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ALGEBRAIC CONSTRUCTIONS OF GROUPOIDS FOR METRIC SPACES

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Abstract. Given a groupoid $(X, *)$ and a real-valued function $d : X \to \mathbf{R}$, a new (derived) function $\Phi(X, *)(d)$ is defined as $[\Phi(X, *)(d)](x, y) := d(x * y) + d(y * x)$ and thus $\Phi(X, *) : \mathbf{R}^X \to \mathbf{R}^{X^2}$ as well, where $\mathbf{R}$ is the set of real numbers. The mapping $\Phi(X, *)$ is an $\mathbf{R}$-linear transformation also. Properties of groupoids $(X, *)$, functions $d : X \to \mathbf{R}$, and linear transformations $\Phi(X, *)$ interact in interesting ways as explored in this paper. Because of the great number of such possible interactions the results obtained are of necessity limited. Nevertheless, interesting results are obtained. E.g., if $(X, *, 0)$ is a groupoid such that $x * y = 0 = y * x$ if and only if $x = y$, which includes the class of all d/BCK-algebras, then $(X, *)$ is $*$-metrizable, i.e., $\Phi(X, *)(d) : X^2 \to X$ is a metric on X for some $d : X \to \mathbf{R}$.

1. Introduction

Given $(Bin(X), \square)$, the set of all groupoids $(X, *)$ defined on the set X, it is the case that if for groupoids $(X, *)$ and $(X, \bullet)$, the groupoid $(X, \square)$ is defined by $x \square y := (x * y) \bullet (y * x)$, then $(Bin(X), \square)$ itself is a semigroup, with identity the left-zero-semigroup $(X, *)$, where $x * y = x$ for all $x, y \in X$. The study of $(Bin(X), \square)$ can take many forms. Thus, e.g., if $Ah(X)$ represents the set of all commutative groupoids $(X, *)$, where $x * y = y * x$ for all $x, y \in X$, then $(Ab(X), \square)$ is a two-sided ideal of $(Bin(X), \square)$, thus explaining why commutativity is such a "strong" property as compared to other properties. Other properties may be associated with subsemigroups, left-ideals, right-ideals, and with other relations, such as partial orders $\leq$ for example. Other perspectives are gained by modeling certain types of mathematical structures, e.g., digraphs, as groupoids $(X, *)$ where the product $x * y \in \{x, y\}$, represents the arrows in the digraph $x \to y$ if $x * y = y$ and $x * y = x$ if there is no such arrow in the digraph. In the existing literature the ideas mentioned above have already been considered. Analytic observations based on groupoids and semigroups have been developed. Sastry *et al.* [10] discussed a fixed point theorem in a lattice ordered semigroup cone valued cone metric spaces, and Moghaddasi [4] studied sequentially injective and complete acts over semigroups.

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In the paper we address the following issue. Suppose that $(X, *)$ is a groupoid, and that $d : X \to \mathbf{R}$ is a real-valued function on X. We would like to know more about the interaction of functions d with groupoids $(X, *)$. In order to do so, we consider a function $\Phi : Bin(X) \to (\mathbf{R}^{X^2})^{\mathbf{R}^X}$ where $\Phi(X, *)(d) : X^2 \to \mathbf{R}$ is defined for $d : X \to \mathbf{R}$ by $[\Phi(X, *)(d)](x, y) := d(x*y)+d(y*x)$ for all $x, y \in X$. We consider the function $\Phi(X, *)(d)$ to be the derived function of d on $(X, *)$. Since $\mathbf{R}^X$ and $\mathbf{R}^{X^2}$ are vector spaces over $\mathbf{R}$, $\Phi(X, *) : \mathbf{R}^X \to \mathbf{R}^{X^2}$ is a linear transformation also with associated $Ker\Phi(X, *)$ and $In\Phi(X, *)$ among the objects of interest as will be seen below.

2. Preliminaries

Among the various types of groupoids referenced in what follows along with those newly defined there are the d-algebras [1, 6, 7] and the BCK/BCI-algebras [2, 3, 12].

A *d-algebra* ([6, 7]) is a non-empty set X with a constant 0 and a binary operation "$*$" satisfying the following axioms:

(A) $x * x = 0$,
(B) $0 * x = 0$,
(C) $x * y = 0$ and $y * x = 0$ imply $x = y$

for all $x, y \in X$.

A BCK-algebra is a d-algebra X satisfying the following additional axioms:

(D) $(x * (x * y)) * y = 0$,
(E) $((x * y) * (x * z)) * (z * y) = 0$

for all $x, y, z \in X$.

The class of algebras fails to be a variety because of the presence axiom (C) which is not of the type $P(x_1, x_2, \cdots) = 0$ for all $x_1, x_2, \cdots \in X$ [11]. Given that groups $(X, *, 0)$ are a well-known class of groupoids, a closely related class of groupoids is the class of B-algebras $(X, *, 0)$ [8, 9] subject to the axioms (A) and

(F) $x * 0 = x$,
(G) $(x * y) * z = x * (z * (0 * y))$

for all $x, y, z \in X$.

Given a B-algebra $(X, *, 0)$, if we define $x \bullet y := x * (0 * y)$, then $(X, \bullet, 0)$ becomes a group. Also, for a group $(X, \bullet, 0)$, $x * y := x \bullet y^{-1}$ yields a B-algebra $(X, *, 0)$. Given a field $K := X$, if we set $x * y := a + bx + cy$, with $a, b, c \in K$, then $(X, *)$ is a linear groupoid over K, and again the linear groupoids form a large class of groupoids whose properties have been of interest [8] and continue to be of interest below. A final class of groupoids on a set X which comes up in the text below is the class of selective groupoids $(X, *)$ such that $x * y \in \{x, y\}$ which can be modeled as digraphs. Many other types are also mentioned and some new classes are discussed [5]. A desirable property for a class K of groupoids on a set X is that if $(X, *), (X, \bullet)$ are elements of K then so is $(X, \Box) = (X, *)\Box(X, \bullet)$, where $x\Box y = (x * y) \bullet (y * x)$ for all $x, y \in X$. If this is the case, then $(\mathrm{K}, \Box)$ is a subsemigroup of the semigroup $(Bin(X), \Box)$.

