$ satisfies:
$(a \times b) \times c = \langle a, c\rangle b - \langle b, c\rangle a$

Example 8: $(e_i \times^\gamma e_j) \times^\gamma e_k = \cos\left(\frac{\gamma\pi}{2}\right)\left[e_i(\cos\left(\frac{\gamma\pi}{2}\right) + \sin\left(\frac{\gamma\pi}{2}\right)) - e_j(\cos\left(\frac{\gamma\pi}{2}\right) + \sin\left(\frac{\gamma\pi}{2}\right))\right]$

For $\gamma = 0$, $(e_i \times^0 e_j) \times^0 e_k = e_i - e_j$

For $\gamma = 1$, $(e_i \times e_j) \times e_k = 0$

5. Divergence and Curl of standard fractional vector cross product

5.1. Curl of a Curl of standard fractional vector cross product. Using eqn (5) we get,

$$\begin{aligned} a \times^\gamma b = \sin\left(\frac{\gamma\pi}{2}\right) &\begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ + \cos\left(\frac{\gamma\pi}{2}\right) &\begin{pmatrix} a_2 + a_3 & 0 & 0 \\ 0 & a_3 + a_1 & 0 \\ 0 & 0 & a_1 + a_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \\ - \cos\left(\frac{\gamma\pi}{2}\right) &\begin{pmatrix} 0 & -a_1 & -a_1 \\ -a_2 & 0 & -a_2 \\ -a_3 & -a_3 & 0 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} \end{aligned} \tag{26}$$

Alternatively

$$\begin{aligned} a \times^\gamma b = \sin\left(\frac{\gamma\pi}{2}\right) &\begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\ + \cos\left(\frac{\gamma\pi}{2}\right) &\begin{pmatrix} 0 & b_1 & b_1 \\ b_2 & 0 & b_2 \\ b_3 & b_3 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \\ - \cos\left(\frac{\gamma\pi}{2}\right) &\begin{pmatrix} b_2 + b_3 & 0 & 0 \\ 0 & b_3 + b_1 & 0 \\ 0 & 0 & b_1 + b_2 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \end{aligned} \tag{27}$$

From eqn (26) and [5] for a vector field $F = F_1e_1 + F_2e_2 + F_3e_3$ the fractional curl is:

$$\nabla \times^{\gamma} F = \sin\left(\frac{\gamma\pi}{2}\right) \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial^{\gamma}}{\partial x} & \frac{\partial^{\gamma}}{\partial y} & \frac{\partial^{\gamma}}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} + \cos\left(\frac{\gamma\pi}{2}\right) \begin{pmatrix} \frac{\partial^{\gamma}}{\partial y} + \frac{\partial^{\gamma}}{\partial z} & 0 & 0 \\ 0 & \frac{\partial^{\gamma}}{\partial z} + \frac{\partial^{\gamma}}{\partial x} & 0 \\ 0 & 0 & \frac{\partial^{\gamma}}{\partial x} + \frac{\partial^{\gamma}}{\partial y} \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} - \cos\left(\frac{\gamma\pi}{2}\right) \begin{pmatrix} 0 & -\frac{\partial^{\gamma}}{\partial x} & -\frac{\partial^{\gamma}}{\partial x} \\ -\frac{\partial^{\gamma}}{\partial y} & 0 & -\frac{\partial^{\gamma}}{\partial y} \\ -\frac{\partial^{\gamma}}{\partial z} & -\frac{\partial^{\gamma}}{\partial z} & 0 \end{pmatrix} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} \tag{28}$$

Alternatively, from eqn (27) we get

$$\nabla \times^{\gamma} F = \sin\left(\frac{\gamma\pi}{2}\right) \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial^{\gamma}}{\partial x} & \frac{\partial^{\gamma}}{\partial y} & \frac{\partial^{\gamma}}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} + \cos\left(\frac{\gamma\pi}{2}\right) \begin{pmatrix} 0 & F_1 & F_1 \\ F_2 & 0 & F_2 \\ F_3 & F_3 & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial^{\gamma}}{\partial x} \\ \frac{\partial^{\gamma}}{\partial y} \\ \frac{\partial^{\gamma}}{\partial z} \end{pmatrix} - \cos\left(\frac{\gamma\pi}{2}\right) \begin{pmatrix} (F_2 + F_3) & 0 & 0 \\ 0 & (F_3 + F_1) & 0 \\ 0 & 0 & (F_1 + F_2) \end{pmatrix} \begin{pmatrix} \frac{\partial^{\gamma}}{\partial x} \\ \frac{\partial^{\gamma}}{\partial y} \\ \frac{\partial^{\gamma}}{\partial z} \end{pmatrix} \tag{29}$$

