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CESÀRO TYPE UNCERTAIN VARIABLES

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Abstract. The main purpose of this study is to shed light on whether new types of uncertain variable sequences can be defined with the help of an infinite matrix. For this purpose, the first-order Cesàro matrix was used as an infinite matrix, and new types of uncertain variable sequences, called Cesàro-type uncertain variable sequences, were obtained. Theorems about uncertain variable sequences of Cesàro type have been included in this study, and some comparisons have been made. Thus, the gaps in the existing literature were filled.

1. Introduction

After research conducted by various authors, especially Baoding Lui, regarding uncertain variables and their properties, it seems there is increasing interest in convergence types of uncertain variable sequences in probability theory. For instance, in [1], Chen et al. investigated the convergence of complex uncertain sequences and obtained noteworthy results. Tripathy and others, in [14], defined the Nörlund and Riesz means of sequences of complex uncertain variables. Additionally, in [11] and [12], Nath and Tripathy investigated some valuable properties of uncertain sequences defined by the Orlicz function. On the other hand, Wu and Xia, in [16], explored the relationships among convergence concepts of uncertain sequences. Saha and Tripathy, in [13], studied the statistical Fibonacci convergence of complex uncertain variables.

Perhaps more crucial than the aforementioned research is the article titled "On the Convergence of Uncertain Sequences" by You in [17]. Additionally, Das et al., in [3], investigated the matrix transformation between complex uncertain sequences in terms of mean and characterized the associated matrix transformations. Further results on the matrix transformation of complex uncertain sequences were presented by Das and Tripathy in [2].

Now let's provide the basic definitions and notations that we will use in this study:

2. Preliminaries and Notations

In this section, some important concepts and theorems in uncertainty theory will be given, which are used throughout this paper. Afterward, we will provide the definition and related theorems of Cesàro convergent sequences of uncertain variables.

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Let Γ be a nonempty set and a collection of subsets of Γ be $\mathscr{L}$. The $\mathscr{L}$ is called a σ-algebra if

(**i**) $\Gamma \in \mathscr{L}$

(**ii**) if $\Lambda \in \mathscr{L}$ then $\Lambda^c \in \mathscr{L}$ and

(**iii**) if $\Lambda_1, \Lambda_2, ... \in \mathscr{L}$, then $\Lambda_1 \cup \Lambda_2 \cup ... \in \mathscr{L}$.

An event is referred to as each element Λ within the σ-algebra $\mathscr{L}$. A function $\mathscr{M}$, denoted as uncertain measure, operates from $\mathscr{L}$ to the interval $[0, 1]$. To construct an axiomatic definition of uncertain measure, the assignment of a number $\mathscr{M}\{\Lambda\}$ to each event Λ becomes essential, representing the degree of belief in the occurrence of Λ. Lui [9] introduced the following four properties to articulate certain mathematical characteristics of $\mathscr{M}\{\Lambda\}$.

(**iv**) Condition of normality. That is $\mathscr{M}\{\Lambda\} = 1$ for $\Lambda \in \Gamma$,

(**v**) Condition of monotonicity. That is if $\Lambda_1 \subset \Lambda_2$ then $\mathscr{M}\{\Lambda_1\} \leq \mathscr{M}\{\Lambda_2\}$,

(**vi**) Condition of self duality. That is $\mathscr{M}\{\Lambda\} + \mathscr{M}\{\Lambda^c\} = 1$,

(**vii**) Condition of countable subadditivity. That is $\mathscr{M}\{\bigcup_{i=1}^{\infty} \Lambda_i\} \leq \sum_{i=1}^{\infty} \mathscr{M}(\Lambda_i)$ for every countable sequence of events $\{\Lambda_n\}$.

The concept of the uncertain variable was defined by Liu [9] as a measurable function ξ from an uncertainty space $U = (\Gamma, \mathscr{M}, \mathscr{L})$ to the set of real numbers, i.e., for any Borel set B of real numbers, the set $\{\xi \in B\} = \{\gamma \in \Gamma : \xi(\gamma) \in B\}$ is a event. Hybrid variable was introduced by Liu [8] as a measurable function from a chance space to the set of real numbers. Accordingly, a concept of chance measure was introduced by Li and Liu [5].

A crisp number c may be regarded as a special uncertain variable. In fact, it is the constant function $\xi(\gamma) = c$ on the uncertainty space $(\Gamma, \mathscr{L}, \mathscr{M})$, [9].