3. $(X,*)$-derived functions

Given a groupoid $(X,*)$, let $d : X \to \mathbf{R}$ be any real-valued function. Then a function $\Phi(X,*)(d) : X \times X \to \mathbf{R}$ is said to be an $(X,*)$-*derived function* from d if

$$\Phi(X,*)(d)(x,y) := d(x*y) + d(y*x)$$

for all $x, y \in X$. If we let $\mathbf{R}^X := \{f|f : X \to \mathbf{R}\}$ and $\mathbf{R}^{X^2} := \{g|g : X^2 \to \mathbf{R}\}$, and if we define $(f+g)(x) := f(x) + f(y)$ and $(\alpha f)(x) := \alpha f(x)$ for all $x \in X$ and for all $f, g \in \mathbf{R}^X$ (or $\mathbf{R}^{X^2}$) and for all $\alpha \in \mathbf{R}$, then $\mathbf{R}^X$ and $\mathbf{R}^{X^2}$ are vector spaces over $\mathbf{R}$.

Proposition 3.1. *Let $(X,*)$ be a groupoid and let $d_i : X \to \mathbf{R}$ be functions ($i = 1, 2$). If $\Phi(X,*)(d_i)$ are $(X,*)$-derived functions, then $\alpha\Phi(X,*)(d_1) + \beta\Phi(X,*)(d_2)$ is also an $(X,*)$-derived function for all $\alpha, \beta \in \mathbf{R}$.*

Proof. Given $x, y \in X$, we have

$$\begin{aligned} &[\alpha\Phi(X,*)(d_1) + \beta\Phi(X,*)(d_2)](x,y) \\ &= \alpha\Phi(X,*)(d_1)(x,y) + \beta\Phi(X,*)(d_2)(x,y) \\ &= \alpha(d_1(x*y) + d_1(y*x)) + \beta(d_2(x*y) + d_2(y*x)) \\ &= (\alpha d_1 + \beta d_2)(x*y) + (\alpha d_2 + \beta d_2)(y*x) \\ &= \Phi(X,*)(\alpha d_1 + \beta d_2)(x,y), \end{aligned}$$

which proves the proposition. □

Proposition 3.1 shows that $\Phi(X,*)(\alpha d_1 + \beta d_2) = \alpha\Phi(X,*)(d_1) + \beta\Phi(X,*)(d_2)$, i.e., $\Phi(X,*)$ is an $\mathbf{R}$-linear transformation from $\mathbf{R}^X$ to $\mathbf{R}^{X^2}$. Note that $\Phi(X,*)(x,y) = d(x*y) + d(y*x) = \Phi(X,*)(y,x)$ for any groupoid $(X,*)$. Given $(X,*) \in Bin(X)$, since $dim(\mathbf{R}^X) = |X|$, if $|X| < \infty$, then $dimKer\Phi(X,*) + dimIm\Phi(X,*) = |X|$, while otherwise $Ker\Phi(X,*) \oplus Im\Phi(X,*) \cong \mathbf{R}^X$. Given $(X,*) \in Bin(X)$, we define the *kernel* of $\Phi(X,*)$ by

$$Ker\Phi(X,*) := \{d \in \mathbf{R}^X \,|\, d(x*y) + d(y*x) = 0, \text{ for all } x, y \in X\}.$$

Proposition 3.2. *Let $(X,*) \in Bin(X)$. If $(X,*)$ has an identity e, i.e., $x * e = x = e * x$ for all $x \in X$, then $Im\Phi(X,*) \cong \mathbf{R}^X$.*

Proof. If $d \in Ker\Phi(X,*)$, then $d(x*y) + d(y*x) = 0$ for all $x, y \in X$. It follows that $d(x*e) + d(e*x) = 0$ for all $x \in X$. Hence $2d(x) = 0$ for all $x \in X$, i.e., $d = 0$ and $Ker\Phi(X,*) = \{0\}$, proving the proposition. □

Proposition 3.3. *$(X,*) \in Bin(X)$ is commutative, i.e., $x * y = y * x$ for all $x, y \in X$ and $X * X = X$, then $Im\Phi(X,*) \cong \mathbf{R}^X$.*

Proof. If $d \in Ker\Phi(X,*)$, then $d(x*y) + d(y*x) = 0$ for all $x, y \in X$. Since $(X,*)$ is commutative, we have $d(x*y) = 0$ for all $x, y \in X$. Since $X * X = X$, we obtain $d(x) = 0$ for all $x \in X$, proving $Ker\Phi(X,*) = \{0\}$. □

In Proposition 3.3, the condition $X * X = X$ is necessary. In fact, let $(X,*)$ be a commutative groupoid with $X * X \subsetneq X$. Then there exists an $u \in X \setminus X * X$. Define a function $d : X \to \mathbf{R}$ by $d(x) := 1$ when $x = u$, $d(x) := 0$ otherwise. Then $\Phi(X,*)(d)(x,y) = d(x*y) + d(y*x) = 0$ for all $x, y \in X$, i.e., d is a non-zero element of $Ker\Phi(X,*)$.

PROPOSITION 3.4. *If* $(X,*)$ *is the left-zero-semigroup, i.e.,* $x*y = x$ *for all* $x, y \in X$, *then* $Im\Phi(X,*) \cong \mathbf{R}^X$.