By theorem 3.8 we get,

$$
\begin{aligned}
(a \times^{\gamma} b) \times^{\gamma} c = {} & \sin\big(\frac{\gamma\pi}{2}\big)^2 (\langle a, c\rangle b - \langle a, b\rangle c) \\
& + \cos\big(\frac{\gamma\pi}{2}\big)\Big\{ \sin\big(\frac{\gamma\pi}{2}\big)(a_1 + a_2 + a_3) \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
& + \sin\big(\frac{\gamma\pi}{2}\big)(b_1 + b_2 + b_3) \begin{vmatrix} e_1 & e_2 & e_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} \\
& + \sin\big(\frac{\gamma\pi}{2}\big)(c_1 + c_2 + c_3) \begin{vmatrix} e_1 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \Big\} \\
& + \cos\big(\frac{\gamma\pi}{2}\big) a \begin{vmatrix} 1 & 1 & 1 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + (a_1 + a_2 + a_3)\cos^2\big(\frac{\gamma\pi}{2}\big)[(b_1 + b_2 + b_3)c \\
& - (c_1 + c_2 + c_3)b]
\end{aligned}
\tag{30}
$$

From (28) and (30) we get

$$
\begin{aligned}
\nabla \times^{\gamma} (\nabla \times^{\gamma} F) = {} & \sin\big(\frac{\gamma\pi}{2}\big)^2 (\nabla\langle \nabla, F\rangle - F\langle \nabla, \nabla\rangle) \\
& + \cos\big(\frac{\gamma\pi}{2}\big)\Big\{ \sin\big(\frac{\gamma\pi}{2}\big)\big(\frac{\partial^{\gamma}}{\partial x} + \frac{\partial^{\gamma}}{\partial y} + \frac{\partial^{\gamma}}{\partial z}\big) \begin{vmatrix} e_1 & e_2 & e_3 \\ F_1 & F_2 & F_3 \\ \frac{\partial^{\gamma}}{\partial x} & \frac{\partial^{\gamma}}{\partial y} & \frac{\partial^{\gamma}}{\partial z} \end{vmatrix} \\
& + \cos\big(\frac{\gamma\pi}{2}\big)\nabla \begin{vmatrix} 1 & 1 & 1 \\ \frac{\partial^{\gamma}}{\partial x} & \frac{\partial^{\gamma}}{\partial y} & \frac{\partial^{\gamma}}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \Big\} \\
& + \cos^2\big(\frac{\gamma\pi}{2}\big)\big(\frac{\partial^{\gamma}}{\partial x} + \frac{\partial^{\gamma}}{\partial y} + \frac{\partial^{\gamma}}{\partial z}\big)\big[\big(\frac{\partial^{\gamma}}{\partial x} + \frac{\partial^{\gamma}}{\partial y} + \frac{\partial^{\gamma}}{\partial z}\big)F \\
& - \nabla(F_1 + F_2 + F_3)\big]
\end{aligned}
\tag{31}
$$

Particular case:

Case 1: The standard fractional vector cross product for $\gamma = 0$ satisfies:

$$
\begin{aligned}
\nabla \times^{0} (\nabla \times^{0} F) = \nabla \begin{vmatrix} 1 & 1 & 1 \\ \frac{\partial^{0}}{\partial x} & \frac{\partial^{0}}{\partial y} & \frac{\partial^{0}}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} + \big(\frac{\partial^{0}}{\partial x} + \frac{\partial^{0}}{\partial y} + \frac{\partial^{0}}{\partial z}\big)\big[\big(\frac{\partial^{0}}{\partial x} + \frac{\partial^{0}}{\partial y} + \frac{\partial^{0}}{\partial z}\big)F \\
-\nabla(F_1 + F_2 + F_3)\big]
\end{aligned}
\tag{32}
$$

Case 2: The SFVCP for $\gamma = 1$ satisfies:

$$\nabla \times (\nabla \times F) = \nabla\langle \nabla, F\rangle - F\langle \nabla, \nabla\rangle \tag{33}$$

5.2. Divergence of a Curl of standard fractional vector cross product.

Using eqn (6), (23) and (29) we get,

$$\nabla.(\nabla \times^{\gamma} F) = 0 \tag{34}$$

Particular case:
Case 1: The SFVCP for $\gamma = 0$ satisfies:

$$\nabla.(\nabla \times^{0} F) = 0 \tag{35}$$

Case 2: The SFVCP for $\gamma = 1$ satisfies:

$$\nabla.(\nabla \times F) = 0 \tag{36}$$

5.3. Curl of a Divergence of standard fractional vector cross product.

$$\nabla.F = \frac{\partial^{\gamma}}{\partial x}F_1 + \frac{\partial^{\gamma}}{\partial y}F_2 + \frac{\partial^{\gamma}}{\partial z}F_3 \tag{37}$$