The expected value operator of an uncertain variable is defined as follows:

$$E[\xi] = \int_0^{\infty} \mathscr{M}(\xi \geq r)dr - \int_{-\infty}^{0} \mathscr{M}(\xi \leq r)dr$$

provided that at least one of the two integrals is finite, [9]. It is clear that, the convergence is a important concept; therefore some concepts of convergence for uncertain sequences were introduced in [9] as follows:

The sequence of uncertain variables is a sequence of uncertain variables indexed by integers. Let suppose that the set of all sequences of uncertain variables be

$$w = \{(\xi_n(\gamma)) : \xi_n(\gamma) \text{ is a uncertain variable for all } n \in \mathbb{N}\}.$$

There are four convergence concepts about the sequences of uncertain variables. These are convergence almost surely, convergence in measure, convergence in mean, and convergence in distribution [6]. These definitions can be given as follows with the notation of Liu:

The set

$$f = \{(\xi_n(\gamma)) \in w : \lim_n |\xi_n(\gamma) - \xi(\gamma)| = 0, \gamma \in \Lambda, \mathscr{M}\{\Lambda\} = 1\} \tag{1}$$

is called convergent almost surely sequences of uncertain variables. Similarly, the uncertain sequence $(\xi_n(\gamma))$ is said to be convergent in measure to $\xi(\gamma)$ if it is element of the set

$$m = \{(\xi_n(\gamma)) \in w : \lim_n \mathscr{M}\{|\xi_n(\gamma) - \xi(\gamma)| \geq \epsilon\} = 0, \epsilon > 0, \gamma \in \Lambda, \mathscr{M}\{\Lambda\} = 1\}. \tag{2}$$

Furthermore, let $\xi_0(\gamma), \xi_1(\gamma), \xi_2(\gamma), ...$ be uncertain variables with finite expected values. Then the sequence $(\xi_n(\gamma))$ is called convergence in mean to $\xi(\gamma)$ if it is in

$$e = \{(\xi_n(\gamma)) \in w : \lim_n E[|\xi_n(\gamma) - \xi(\gamma)|] = 0, \gamma \in \Lambda, \mathcal{M}\{\Lambda\} = 1\}. \tag{3}$$

In [9], it is proved that if the uncertain sequence $(\xi_n(\gamma))$ converges in mean to $\xi(\gamma)$, then $(\xi_n(\gamma))$ converges in measure to $\xi(\gamma)$, that is $e \subset m$.

The uncertainty distribution Φ of an uncertain variable ξ is defined by $\Phi(x) = \mathcal{M}\{\gamma \in \Gamma : \xi(\gamma) \leq x\}$ for any real number x, [6].

Let $\Phi_0, \Phi_1, \Phi_2, ...$ be the uncertain distributions of uncertain variables $\xi_0(\gamma), \xi_1(\gamma), \xi_2(\gamma), ...$ respectively. The sequence $(\xi_n(\gamma))$ is called convergence in distribution to $\xi(\gamma)$ if $\Phi_n \to \Phi$ at any continuity point of Φ and the set of all sequences of convergent in distribution denotes with d.

Let $\xi_n(\gamma)$ and $\xi(\gamma)$ be uncertain variables defined on the uncertainty space $(\Gamma, \mathcal{L}, \mathcal{M})$ where $(n \in \mathbb{N})$. The sequence $(\xi_n(\gamma))$ is said to be convergent uniformly almost sure to $\xi(\gamma)$ if there exists $\{E_k\}$ so that $\mathcal{M}(E_k) \to 0$, so the sequence $(\xi_n(\gamma))$ converges uniformly to $\xi(\gamma)$ in $\Gamma - E_k$, for any fixed k, [17]. The set of all sequences of convergent uniformly almost sure denotes with u.

In [5], You examined the relationships between the convergence concepts we have defined above and revealed some interesting results.

The sequence $(\xi_n(\gamma))$ of uncertain variables in the space $(\Gamma, \mathcal{L}, \mathcal{M})$ is said to be slowly oscillating if $|\mathcal{M}(\xi_n(\gamma)) - \mathcal{M}(\xi_m(\gamma))| \to 0$, as $m, n \to \infty$ with $1 \leq \frac{m}{n} \to 1$, [15].

The following lemma have given together with notation of Lui [7].