Proof. If $d \in \mathbf{R}^X$, then $\Phi(X,*)(d)(x,y) = d(x*y) + d(y*x) = d(x) + d(y)$, since $(X,*)$ is the left-zero-semigroup. It follows that $2d(x) = 0$ for all $x \in X$, proving that $Ker\Phi(X,*) = \{0\}$. □

THEOREM 3.5. *Let* $(\mathbf{R},*)$ *be a groupoid defined by* $x * y := a + bx + cy$ *for all* $x, y \in \mathbf{R}$ *where* $a, b, c \in \mathbf{R}$. *If* $Ker\Phi(X,*) \neq \{0\}$, *then* $b + c = 0$.

Proof. Let $d \in Ker\Phi(X,*)$. Then $0 = \Phi(X,*)(d)(x,y) = d(x*y) + d(y*x) = d(a + bx + cy) + d(a + by + cx)$ for all $x, y \in \mathbf{R}$. If $b = c = 0$, i.e., $x * y := a$ for all $x, y \in \mathbf{R}$, then $0 = d(x*y) + d(y*x) = 2d(a)$. Hence $d(x*y) = d(a) = 0$ for all $x, y \in \mathbf{R}$, i.e., $d(\mathbf{R}\times\mathbf{R}) = \{0\}$, which shows that $Ker\Phi(X,*) = \{0\}$.

If $b \neq 0, c = 0$, i.e., $x * y := a + bx$ for all $x, y \in \mathbf{R}$, then $0 = d(x*y) + d(y*x) = d(a + bx) + d(a + by)$ for all $x, y \in \mathbf{R}$. If we take $u := a + bx, v := a + by$, then $x = \frac{u-a}{b}, y = \frac{v-a}{b}$. It follows that $0 = d(u) + d(v)$ for all $u, v \in \mathbf{R}$, and hence $0 = 2d(u)$ for all $u \in \mathbf{R}$. Hence $Ker\Phi(X,*) = \{0\}$.

The case $b = 0, c \neq 0$ is similar to the above case, and we omit the proof.

Consider the case $b \neq 0 \neq c$. If $b^2 - c^2 \neq 0$. Given $u, v \in \mathbf{R}$, if we let

$$x := \frac{(bu - cv) + (c - b)a}{b^2 - c^2}, \quad y := \frac{(bv - cu) + (c - b)a}{b^2 - c^2}$$

then

$$\begin{aligned} 0 &= d(x*y) + d(y*x) \\ &= d(a + bx + cy) + d(a + by + cx) \\ &= d(u) + d(v) \end{aligned}$$

for all $u, v \in \mathbf{R}$. It follows that $Ker\Phi(X,*) = \{0\}$. If $b^2 - c^2 = 0$, then $b = \pm c$. If $b = c$, i.e., $x * y := a + b(x + y)$ for all $x, y \in \mathbf{R}$, then $x * y = y * x$ and hence $0 = \Phi(X,*)(d)(x,y) = d(x*y) + d(y*x) = 2d(x*y)$ for all $x, y \in \mathbf{R}$. If $u \in \mathbf{R}$, then there exist $x_0, y_0 \in \mathbf{R}$ such that $u = a + b(x_0 + y_0)$, i.e., $u = x_0 * y_0$. This shows that $d(u) = d(x_0 * y_0) = 0$ for all $u \in \mathbf{R}$, proving that $Ker\Phi(X,*) = \{0\}$. If $b = -c$, i.e., $x * y := a + b(x - y)$ for all $x, y \in \mathbf{R}$, define a function $\widehat{d} : \mathbf{R} \to \mathbf{R}$ by $\widehat{d}(x) := x - a$. Given $x, y \in \mathbf{R}$, we have $\widehat{d}(x*y) + \widehat{d}(y*x) = \widehat{d}(a + b(x-y)) + \widehat{d}(a + b(y-x)) = b(x-y) + b(y-x) = 0$, proving that $\widehat{d} \neq 0$ and $\widehat{d} \in Ker\Phi(X,*)$. □

4. Φ-injective and richly non-commutative

In Section 3, we obtained a class of groupoids which we shall consider as Φ-*injective groupoids*, i.e., those groupoids $(X,*)$ for which $Ker\Phi(X,*) = \{0\}$, i.e., for which the linear transformation $\Phi : \mathbf{R}^X \to \mathbf{R}^{X^2}$ is an injection. Suppose that a groupoid $(X,*)$ has the property that for any $u, v \in X$ there exist $x, y \in X$ such that $x * y = u$ and $y * x = v$. Then certainly $X * X = X$, but in a stronger fashion. E.g., if $(X,*)$ is a leftoid for f, where $f : X \to X$ is a surjection, then for given $u, v \in x$ there exist $x, y \in X$ such that $f(x) = u$ and $f(y) = v$. It follows that $x * y = f(x) = u, y * x = f(y) = v$. We shall consider groupoids of this type to be *richly non-commutative*.

EXAMPLE 4.1. Let $\mathbf{R}$ be the set of all real numbers. Define a binary operation "$*$" on $\mathbf{R}$ by $x * y := 2x + 3$ for all $x, y \in \mathbf{R}$. Then $(X, *)$ is richly non-commutative. In fact, given $u, v \in \mathbf{R}$, if we let $x := \frac{u-3}{2}, y := \frac{v-3}{2}$, then $x * y = 2(\frac{u-3}{2}) + 3 = u, y * x = 2(\frac{v-3}{2}) + 3 = v$.

PROPOSITION 4.2. *If a groupoid $(X, *)$ is richly non-commutative, then $X * X = X$.*

Proof. The proof is straightforward. $\square$

The converse of Proposition 4.2 need not be true in general.

EXAMPLE 4.3. Let $X := \{0, 1, 2, 3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Then $(X, *, 0)$ is a BCK-algebra [3, p. 245]. It is easy to see that $X * X = X$, but given $1, 3 \in X$ there are no $\alpha, \beta \in X$ such that $\alpha * \beta = 1, \beta * \alpha = 3$.

