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ON VECTOR VALUED DIFFERENCE SEQUENCE SPACES

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Abstract. In the present paper, using the notion of difference sequence spaces, we introduce new kind of Cesàro summable difference sequence spaces of vector valued sequences with the aid of paranorm and modulus function. In addition, we extend the notion of statistical convergence to introduce a new sequence space $SC_1(\Delta, q)$ which coincides with $C_1^1(X, \Delta, \phi, \lambda, q)$ (one of the above defined Cesàro summable difference sequence spaces) under the restriction of bounded modulus function.

1. Introduction and preliminaries

Generalizing the usual notion of convergence of scalar sequences, Zygmund [40] in 1979 paid key attention to statistical convergence. However, Steinhaus [35] and Fast [15] already have formally introduced this concept of statistical convergence in 1951. While dealing with 'convergence in density' it was Buck [9] who get encountered with statistical convergence.

After the remarkable work in the field of statistical convergence by Šalát [32] and Fridy [17], now statistical convergence has become one of the most vibrant field for researcher in summability theory. Later on, statistical convergence was further scrutinized by Mursaleen [26], Şengül and Et [33], Tripathy [36] and many others [7, 10, 12, 14, 16, 20, 21, 23, 30, 38]. The statistical convergence relies upon the definition of natural density of subset of $\mathbb{N}$

Definition 1.1. [28] For $A \subseteq \mathbb{N}$, the natural density $\delta(A)$ is defined as

$$\delta(A) = \lim_{n\to\infty} \frac{1}{n}\text{card}(\{k \le n : k \in A\})$$

provided the limit exists, where card($\cdot$) means numbers of elements in the enclosed set. Obviously, $\delta(A) = 0$, for finite subset A of $\mathbb{N}$. Also $\delta(\mathbb{N} - A) = 1 - \delta(A)$.

Definition 1.2. A scalar sequence (ξ_m) is said to be statistically convergent to $l \in \mathbb{C}$ if for given $\varepsilon > 0$, $\delta(\{m \le n : |\xi_m - l| \ge \epsilon\}) = 0$, i.e., $\lim_{n\to\infty} \frac{1}{n}\text{card}(\{m \le n : |\xi_m - l| > \varepsilon\}) = 0$. And l is referred as statistical limit of the (ξ_m). By S we notate the class of all statistically convergent sequences.

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Motivating from the definition of absolute value function, i.e., $|a|$

$$|a| = \begin{cases} a & if \quad a \geq 0, \\ -a & if \quad a < 0 \end{cases}$$

Nakano [27] in 1953, structured the image of modulus function. By Ruckle [31] and Maddox [24], a modulus function is a map $\phi : [0, \infty) \to [0, \infty)$ such that the following holds:

(M_1) $\phi(\xi) = 0$ iff $\xi = 0$
(M_2) $\phi(\xi + \eta) \leq \phi(\xi) + \phi(\eta)$ for all $\xi \geq 0, \eta \geq 0$
(M_3) ϕ is monotonically increasing
(M_4) $\lim_{\xi \to 0^+} \phi(\xi) = \phi(0)$.

As an example, $\phi_1(\xi) = \dfrac{\xi}{1+\xi}$ and $\phi_2(\xi) = \xi^p$, $(0 < p \leq 1)$ are modulus functions where ϕ_1 is bounded and ϕ_2 is unbounded. It is observed that sum of two modulus functions is again a modulus function. Moreover, composition of a modulus function over itself is also a modulus function.

Kizmaz [22] in 1981, introduced the idea of difference sequence space by introducing the following difference sequence spaces :

$$c_0(\Delta) = \{(\xi_m) \in s : (\Delta\xi_m) = (\xi_m - \xi_{m+1}) \in c_0\}$$
$$c(\Delta) = \{(\xi_m) \in s : (\Delta\xi_m) = (\xi_m - \xi_{m+1}) \in c\}$$
$$\ell_\infty(\Delta) = \{(\xi_m) \in s : (\Delta\xi_m) = (\xi_m - \xi_{m+1}) \in \ell_\infty\}$$

where s, ℓ_∞, c and c_0 respectively denote the linear spaces of all, bounded, convergent and null sequences of scalars.

Adding the flavour of statistical convergence, modulus function and difference sequence, many more mathematician, for instance, Connor [13], Ghosh and Srivastva [18], Çolak [11], Altin and Et [2], Bhardwaj and Singh [8] and some others have enriched the theory of sequence space by introducing some new sequence spaces. For many more refrences one may refer to [1, 3–6, 19, 29, 34, 37, 39].