Lemma 2.1. *[7] Suppose $\xi_0(\gamma), \xi_1(\gamma), \xi_2(\gamma), ...$ are uncertain variables. $(\xi_n(\gamma))$ converges in measure to $\xi(\gamma)$ if and only if there exists subsequence $(\xi_{n'_k}(\gamma))$ of $(\xi_{n'}(\gamma))$ such that $(\xi_{n'_k}(\gamma))$ converges uniformly almost sure to $\xi(\gamma)$, $(k \to \infty)$, for any subsequence $(\xi_{n'}(\gamma))$ of $(\xi_n(\gamma))$.*

3. Main Definitions and Theorems

Let us consider the row and column elements of any infinite matrix of real numbers as uncertain variables since a crisp number c may be regarded as a special uncertain variable. If $A = (a_{nk})$ is an infinite matrix of real numbers and $(\xi_k(\gamma))$ is a sequence of uncertain variables ($n, k \in \mathbb{N}$) then the product $a_{nk}\xi_k(\gamma)$ is also an uncertain variable defined as $\eta(\gamma) = a_{nk}\xi_k(\gamma)$ (see, [9]); thus the product $a_{nk}\xi_k(\gamma)$ is meaningful. Let ξ_1 and ξ_2 be two uncertain variables. Then the sum $\xi = \xi_1 + \xi_2$ and the product $\xi = \xi_1\xi_2$ are uncertain variables and defined by $\xi(\gamma) = \xi_1(\gamma) + \xi_2(\gamma)$ and $\xi(\gamma) = \xi_1(\gamma)\xi_2(\gamma)$ for all $\gamma \in \Gamma$, respectively [6].

These operations can be extended to infinite sequences of uncertain variables as follows:

Consider $(\xi_n(\gamma))$ and $(\eta_n(\gamma))$ as two sequences of uncertain variables. Then the sum and product of the sequences $(\xi_n(\gamma))$ and $(\eta_n(\gamma))$ of uncertain variables are

$$(\xi_n(\gamma) + \eta_n(\gamma)) = (\xi_0(\gamma) + \eta_0(\gamma), \xi_1(\gamma) + \eta_1(\gamma), ..., \xi_n(\gamma) + \eta_n(\gamma), ...)$$

and

$$(\xi_n(\gamma)\eta_n(\gamma)) = (\xi_0(\gamma)\eta_0(\gamma), \xi_1(\gamma)\eta_1(\gamma), ..., \xi_n(\gamma)\eta_n(\gamma), ...);$$

respectively.

There have been many studies in the literature about the convergence of sequences of uncertain variables. Upon examination of these studies, it is observed that there is no new set of uncertain variables defined as the domain of an infinite matrix. To define such a set of sequences of uncertain variables, I personally believe that there is only one way: to use the domain of an infinite matrix for sequences of uncertain variables. This can be explained as follows:

Let $A = (a_{nk})$ be an infinite matrix of real numbers, and assume that λ is one of the f, m and e.

The *matrix domain* λ_A of an infinite matrix A with respect to λ is defined by

$$\lambda_A = \left\{ (\xi_k(\gamma)) \in w \ : \ \sum_k a_{nk}\xi_k(\gamma) \in \lambda \text{ for all } n \in \mathbb{N} \text{ and } \gamma \in \Lambda \right\}. \tag{4}$$

In this case, we can define the following new set as the matrix domain of the related sets. The set f_A defined as

$$f_A = \{(\xi_i(\gamma)) : \lim_n |\sum_k a_{nk}\xi_k(\gamma) - \xi(\gamma)| = 0, \gamma \in \Lambda, \mathcal{M}\{\Lambda\} = 1\} \tag{5}$$

is called A type convergent almost surely sequences of uncertain variables. Similarly, the sets A type convergent in measure of all uncertain sequences and A type convergence in mean of uncertain sequences are defined, respectively, as follows:

$$m_A = \{(\xi_i(\gamma)) : \lim_n \mathcal{M}\{|\sum_k a_{nk}\xi_k(\gamma) - \xi(\gamma)| \geq \epsilon\} = 0, \epsilon > 0, \gamma \in \Lambda, \mathcal{M}\{\Lambda\} = 1\} \tag{6}$$

and

$$e_A = \{(\xi_i(\gamma)) : \lim_n E[|\sum_k a_{nk}\xi_k(\gamma) - \xi(\gamma)|] = 0, \gamma \in \Lambda, \mathcal{M}\{\Lambda\} = 1\}. \tag{7}$$