PROPOSITION 4.4. *If a groupoid $(X, *)$ is richly non-commutative, then it is Φ-injective.*

Proof. Let $d \in Ker\Phi(X, *)$. Since $(X, *)$ is richly non-commutative, given $u, v \in X$, there exist $x, y \in X$ such that $u = x*y, v = y*x$. It follows that $0 = d(x*y)+d(y*x) = d(u) + d(v)$ for all $u, v \in X$, and hence $0 = d(u) + d(u) = 2d(u)$ for all $u \in X$, proving that $Ker\Phi(X, *) = \{0\}$, i.e., $(X, *)$ is Φ-injective. $\square$

THEOREM 4.5. *Let $(X, *)$ be a richly non-commutative groupoid and let $(X, \bullet)$ be a Φ-injective groupoid. If $(X, \Box) := (X, *)\Box(X, \bullet)$, then $(X, \Box)$ is Φ-injective.*

Proof. If $d \in Ker\Phi(X, \Box)$, then for all $x, y \in X$ we have

$$\begin{aligned} 0 &= d(x\Box y) + d(y\Box x) \\ &= d((x * y) \bullet (y * x)) + d((y * x) \bullet (x * y)) \end{aligned} \tag{1}$$

Since $(X, *)$ is richly non-commutative, given $u, v \in X$, there exist $x, y \in X$ such that $u = x * y, v = y * x$. By (1), we have $\Phi(X, \bullet)(d)(u, v) = d(u \bullet v) + d(v \bullet u) = 0$, i.e., $d \in Ker\Phi(X, \bullet) = \{0\}$, which shows that $d = 0$. This shows that $(X, \Box)$ is Φ-injective. $\square$

THEOREM 4.6. *Let $(X, *), (X, \bullet)$ be richly non-commutative groupoids. If $(X, \Box) := (X, *)\Box(X, \bullet)$, then $(X, \Box)$ is also richly non-commutative.*

Proof. Given $u, v \in X$, since $(X, \bullet)$ is richly non-commutative, there exist $a, b \in X$ such that $u = a \bullet b, v = b \bullet a$. Since $(X, *)$ is richly non-commutative, there exist $x, y \in X$ such that $a = x * y, b = y * x$. It follows that $u = a \bullet b = (x * y) \bullet (y * x) = x\Box y$ and $v = b \bullet a = (y * x) \bullet (x * y) = y\Box x$, proving that $(X, \Box)$ is richly non-commutative. $\square$

COROLLARY 4.7. *If $RNC(X)$ consists of all richly non-commutative groupoids defined on X, then $(RNC(X), \Box)$ is a subsemigroup of $(Bin(X), \Box)$.*

5. Diagonal groupoids

Given $(X, *) \in Bin(X)$, we define a set $D(X, *)$ by

$$D(X, *) := \{x * x \,|\, x \in X\}$$

A groupoid $(X, *)$ is said to be a *diagonal groupoid* if $D(X, *) = X$. For example, if "+" is the usual addition on $\mathbf{R}$, then $D(\mathbf{R}, +) = \{x + x | x \in \mathbf{R}\} = \{2x | x \in \mathbf{R}\} = \mathbf{R}$, i.e., $(\mathbf{R}, +)$ is a diagonal groupoid.

Proposition 5.1. *Every diagonal groupoid $(X, *)$ is Φ-injective.*

Proof. If $d \in Ker\Phi(X, *)$, then $0 = d(x*x)+d(x*x) = 2d(x*x)$ for all $x \in X$, i.e., $d(x * x) = 0$. Since $(X, *)$ is diagonal, for any $x \in X$, there exists $\alpha \in X$ such that $x = \alpha * \alpha$. It follows that $d(x) = d(\alpha * \alpha) = 0$, proving that $(X, *)$ is Φ-injective. □

Proposition 5.2. *Let $(X, *), (X, \bullet)$ be diagonal groupoids. If $(X, \Box) := (X, *)\Box(X, \bullet)$, then $(X, \Box)$ is also diagonal.*

Proof. The proof is similar to the proof of Theorem 4.6, and we omit it. □

Corollary 5.3. *If $Diag(X)$ consists of all diagonal groupoids defined on X, then $(Diag(X), \Box)$ is a subsemigroup of $(Bin(X), \Box)$.*

6. $(X, *)$-metric on a groupoid

Given $(X, *) \in Bin(X)$ and a function $d : X \to \mathbf{R}$, the $(X, *)$-derived function $\Phi(X, *)(d)$ may have the following conditions:
(I) $\Phi(X, *)(d)(x, y) = \Phi(X, *)(d)(y, x)$,
(II) $\Phi(X, *)(d)(x, y) \geq 0$,
(III) $\Phi(X, *)(d)(x, y) = 0$ if and only if $x = y$,
(IV) $\Phi(X, *)(d)(x, z) \leq \Phi(X, *)(d)(x, y) + \Phi(X, *)(d)(y, z)$
for all $x, y, z \in X$. The $(X, *)$-derived function $\Phi(X, *)(d)$ is said to be an $(X, *)$-*metric* on a groupoid $(X, *)$ if it satisfies the conditions (I) $\sim$ (IV). In this case, the function d is said to be an $(X, *)$-*pre-metric* over $(X, *)$.

Example 6.1. Let $X := \mathbf{R}$ be the set of real numbers and "$-$" be the usual subtraction on X. If we define $d(x) := x$ for all $x \in X$, then $\Phi(X, -)(|d|)$ is an $(\mathbf{R}, -)$-metric on $(\mathbf{R}, -)$. In fact, given $x, y \in X$, we have $\Phi(X, -)(|d|)(x, y) = |d|(x-y)+|d|(y-x) = |x-y|+|y-x| = \Phi(X, -)(|d|)(y, x)$ and $\Phi(X, -)(|d|)(x, y) = |x-y|+|y-x| \geq 0$. Clearly, $\Phi(X, -)(|d|)(x, y) = 0$ if and only if $|x-y| = 0 = |y-x|$ if and only if $x = y$. We know that $\Phi(X, -)(|d|)(x, z) = |x - z| \leq |x - y| + |y - z| = \Phi(X, -)(|d|)(x, y) + \Phi(X, -)(|d|)(y, z)$.