Using eqn (28) we get,

$$\nabla \times^{\gamma} (\nabla.F) = \sin\left(\frac{\gamma\pi}{2}\right) \begin{vmatrix} e_1 & e_2 & e_3 \\ \frac{\partial^{\gamma}}{\partial x} & \frac{\partial^{\gamma}}{\partial y} & \frac{\partial^{\gamma}}{\partial z} \\ \frac{\partial^{\gamma}}{\partial x}F_1 & \frac{\partial^{\gamma}}{\partial y}F_2 & \frac{\partial^{\gamma}}{\partial z}F_3 \end{vmatrix}$$

$$+ \cos\left(\frac{\gamma\pi}{2}\right) \begin{pmatrix} \frac{\partial^{\gamma}}{\partial y} + \frac{\partial^{\gamma}}{\partial z} & 0 & 0 \\ 0 & \frac{\partial^{\gamma}}{\partial z} + \frac{\partial^{\gamma}}{\partial x} & 0 \\ 0 & 0 & \frac{\partial^{\gamma}}{\partial x} + \frac{\partial^{\gamma}}{\partial y} \end{pmatrix} \begin{pmatrix} \frac{\partial^{\gamma}}{\partial x}F_1 \\ \frac{\partial^{\gamma}}{\partial y}F_2 \\ \frac{\partial^{\gamma}}{\partial z}F_3 \end{pmatrix}$$

$$-\cos\left(\frac{\gamma\pi}{2}\right)\begin{pmatrix} 0 & -\frac{\partial^\gamma}{\partial x} & -\frac{\partial^\gamma}{\partial x} \\ -\frac{\partial^\gamma}{\partial y} & 0 & -\frac{\partial^\gamma}{\partial y} \\ -\frac{\partial^\gamma}{\partial z} & -\frac{\partial^\gamma}{\partial z} & 0 \end{pmatrix}\begin{pmatrix} \frac{\partial^\gamma}{\partial x}F_1 \\ \frac{\partial^\gamma}{\partial y}F_2 \\ \frac{\partial^\gamma}{\partial z}F_3 \end{pmatrix} \tag{38}$$

Particular case:

Case 1: The SFVCP for $\gamma = 0$ satisfies:

$$\nabla \times^0 (\nabla.F) == 0 \tag{39}$$

Case 2: The SFVCP for $\gamma = 1$ satisfies:

$$\nabla \times (\nabla.F) = 0 \tag{40}$$

6. Conclusion

Fractional cross product is one of the important property in the study of fractional calculus. In this research we have defined a new fractional cross product named as standard fractional vector cross product (SFVCP) and further properties for that are discussed. Later we have discussed fractional curl and divergence of an electromagnetic vector field. Geometrical meaning of all the results are also discussed for $\gamma = 0$ and $\gamma = 1$. It is noticed that at $\gamma = 1$ all the results are similar to the results of standard vector cross product.

Conflicts of interest : The authors declare no conflict of interest.

Data availability : Not applicable

References

1. M.J. Crowe, *A history of vector analysis*, The evolution of the idea of a vectorial system, University of Notre Dame Press, Notre Dame, Ind.-London, 1967.
2. M.M. Kankarej and J.P. Singh, *Fractional Vector Cross Product in Euclidean 3-Space*, International Journal of Analysis and Applications (2023). https://doi.org/10.28924/2291-8639-21-2023-44
3. M.M. Tripathi and J.R. Kim, *Fractional vector cross product*, J. Kor. Soc. Math. educ. Ser B: Pure Appl. Math. **29** (2022), 103-112.
4. R.A. Horn, C.R. Johnson, *Matrix Analysis*, Cambridge Univ. Press, UK, 1985.
5. S. Das, *Geometrically deriving fractional cross product and fractional curl*, Int. J. Math. Comput. **20** (2013), 1-29.

6. S. Mishra, R.K. Mishra and S. Patnaik, *Fractional Cross product applied in radiation characteristic in micro-strip antenna: A simulation approach*, Test Engineering and Management **81** (2019), 1392-1401.

Manisha M. Kankarej received M.Sc. and Ph.D. from Pt. Ravishankar Shukla University and DDU Gorakhpur University, India. She is currently Assistant Professor at Rochester Institute of Technology, Dubai, UAE. Her research interest includes differential geometry and application of mathematics in hydrogeology.

Department of Mathematics, Rochester Institute of Technology, Dubai, UAE.
e-mail: manisha.kankarej@gmail.com

Jai P. Singh received M.Sc. and Ph.D. from Lucknow University, India. He is currently a Professor at B. S. N. V. P. G. College, Lucknow University, India. His research interests is differential geometry.

Department of Mathematics, B. S. N. V. P. G. College, Lucknow University, India.
e-mail: jaisinghjs@gmail.com