The matrix $C = (c_{nk})$ defined by

$$c_{nk} = \begin{cases} \frac{1}{n}, & n \leq k \\ 0, & \text{otherwise} \end{cases} ; \ (n, \ k \in \mathbb{N}) \tag{8}$$

is called Cesàro matrix of order one, [4]. Now let's define the sequence $(\nu_n(\gamma))$ which will be frequently used, as the Cesàro transform, of a sequence $(\xi_i(\gamma))$ i.e.,

$$\nu_n(\gamma) = \frac{1}{n}\sum_{i=1}^{n} \xi_i(\gamma). \tag{9}$$

For every i, if the $\xi_i(\gamma)$ is an uncertain variable then clearly $\frac{1}{n}\sum_{i=1}^{n} \xi_i(\gamma)$ is also an uncertain variable for all $n \in \mathbb{N}$.

Definition 3.1. Let $\nu_n(\gamma) = \frac{1}{n}\sum_{i=1}^{n} \xi_i(\gamma)$. Then, $\nu_n(\gamma)$ is called the Cesàro type of uncertain variable for $n = 1, 2, 3, ...$

In (5), if we take the matrix C instead of the matrix A then we can define the set of all convergent almost surely sequences of Cesàro type uncertain variables as follows:

$$f_C = \{\xi \in w : \lim_n |\frac{1}{n}\sum_{i=1}^{n} \xi_i(\gamma) - \xi(\gamma)| = 0, \gamma \in \Lambda, \mathcal{M}\{\Lambda\} = 1\}. \tag{10}$$

Similarly if we take the matrix C instead of the matrix A in (6) then the set of all convergent in measure of all Cesàro type uncertain variables sequences will be

$$(11)\quad m_C = \{\xi \in w : \lim_n \mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n} \xi_i(\gamma) - \xi(\gamma)| \geq \epsilon\} = 0, \epsilon > 0, \gamma \in \Lambda, \mathcal{M}\{\Lambda\} = 1\}.$$

Finally, we define the set of all convergence in mean of Cesàro type uncertain variables sequences with

$$(12)\quad e_C = \{\xi \in w : \lim_n E[|\frac{1}{n}\sum_{i=1}^{n} \xi_i(\gamma) - \xi(\gamma)|] = 0, \gamma \in \Lambda, \mathcal{M}\{\Lambda\} = 1\}.$$

It is obviously that the equality (10), (11) and (12) are special case of (5), (6) and (7) for $A = C$, respectively.

Now, in parallel with the definition made by You and Yan, we will give a definition about the Cesàro convergent in $p-$distance [18].

Definition 3.2. The uncertain sequence $(\xi_n(\gamma))$ is said to be Cesàro convergent in $p-$distance to $\xi(\gamma)$ if

$$\lim_{n\to\infty} d_C^p\left(\frac{1}{n}\sum_{i=1}^{n} \xi_i(\gamma), \xi(\gamma)\right) = \lim_{n\to\infty}\left(E\left[\left|\frac{1}{n}\sum_{i=1}^{n} \xi_i(\gamma) - \xi(\gamma)\right|^p\right]\right)^{\frac{1}{p}} = 0.$$

The following Theorem 3.4 gives a comparison between Cesàro convergent almost surely sequences of uncertain variables and the convergent sequences of uncertain variables. But, firstly, we will give a lemma.

Lemma 3.3. *Let $(\xi_n(\gamma))$ be a sequence Cesàro type of uncertain variable. Then for any given numbers $t > 0$ and $p > 0$ we have $\mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n} \xi_i(\gamma)| \geq t\} \leq E[|\frac{1}{n}\sum_{i=1}^{n} \xi_i(\gamma)|^p]t^{-p}$.*

Proof. Let $f(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$. It is clear that the function f is even and increasing on $[0, \infty)$. The function $f(x)$ is also satisfies the conditions of Theorem 2.44 of Lui ([10], p.56). If take $f(x) = |\frac{1}{n}\sum_{i=1}^{n} x_i|^p$ in the Theorem 2.44 of Lui then the proof is easily obtained so we omit it. □

Theorem 3.4. *If the sequence the $(\xi_n(\gamma))$ is convergent in almost sure to $\xi(\gamma)$ then $(\xi_n(\gamma))$ is also Cesàro type convergent in almost sure to $\xi(\gamma)$ but conversely not true, generally.*