Proposition 6.2. *Let $(X, *) \in Bin(X)$ and let $d : (X, *) \to \mathbf{R}$ be a function. If $|\,d\,|$ is an absolute function from $\mathbf{R}$ to $\mathbf{R}$, i.e., $|d|(x) := |d(x)|$ for all $x \in \mathbf{R}$, then $|\Phi(X, *)(d)| \leq \Phi(X, *)(|d|)$.*

Proof. Given $x, y \in X$, we have

$$\begin{aligned} |\Phi(X, *)(d)|(x, y) &= |d(x * y) + d(y * x)| \\ &\leq |d(x * y)| + |d(y * x)| \\ &= |d|(x * y) + |d|(y * x) \\ &= \Phi(X, *)(|d|)(x, y), \end{aligned}$$

proving the proposition. □

THEOREM 6.3. *Let $(X,*) \in Bin(X)$ and let $d_1, d_2, \cdots, d_n$ be $(X,*)$-pre-metrics on $(X,*)$ $(i = 1, \cdots, n)$. If we define $d := \alpha_1 d_2 + \cdots + \alpha_n d_n$ where $\alpha_i \geq 0$ with $\alpha_1 + \cdots + \alpha_n > 0$, then d is an $(X,*)$-pre-metric on $(X,*)$.*

Proof. (I) Given $x, y \in X$, we have

$$\begin{aligned}
\Phi(X,*)(d)(x,y) &= \Phi(X,*)(\sum_{i=1}^{n} \alpha_i d_i)(x,y) \\
&= (\sum_{i=1}^{n} \alpha_i d_i)(x*y) + (\sum_{i=1}^{n} \alpha_i d_i)(y*x) \\
&= \sum_{i=1}^{n} \alpha_i d_i(x*y) + \sum_{i=1}^{n} \alpha_i d_i(y*x) \\
&= \Phi(X,*)(\sum_{i=1}^{n} \alpha_i d_i)(y,x) \\
&= \Phi(X,*)(d)(y,x).
\end{aligned}$$

(II) It follows from (I) that

$$\Phi(X,*)(d)(x,y) = \sum_{i=1}^{n} \alpha_i (d_i(x*y) + d_i(y*x)) = \sum_{i=1}^{n} \alpha_i \Phi(X,*)(d_i)(x,y) \geq 0.$$

(III) Assume $\Phi(X,*)(d)(x,y) = 0$. Then $0 = \sum_{i=1}^{n} \alpha_i \Phi(X,*)(d_i)(x,y)$. Without loss of generality, we let $\alpha_{i_0} \neq 0$. Then $\alpha_{i_0} \Phi(X,*)(d_{i_0})(x,y) = 0$. Since $\Phi(X,*)(d_{i_0})$ is a metric, we obtain $x = y$. The converse is trivial, and we omit the proof.

(IV) Given $x, y, z \in X$, we have

$$\begin{aligned}
\Phi(X,*)(d)(x,z) &= \sum_{i=1}^{n} a_i \Phi(X,*)(d_i)(x,z) \\
&\leq \sum_{i=1}^{n} a_i [\Phi(X,*)(d_i)(x,y) + \Phi(X,*)(d_i)(y,z)] \\
&= \Phi(X,*)(d)(x,y) + \Phi(X,*)(d)(y,z)
\end{aligned}$$

Hence $\Phi(X,*)(d)$ is an $(X,*)$-metric on $(X,*)$ and hence d is an $(X,*)$-pre-metric on $(X,*)$. □

PROPOSITION 6.4. *Let $(X,*) \in Bin(X)$ and let d be an $(X,*)$-pre-metric on $(X,*)$. If $d_0 \in Ker\Phi(X,*)$, then $\Phi(X,*)(d + d_0) = \Phi(X,*)(d)$.*

Proof. Given $x, y \in X$, we have

$$\begin{aligned}
\Phi(X,*)(d+d_0)(x,y) &= (d+d_0)(x*y) + (d+d_0)(y*x) \\
&= [d(x*y) + d_0(x*y)] + [d(y*x) + d_0(y*x)] \\
&= \Phi(X,*)(d)(x,y) + \Phi(X,*)(d_0)(x,y) \\
&= \Phi(X,*)(d)(x,y),
\end{aligned}$$

proving the proposition. □

In Proposition 6.4, $\Phi(X,*)(d_0)$ need not be an $(X,*)$-metric on $(X,*)$ unless $|X| = 1$. In fact, assume $\Phi(X,*)(d_0)$ is an $(X,*)$-metric on $(X,*)$ and $|X| \geq 2$. Then there exist $x, y \in X$ such that $x \neq y$. Since $d_0 \in Ker\Phi(X,*)$, we have $\Phi(X,*)(d_0)(x,y) = 0$ and hence $x = y$, a contradiction.

Proposition 6.5. *Let $\varphi : (X,*) \to (Y,\bullet)$ be a homomorphism of groupoids and let $d : Y \to \mathbf{R}$ be a function. Then there exists a map $d_\varphi : X \to \mathbf{R}$ such that*

$$\Phi(X,*)(d_\varphi)(x,y) = \Phi(Y,\bullet)(d)(\varphi(x),\varphi(y))$$

for all $x, y \in X$.