Proof. Let us suppose that $(\xi_n) \in f$. Then from (1), for $\gamma \in \Lambda, \mathcal{M}\{\Lambda\} = 1$ then there is a constant H such that $|\xi_n(\gamma) - \xi(\gamma)| \leq H$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$ be given. Then there is exists an integer n_0 such that $|\xi_n(\gamma) - \xi(\gamma)| \leq \epsilon$ for $n \geq n_0$. Next, choose an integere $N \geq n_0$ such that $N > 2Hn_0\epsilon^{-1}$. Then for $n > N$

$$\begin{aligned}|\frac{1}{n}\sum_{i=1}^{n} \xi_i(\gamma) - \xi(\gamma)| &= \frac{1}{n}|\xi_1(\gamma) - \xi(\gamma) + \xi_2(\gamma) - \xi(\gamma) + \cdots + \xi_n(\gamma) - \xi(\gamma)| \\ &\leq \frac{1}{n}\sum_{i=1}^{n_0} |\xi_i(\gamma) - \xi(\gamma)| + \frac{1}{n}\sum_{i=n_0+1}^{n} |\xi_i(\gamma) - \xi(\gamma)| \\ &\leq \frac{1}{n}(2Hn_0) + \frac{1}{n}(n - n_0)\epsilon < \epsilon + \epsilon = 2\epsilon.\end{aligned}$$

This proves that $\frac{1}{n}\sum_{i=1}^{n} \xi_i(\gamma) \to \xi(\gamma)$ for $n \to \infty$.

Now, we define

$$\xi_j(\gamma_i) = \begin{cases} 2j, & i = j \\ 0, & \text{otherwise} \end{cases} \tag{13}$$

then the sequence of uncertain variables

$$(\eta_k(\gamma)) = (\sum_{k=n-1}^{n} (-1)^{n-k} k\xi_k(\gamma)) \tag{14}$$

is in f_C but the sequence $(\eta_k(\gamma))$ is not in f. This is also the proof of the Theorem 3.4. □

THEOREM 3.5. *If $(\xi_n(\gamma)) \in m_C$ then $(\xi_n(\gamma)) \in e_C$ that is the inclusion $m_C \subseteq e_C$ is valid.*

Proof. Let us suppose that $(\xi_n(\gamma)) \in m_C$. Then we have $E[|\frac{1}{n}\sum_{i=1}^{n} \xi_n(\gamma) - \xi(\gamma)|] \to 0$ as $n \to \infty$. It is clear that from definition of expected value that

$$\begin{aligned} E[|\frac{1}{n}\sum_{i=1}^{n} \xi_n(\gamma) - \xi(\gamma)|] &= \int_0^{+\infty} \mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n} \xi_n(\gamma) - \xi(\gamma)| \geq x\}dx \\ &\geq \int_0^{\epsilon} \mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n} \xi_n(\gamma) - \xi(\gamma)| \geq x\}dx \\ &\geq \int_0^{\epsilon} \mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n} \xi_n(\gamma) - \xi(\gamma)| \geq \epsilon\}dx \\ &= \epsilon\mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n} \xi_n(\gamma) - \xi(\gamma)| \geq \epsilon\} \end{aligned}$$

for any given $\epsilon > 0$. Thus we can write

$$\mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n} \xi_n(\gamma) - \xi(\gamma)| \geq \epsilon\} \leq \frac{E[|\frac{1}{n}\sum_{i=1}^{n} \xi_n(\gamma) - \xi(\gamma)|]}{\epsilon}. \tag{15}$$

If we take consider of the definition of e_C; we see that

$$\lim_n \mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n} \xi_n(\gamma) - \xi(\gamma)| \geq \epsilon\} = 0$$

that is $(\xi_n(\gamma)) \in e_C$. This is last step of the proof. □

Let $(\xi_n(\gamma))$ be a sequence uncertain variables and define the set $\tilde{m}$ as follows:

$$\tilde{m} = \{(\xi_n(\gamma)) : \mathcal{M}\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\{\gamma \in \Gamma : |\frac{1}{n}\sum_{i=1}^{n} \xi_i(\gamma) - \xi(\gamma)| \geq \epsilon\}\right) = 0, \text{ for all } \epsilon > 0\}.$$

PROPOSITION 3.6. *The condition $(\xi_n(\gamma)) \in f_C$ if and only if $(\xi_n(\gamma)) \in \tilde{m}$ is holds.*

Proof. The proof is easily obtained from definition of Cesàro type convergent almost sure in (10). □

Definition 3.7. Let the $\xi_n(\gamma)$ and $\xi(\gamma)$ be uncertain variables defined on uncertainty space $(\Gamma, \mathscr{L}, \mathscr{M})$ for all $n \geq 0$. The sequence $(\xi_n(\gamma))$ is said to be Cesàro type convergent uniformly almost sure to $\xi(\gamma)$ if there exists (E_k), $\mathscr{M}(\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)) \to 0$ such that the sequence $(\xi_n(\gamma))$ converges uniformly to $\xi(\gamma)$ in $\Gamma \setminus E_k$ for any fixed k. The set of all sequences of Cesàro type convergent uniformly almost sure to $\xi(\gamma)$ is denoted with u_C.

Let define the set $\tilde{u}_C$ as follows:

$$\tilde{u}_C = \{(\xi_n(\gamma)) : \lim_m \mathscr{M}\left(\bigcup_{n=m}^{\infty}\{\gamma \in \Gamma : |\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)| \geq \epsilon\}\right) = 0\}.$$

Theorem 3.8. *Let consider the sequence $(\xi_n(\gamma))$ of uncertain variables. Then $(\xi_n(\gamma)) \in u_C$ if and only if $(\xi_n(\gamma)) \in \tilde{u}_C$.*

Proof. Let us suppose that the sequence $(\xi_n(\gamma)) \in u_C$ that is Cesàro type convergent uniformly almost sure to $\xi(\gamma)$. Then for any $K > 0$ there exists a set B such that $\mathscr{M}(B) < K$ and the sequence $(\xi_n(\gamma))$ is convergent uniform converges to $\xi(\gamma)$ on $\Gamma \setminus B$. This means that, for any $\epsilon > 0$ and for all $n \geq m$, $\gamma \in \Gamma \setminus B$ there exists an integer $m > 0$ such that

$$|\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)| < \epsilon.$$

From here, we see that $\bigcup_{n=m}^{\infty}\{\gamma \in \Gamma : |\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)| \geq \epsilon\} \subset B$ and in this case

$$\mathscr{M}(\bigcup_{n=m}^{\infty}\{\gamma \in \Gamma : |\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)| \geq \epsilon\}) = 0.$$

Thus

$$\lim_m \mathscr{M}(\bigcup_{n=m}^{\infty}\{\gamma \in \Gamma : |\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)| \geq \epsilon\}) = 0.$$

On the contrary, let us suppose $(\xi_n(\gamma)) \in \tilde{u}_C$ that for any $\epsilon > 0$ then we have

$$\lim_m \mathscr{M}(\bigcup_{n=m}^{\infty}\{\gamma \in \Gamma : |\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)| \geq \epsilon\}) = 0.$$

In this case, for any given $K > 0$ and $k \geq 1$ there exists m_k such that

$$\mathscr{M}(\bigcup_{n=m}^{\infty}\{\gamma \in \Gamma : |\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)| \geq \frac{1}{k}\}) < \frac{K}{2^k}.$$

Let

$$B = \bigcup_{k=1}^{\infty}\bigcup_{n=m_k}^{\infty}\{\gamma \in \Gamma : |\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)| \geq \frac{1}{k}\}.$$

Then $\mathscr{M}(B) < K$. Furthermore any $k \in \mathbb{N}$ and $n \geq m_k$ we have

$$\sup_{\gamma \in \Gamma}|\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)| \geq \frac{1}{k}.$$

Thus the Theorem 3.8 is proved. □

Theorem 3.9. *If the sequence* $(\xi_n(\gamma)) \in f_C$ *then* $(\xi_n(\gamma)) \in m_C$, *that is the inclusion* $f_C \subseteq m_C$ *is hold.*

Proof. Let suppose that the sequence $(\xi_n(\gamma))$ of uncertain variables be Cesàro type converges uniform sense to $\xi(\gamma)$ then from Theorem 3.8 it follows that

$$\lim_m \mathcal{M}\left(\bigcup_{n=m}^{\infty}\{\gamma\in\Gamma : |\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)-\xi(\gamma)|\geq\epsilon\}\right)=0. \tag{16}$$