Proof. Given a homomorphism $\varphi : (X,*) \to (Y,\bullet)$ and a function $d : Y \to \mathbf{R}$, we define a function $d_\varphi : X \to \mathbf{R}$ by $d_\varphi(x) := d(\varphi(x))$ for all $x \in X$. Then

$$\begin{aligned}
\Phi(X,*)(d_\varphi)(x,y) &= d_\varphi(x*y) + d_\varphi(y*x) \\
&= [d(x*y) + d_0(x*y)] + [d(y*x) + d_0(y*x)] \\
&= d(\varphi(x*y)) + d(\varphi(y*x)) \\
&= d(\varphi(x)\bullet\varphi(y)) + d(\varphi(y)\bullet\varphi(x)) \\
&= \Phi(Y,\bullet)(d)(\varphi(x),\varphi(y))
\end{aligned}$$

proving the proposition. □

Corollary 6.6. *Let $\varphi : (X,*) \to (Y,\bullet)$ be a homomorphism of groupoids and let $d : Y \to \mathbf{R}$ be a function. If $\Phi(Y,\bullet)(d)$ is a $(Y,\bullet)$-metric over $(Y,\bullet)$ and if φ is one-one, then $\Phi(X,*)(d_\varphi)$ is an $(X,*)$-metric over $(X,*)$.*

Proof. The proofs of (I), (II) and (IV) are routine, and we omit its proof. Consider (III). Suppose that $\Phi(X,*)(d_\varphi)(x,y) = 0$. Then $\Phi(Y,\bullet)(d)(\varphi(x),\varphi(y)) = 0$. Since $\Phi(Y,\bullet)(d)$ is a $(Y,\bullet)$-metric, we obtain $\varphi(x) = \varphi(y)$. Since φ is one-one, we obtain $x = y$.

Assume that $x = y$. Then $\varphi(x) = \varphi(y)$. Since $\Phi(Y,\bullet)(d)$ is a $(Y,\bullet)$-metric, we have $\Phi(Y,\bullet)(d)(\varphi(x),\varphi(y)) = 0$. By Proposition 6.5, we obtain $\Phi(X,*)(d_\varphi)(x,y) = 0$. □

7. Metrizable groupoids

A groupoid $(X,*)$ is said to be $*$-*metrizable* if, for some function $d : X \to \mathbf{R}$, the $(X,*)$-derived function $\Phi(X,*)(d) : X^2 \to \mathbf{R}$ with $\Phi(X,*)(d)(x,y) := d(x*y)+d(y*x)$ is a non-zero $(X,*)$-metric over $(X,*)$.

Example 7.1. Let $X := \mathbf{R}$ be the set of real numbers and let $x \otimes y := x - y$ for all $x, y \in \mathbf{R}$. If we define $d : X \to \mathbf{R}$ by $d(x) := |x|$ for all $x \in X$, then it is easy to see that $(X,\otimes)$ is $\otimes$-metrizable.

Note that $(\mathbf{R},+)$ is not $+$-metrizable where $+$ is the usual addition on $\mathbf{R}$. In fact, assume $\Phi(\mathbf{R},+)(d)$ is non-zero $+$-metrizable over $(\mathbf{R},+)$ for some function $d : \mathbf{R} \to \mathbf{R}$. Then $0 = \Phi(\mathbf{R},+)(d)(x,x) = 2d(2x)$ for all $x \in \mathbf{R}$. It follows that $d(x) = 0$ for all $x \in \mathbf{R}$. This shows that $\Phi(\mathbf{R},*)(d)(x,y) = d(x+y) + d(y+x) = 0$ for all $x, y \in \mathbf{R}$, a contradiction.

Note that the group $(\mathbf{R},+)$ has the associated B-algebra $(\mathbf{R},*)$ where $x * y := x + (-y) = x - y$. The fact that $(\mathbf{R},+)$ and $(\mathbf{R},*)$ are quite distinct in many ways is illustrated once again in this setting when we note that $(\mathbf{R},+)$ is not $+$-metrizable and that $(\mathbf{R},*)$ is $*$-metrizable.

Proposition 7.2. *Every left-zero-semigroup* $(X,*)$ *has no non-* $(X,*)$*-metric over* $(X,*)$.

Proof. Assume $\Phi(X,*)(d)$ is a non-zero metric over $(X,*)$ for some $d : X \to \mathbf{R}$. Then $\Phi(X,*)(d)(x,y) = d(x*y)+d(y*x) = d(x)+d(y) \geq 0$ for all $x, y \in X$. It follows that $d(x) \geq 0$ for all $x \in X$. Since $\Phi(X,*)(d)(x,x) = 0$, we obtain $2d(x) = 0$ for all $x \in X$. This shows that $\Phi(X,*)(d)(x,y) = 0$ for all $x, y \in X$, a contradiction. □

Proposition 7.3. *Let* $(X,*,0)$ *be a* BCK*-algebra and let* $\Phi(X,*)(d)$ *be an* $(X,*)$*-metric over* $(X,*)$*. Then*

(i) $d(0) = 0$,
(ii) $\Phi(X,*)(d)(x,0) = d(x) \geq 0$,
(iii) $\Phi(X,*)(d)(x,y) \leq d(x) = d(y)$,
(iv) $d(x) = d(y) = 0$ *implies* $x = y$,

for all $x, y \in X$.