On the other hand

$$\mathcal{M}\{\gamma\in\Gamma : |\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)-\xi(\gamma)|\geq\epsilon\}\leq\mathcal{M}(\bigcup_{n=m}^{\infty}\{\gamma\in\Gamma : |\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)-\xi(\gamma)|\geq\epsilon\})$$

which it shows that to us the sequence $(\xi_n(\gamma))$ of uncertain variables Cesàro type convergent in measure to $\xi(\gamma)$. □

Proposition 3.10. *Let suppose that* $(\xi_n(\gamma))$ *be Cesàro type convergent uniform almost sure to* $\xi(\gamma)$. *Then sequence* $(\xi_n(\gamma))$ *of uncertain variables is Cesàro type convergent in distribution to* $\xi(\gamma)$.

Proof. The proof is obtain from Theorem 3.8. So we omit it. □

Theorem 3.11. *The sequence* $(\xi_n(\gamma))$ *of uncertain variables Cesàro type convergent in measure to* $\xi(\gamma)$ *if and only if there exists subsequence* $(\xi_{n'_k}(\gamma))$ *of* $(\xi_{n'}(\gamma))$ *such that* $(\xi_{n'_k}(\gamma))$ *is Cesàro type converges uniformly almost sure to* $\xi(\gamma)$, $(k\to\infty)$, *for any subsequence* $(\xi_{n'}(\gamma))$ *of* $(\xi_n(\gamma))$.

Proof. Suppose that the sequence $(\xi_n(\gamma))$ of uncertain variables Cesàro type convergent in measure to $\xi(\gamma)$. Then the sequence $(\xi_{n'}(\gamma))$ of uncertain variables Cesàro type convergent in measure to $\xi(\gamma)$. It follows from the definition of Cesàro type converges in measure that there exists a subsequence $(\xi_{n'_k}(\gamma))$ of $(\xi_{n'}(\gamma))$ such that

$$\mathcal{M}\{\gamma\in\Gamma : |\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)-\xi(\gamma)|\geq\frac{1}{k}\}\leq\frac{1}{2^k}\text{ for any }k\geq 1.$$

From here we see that

$$\mathcal{M}(\bigcup_{k=m}^{\infty}\{\gamma\in\Gamma : |\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)-\xi(\gamma)|\geq\frac{1}{k}\})\leq\sum_{k=m}^{\infty}\frac{1}{2^k}=\frac{1}{2^{m-1}}$$

for any $m\geq 1$. Thus

$$\lim_m \mathcal{M}(\bigcup_{k=m}^{\infty}\{\gamma\in\Gamma : |\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)-\xi(\gamma)|\geq\frac{1}{k}\})=0$$

This means that $(\xi_{n'_k}(\gamma))$ is Cesàro type converges uniformly almost sure convergent $\xi(\gamma)$.

Now let us suppose that the sequence $(\xi_n(\gamma))$ of uncertain variables does not Cesàro type convergent in measure to $\xi(\gamma)$. Then there exists a $\epsilon>0$ such that

$$\lim_m \mathcal{M}(\bigcup_{k=m}^{\infty}\{\gamma\in\Gamma : |\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)-\xi(\gamma)|\geq\frac{1}{k}\})>K>0.$$

Then there exists a subsequence $(\xi_{n'}(\gamma))$ of $(\xi_n(\gamma))$ such that

$$\mathcal{M}(\bigcup_{k=m}^{\infty}\{\gamma\in\Gamma : |\frac{1}{n}\sum_{i=1}^{n'}\xi_i(\gamma)-\xi(\gamma)|\geq\epsilon\})>K.$$

It is clear that $(\xi_{n'}(\gamma))$ has no subsequence Cesàro type converges uniformly almost sure convergent to $\xi(\gamma)$ then $(\xi_n(\gamma))$ Cesàro type convergent in measure to $\xi(\gamma)$. □

The following theorem is describe some important properties of Cesàro type convergent in measure and Cesàro type converges in $p-$distance.