Proof. (i) Since $(X,*,0)$ is a BCK-algebra, we have

$$0 \leq \Phi(X,*)(d)(x,0) = d(x*0) + d(0*x) = d(x) + d(0) \tag{2}$$

for all $x \in X$. If we let $x := 0$ in (2), then by (III) we have

$$0 = \Phi(X,*)(d)(0,0) = d(0) + d(0) = 2d(0),$$

i.e., $d(0) = 0$.
(ii) It follows from (i) and (2) immediately.
(iii) By (IV), (I) and (2), we have

$$\Phi(X,*)(d)(x,y) \leq \Phi(X,*)(d)(x,y) + \Phi(X,*)(d)(x,y) = d(x) + d(y).$$

(iv) Assume that $d(x) = d(y) = 0$. Then $\Phi(X,*)(d)(x,y) \leq d(x) + d(y) = 0$, i.e., $\Phi(X,*)(d)(x,y) = 0$. By (III), we obtain $x = y$. □

Proposition 7.4. *Let* $(X,*,0)$ *be a standard* BCK*-algebra and let* $\Phi(X,*)(d)$ *be an* $(X,*)$*-metric over* $(X,*)$*. Then*

$$\Phi(X,*)(d)(x,y) := \begin{cases} d(y) & \text{if } x < y, \\ d(x) & \text{if } x < y, \\ d(x) + d(y) & \text{if } x \,||\, y. \end{cases}$$

Proof. If $x < y$, then $x * y = 0, y * x = y$. It follows that $\Phi(X,*)(d)(x,y) = d(x*y) + d(y*x) = d(0) + d(y) = d(y)$ by Proposition 7.3-(i). If $y < x$, then we obtain $\Phi(X,*)(d)(x,y) = d(x)$. Assume that $x \,||\, y$. Then $x * y = x, y * x = y$. It follows that $\Phi(X,*)(d)(x,y) = d(x*y) + d(y*x) = d(x) + d(y)$. □

Theorem 7.5. *Let* $(X,*,0)$ *be a groupoid with the condition:* $x * y = 0 = y * x$ *if and only if* $x = y$*. Then* $(X,*)$ *is* $*$*-metrizable.*

Proof. Let $d : X \to \mathbf{R}$ be a function defined by $d(0) := 0$ and $d(x) := 1$ if $x \neq 0$. Then $\Phi(X,*)(d)(x,y) = d(x*y) + d(y*x) = \Phi(X,*)(d)(y,x) \geq 0$ and $\Phi(X,*)(d)(x,y) \in \{0,1,2\}$ for all $x, y \in X$. Moreover, by assumption, we have

$$\begin{aligned} \Phi(X,*)(d)(x,y) = 0 &\iff d(x*y) + d(y*x) = 0 \\ &\iff d(x*y) = 0 = d(y*x) \\ &\iff x*y = 0 = y*x \\ &\iff x = y. \end{aligned}$$

Assume that the condition (IV) does not hold. Then there exist $x, y, z \in X$ such that $\Phi(X,*)(d)(x,z) > \Phi(X,*)(d)(x,y)+\Phi(X,*)(d)(y,z)$. Since $0 \le \Phi(X,*)(d)(x,z) \le 2$, we have two cases: (i) $\Phi(X,*)(d)(x,y) + \Phi(X,*)(d)(y,z) = 0$; (ii) $\Phi(X,*)(d)(x,y) + \Phi(X,*)(d)(y,z) = 1$. For the case (i), since $\Phi(X,*)(d)(x,y) + \Phi(X,*)(d)(y,z) = 0$, we obtain $x = y = z$, which implies that $0 = \Phi(X,*)(d)(x,z) < \Phi(X,*)(d)(x,y) + \Phi(X,*)(d)(y,z) = 0$, a contradiction.
Consider the case (ii) $\Phi(X,*)(d)(x,y) + \Phi(X,*)(d)(y,z) = 1$. It follows that either $\Phi(X,*)(d)(x,y) = 0$ or $\Phi(X,*)(d)(y,z) = 0$. This shows that either $x = y$ or $y = z$. If $x = y$ holds, then $\Phi(X,*)(d)(x,z) > \Phi(X,*)(d)(x,x) + \Phi(X,*)(d)(x,z) = 0 + \Phi(X,*)(d)(x,z) = \Phi(X,*)(d)(x,z)$, a contradiction. If $y = z$, then it leads to $\Phi(X,*)(d)(x,y) > \Phi(X,*)(d)(x,y)$, a contradiction. Hence $(X,*)$ is $*$-metrizable. □

Corollary 7.6. *Every d/BCK-algebra $(X,*,0)$ is $*$-metrizable.*

Proof. Every d/BCK-algebra has the condition: $x * y = 0 = y * x$ if and only if $x = y$. □

Corollary 7.7. *Every B-algebra $(X,*,e)$ is $*$-metrizable.*

Proof. Let $(X,\bullet,e)$ be a group and let $(X,*,e)$ be the associated B-algebra. Then $x * y = x \bullet y^{-1}$ for all $x, y \in X$. Assume $x * y = e = y * x$. Then $x \bullet y^{-1} = e = y \bullet x^{-1}$ and hence $x = y$. The converse is trivial. □

Example 7.8. Let $\mathbf{R}$ be the set of all real numbers and "+" be the usual addition on $\mathbf{R}$. Define a function $d : \mathbf{R} \to \mathbf{R}$ by $d(0) = 0$ and $d(x) = 1$ if $x \ne 0$. Then

$$\begin{aligned}\Phi(\mathbf{R},+)(d)(x,y) &= d(x+y) + d(y+x) \\ &= \begin{cases} 2 & \text{if } x+y \ne 0, \\ 0 & \text{otherwise.}\end{cases}\end{aligned}$$

The triangle inequality may fail in that $\Phi(\mathbf{R},+)(d)(x,x) = 2 > 0 = \Phi(\mathbf{R},+)(d)(x,-x)+ \Phi(\mathbf{R},+)(d)(-x,x)$ for $x \ne 0$. In fact, the groupoid $(\mathbf{R},+)$ does not satisfies the condition: $x + y = y + x = 0$ if and only if $x = y$.

8. Quasi-logarithm for groupoids

Let $(X,*) \in Bin(X)$. A function $d : X \to \mathbf{R}$ is said to be a *quasi-logarithm* for $(X,*)$ if for all $x, y \in X$, $d(x * y) = \alpha d_1(x) + \beta d_2(y)$ for some functions $d_i : X \to \mathbf{R}$ and for some $\alpha, \beta \in \mathbf{R}$. We denote it by $d := \alpha d_1 + \beta d_2$.

Example 8.1. Let $X := \mathbf{R}$ and let $a, b, c \in X$. Define $x * y := a + bx + cy$ for all $x, y \in X$. Define functions $d(x) := x$ and $d_1(x) := \frac{1}{\alpha}(a+bx)$, $\alpha \ne 0$ and $d_2(x) := 2x$ for all $x \in X$. Then $d(x*y) = x*y = a+bx+cy = \alpha\frac{1}{\alpha}(a+bx)+\frac{c}{2}d_2(y) = \alpha d_1(x)+\frac{c}{2}d_2(y)$, i.e., $d = \alpha d_1 + \frac{c}{2}d_2$ is a quasi-logarithm on $(X,*)$.
Let $(X,*)$ be the left-zero-semigroup. Then $d(x * y) = d(x) = 1d(x) + 0d(y)$, so that $d = d_1 = d_2, \alpha = 1, \beta = 0$ shows that any function $d : X \to \mathbf{R}$ whatsoever is a quasi-logarithm for $(X,*)$.
Let $X := (0,\infty)$ and let $x * y := xy$ be the usual multiplication on real numbers. If we define $d := \log$, then $\log(xy) = \log(x) + \log(y)$ so that $d = d_1 = d_2 = \log$, $\alpha = \beta = 1$ shows that log is a quasi-logarithm for the groupoid $(X,*)$.

PROPOSITION 8.2. *Let* $(X,*) \in Bin(X)$ *and let* $d = \alpha d_1 + \beta d_2$ *be a quasi-logarithm where* $\alpha\beta \neq 0$. *If* $x_0, y_0 \in X$ *such that* $d_1(x_0) = 0$ *and* $d_2(y_0) = 0$, *then* $d(x*y) = d(x*y_0) + d(x_0*y)$ *for all* $x, y \in X$.

Proof. Since $d(x_0*y) = \alpha d_1(x_0) + \beta d_2(y) = \beta d_2(y)$, we obtain $d_2(y) = \frac{1}{\beta}d_2(y)$. Similarly, $d(x*y_0) = \alpha d_1(x) + \beta d_2(y_0) = \alpha d_1(x)$ implies that $d_1(x) = \frac{1}{\alpha}d(x*y_0)$. This shows that $d(x*y) = d(x*y_0) + d(x_0*y)$ for all $x, y \in X$. □

PROPOSITION 8.3. *Let* $\varphi : (X,*) \to (Y,\bullet)$ *be a homomorphism of groupoids and let* $d = \alpha d_1 + \beta d_2$ *be a quasi-logarithm for* $(Y,\bullet)$. *Then* $d_\varphi := d\circ\varphi$ *is a quasi-logarithm for* $(X,*)$.

Proof. Given $x, y \in X$, we have

$$\begin{aligned} d_\varphi(x*y) &= d(\varphi(x*y))) \\ &= d(\varphi(x)\bullet\varphi(y)) \\ &= \alpha d_1(\varphi(x)) + \beta d_2(\varphi(y)) \\ &= \alpha(d_1)_\varphi(x) + \beta(d_2)_\varphi(y), \end{aligned}$$

proving the proposition. □

THEOREM 8.4. *Let* $(X,*), (X,\bullet) \in Bin(X)$ *and let* $(X,\Box) := (X,*)\Box(X,\bullet)$. *If* $d = \alpha d_1 + \beta d_2$ *is a quasi-logarithm for* $(X,\bullet)$, *then*

$$\Phi(X,\Box)(d) = \alpha\Phi(X,*)(d_1) + \beta\Phi(X,*)(d_2).$$

Proof. Given $x, y \in X$, we have $d(x\Box y) = d((x*y)\bullet(y*x)) = \alpha d_1(x*y) + \beta d_2(y*x)$ and $d(y\Box x) = \alpha d_1(y*x) + \beta d_2(x*y)$. It follows that

$$\begin{aligned} \Phi(X,\Box)(d)(x,y) &= d(x\Box y) + d(y\Box x) \\ &= \alpha d_1(x*y) + \beta d_2(y*x) + \alpha d_1(y*x) + \beta d_2(x*y) \\ &= \alpha(d_1(x*y) + d_1(y*x)) + \beta(d_2(x*y) + d_2(y*x)) \\ &= \alpha\Phi(X,*)(d_1)(x,y) + \beta\Phi(X,*)(d_2)(x,y) \\ &= [\alpha\Phi(X,*)(d_1) + \beta\Phi(X,*)(d_2)](x,y), \end{aligned}$$

proving the theorem. □

COROLLARY 8.5. *Let* $(X,*), (X,\bullet) \in Bin(X)$ *and let* $(X,\Box) := (X,*)\Box(X,\bullet)$. *Let* $d = \alpha d_1 + \beta d_2$ *be a quasi-logarithm for* $(X,\bullet)$ *where* $\alpha \geq 0, \beta \geq 0, \alpha\beta > 0$. *If* $\Phi(X,*)(d_i)$ $(i = 1, 2)$ *are an* $(X,*)$-*metric over* $(X,*)$, *then* $\Phi(X,\Box)$ *is an* $(X,\Box)$-*metric over* $(X,\Box)$.

Proof. It follows immediately from Theorem 8.4. □

9. Conclusion

As we have seen above, the idea of a derived function $\Phi(X,*)(d)$ of a function $d : X \to \mathbf{R}$ based on the groupoid structure $(X,*)$ allows consideration of the interaction of the groupoid structure $(X,*)$ with the function structure $d : X \to \mathbf{R}$. Clearly, the amount of information potentially available in this way is very much larger than what has been obtained sofar. Our goal has been to point out a direction in which opportunity lies. In the future we hope and expect to gather larger harvests from further investigation of this subject. For example, if $d : X \to [0, 1]$, then the approach used above will produce a sort of fuzzy theory quite naturally.

Declarations

Conflict of interest

The authors declare that they have no competing interests.

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