THEOREM 3.12. *Suppose that uncertain variables sequences $(\xi_n(\gamma))$ and $(\eta_n(\gamma))$ Cesàro type convergent in measure to $\xi(\gamma)$ and $\eta(\gamma)$, respectively. For some positive numbers M, N, M_1, N_1 and for any n, if $M_1 \leq |\xi_n(\gamma)| < M$ and $N_1 \leq |\eta_n(\gamma)| < N$ then $(\xi_n(\gamma) * \eta(\gamma))$ Cesàro type converges in measure to $\xi(\gamma) * \eta(\gamma)$, where $* \in \{+, -, \cdot, \div\}$.*

Proof. Since the proof similar one to other, the proof will be give only $* = +$.
From subadditivity,

$$\mathcal{M}\{|(\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)+\frac{1}{n}\sum_{i=1}^{n}\eta_i(\gamma))-(\xi(\gamma)+\eta(\gamma))|\geq 2\epsilon\}$$

$$\leq\mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)-\xi(\gamma))|\geq\epsilon\}+\mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n}\eta_i(\gamma)-\eta(\gamma))|\geq\epsilon\}\to 0 \text{ for } n\to\infty.$$

Thus uncertain variables sequence $(\xi_n(\gamma)+\eta_n(\gamma))$ Cesàro type convergent in measure to $(\xi(\gamma)+\eta(\gamma))$. □

THEOREM 3.13. *If uncertain variables sequence $(\xi_n(\gamma))$ Cesàro type convergent to $\xi(\gamma)$ and $\eta(\gamma)$, respectively then $\mathcal{M}\{\xi(\gamma)=\eta(\gamma)\}=1$.*

Proof. If we consider subadditivity axiom the we have

$$\mathcal{M}\{|\xi(\gamma)-\eta(\gamma)+\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)-\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)|\geq\epsilon\}$$

$$\leq\mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)-\xi(\gamma)|\geq\frac{\epsilon}{2}\}+\mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)-\eta(\gamma)|\geq\frac{\epsilon}{2}\}\to 0$$

for $n\to\infty$. □

The following theorem is an interesting theorem comparing the Cesàro type convergent in measure with the Cesàro type convergent in $p-$ distance.

THEOREM 3.14. *Suppose that the sequence of uncertain variables $(\xi_n(\gamma))$ be Cesàro type convergent in $p-$ distance. Then the sequence of uncertain variables $(\xi_n(\gamma))$ is Cesàro type convergent in measure to ξ. But conversely is not true, generally.*

Proof. If uncertain sequence $(\xi_n(\gamma))$ Cesàro type convergent in $p-$ distance then we can write

$$\lim_{n}E[|\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma)-\xi(\gamma)|^p]=0.$$

From Lemma 3.3 it follows that for any $\epsilon > 0$ and for $n \to \infty$ we have,

$$\mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)| \geq \epsilon\} \leq E[|\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)|^p(\epsilon^{-p})] \to 0.$$

That is uncertain sequence $(\xi_n(\gamma))$ Cesàro convergent in measure to $\xi(\gamma)$.

Now, let us consider the sequence which it given in (14) for $\xi \equiv 0$. For $i > 1$ and $\epsilon \geq 0$, we have $\lim_i \mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n}\eta_i(\gamma) - \xi(\gamma)| \geq \epsilon\} = \lim_i \frac{1}{2i} = 0$ which means that the sequence $(\xi_i(\gamma))$ is Cesàro type convergent in measure to ξ. Let us define uncertain distribution of uncertain variable $|\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)|$ by

$$\Phi_i(x) = \begin{cases} 0, & for\ x < 0 \\ 1 - \frac{1}{2i}, & for\ 0 \leq x < 2 \\ 1, & for\ x \geq 2i \end{cases}.$$

Since $E[\xi] = \int_0^\infty (1 - \Phi(x))dx - \int_{-\infty}^0 \Phi(x)dx$ for uncertain variable ξ and uncertain distribution Φ (see, [9]) for each $i > 1$ we can write $E[\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)] = 1$. From monotonicity property, we have

$$\begin{aligned} E[|\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)|^p] &= \int_0^\infty \mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)|^p \\ &\geq x\}dx \geq \int_0^\infty \mathcal{M}\{|\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)| \geq x\}dx \\ &= 1. \end{aligned}$$

That is $1 = \lim_i d_C^p(\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma), \xi(\gamma)) = \lim_i (E[|\frac{1}{n}\sum_{i=1}^{n}\xi_i(\gamma) - \xi(\gamma)|])^{\frac{1}{p+1}} \neq 0$. This is shows to us that conversely of the Theorem 3.14 is not true. □

4. Conclusion

In this study, the Cesàro type convergence in measure, Cesàro type convergence in almost sure and Cesàro type convergence in p- distance of uncertain variables was introduced and some gaps in the literature were filled with basic theorems related to this convergence.

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