Let us recall some definitions and notations, before proceeding further.

A paranorm space (X, q) is a topological linear space whose topology is induced by a paranorm, a real valued sub-additive function on X such that $q(\theta) = 0$, $q(-\xi) = q(\xi)$ and the scalar multiplication is continuous (here θ is the zero element of linear space X).

A seminorm q is real valued function defined on linear space X such that $q(\xi) \geq 0$; $q(\xi + \eta) \leq q(\xi) + q(\eta)$ and $q(a\xi) = |a|q(\xi)$. Every seminorm is a paranorm but not conversely.

A seminorm q_2 is said to be rough than q_1 on X if there exists a constant $\mu > 0$ such that $q_2(\xi) < \mu.q_1(\xi)$, $\xi \in X$.

Throughout the paper, ϕ will denote a modulus function. By X we refer a linear topological and locally convex T_2-space whose topology is induced by continuous seminorm q. The symbol $s(X)$ will denote the space of $X-$valued sequences. Let $\lambda = \langle\lambda_m\rangle$ be a bounded sequence of positive real numbers with $h = \inf_{m\geq1} \lambda_m$, $H = \sup_{m\geq1} \lambda_m$ and $C = \max\{1, 2^{H-1}\}$.

Also for $a_m, b_m \in \mathbb{C}$, we have $|a_m + b_m|^{\lambda_m} \leq C[|a_m|^{\lambda_m} + |b_m|^{\lambda_m}]$ $\forall$ $m \in \mathbb{N}$, and for any $\mu \in \mathbb{C}$, $|\mu|^{\lambda_m} \leq \max\{1, |\mu|^H\}$ (see for instance Maddox [24]).

In the present paper, we get an opportunity to work with vector valued sequences and making use of modulus function, paranorm and Cesàro summability to introduced some generalized Cesàro difference sequence spaces.

2. Main Results

Motivating from the spaces of strongly Cesàro summable sequences of Maddox [25] and exploring the Cesàro means of difference sequences of X, we introduce the following sequence spaces:

$$C_1^0(X,\Delta,\phi,\lambda,q)=\left\{\xi=(\xi_m)\in s(X):\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right)\right]^{\lambda_m}=0\right\}$$

$$C_1^\infty(X,\Delta,\phi,\lambda,q)=\left\{\xi=(\xi_m)\in s(X):\sup_n\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right)\right]^{\lambda_m}<\infty\right\}$$

$$C_1^1(X,\Delta,\phi,\lambda,q)$$
$$=\left\{\xi=(\xi_m)\in s(X):\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i-l\right)\right)\right]^{\lambda_m}=0\text{ for some } l\in X\right\}.$$

If we take $\phi(\xi)=\xi$, then the above defined spaces reduce to $C_1^0(X,\Delta,\lambda,q)$, $C_1^\infty(X,\Delta,\lambda,q)$ and $C_1^1(X,\Delta,\lambda,q)$ respectively.

Throughout the paper Θ will notate $0,1$ or ∞.

THEOREM 2.1. *The sequence sets $C_1^\Theta(X,\Delta,\phi,\lambda,q)$ are linear spaces.*

Proof. It is sufficient to prove the result for $\Theta=0$, as other cases may be proved on similar line. Let $\xi,\eta\in C_1^0(X,\Delta,\phi,\lambda,q)$ and $\alpha,\beta\in\mathbb{C}$. Then there exists positive integers M_α and N_β such that $|\alpha|\le M_\alpha$ and $|\beta|\le N_\beta$. Linear of Δ and sub-additivity of ϕ yields

$$\begin{aligned}&\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta(\alpha\xi_i+\beta\eta_i\right)\right)\right]^{\lambda_m}\\
&\qquad\le\left[\phi\left(|\alpha|q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right)+\phi\left(|\beta|q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\eta_i\right)\right)\right]^{\lambda_m}\\
&\qquad\le C\left[M_\alpha\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right)\right]^{\lambda_m}+C\left[N_\beta\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\eta_i\right)\right)\right]^{\lambda_m}\\
&\qquad\le C.M_\alpha^H\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right)\right]^{\lambda_m}+C.N_\beta^H\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\eta_i\right)\right)\right]^{\lambda_m}\\
&\qquad\longrightarrow 0\ as\ n\to\infty.\end{aligned}$$

Hence $(\alpha\xi+\beta\eta)\in C_1^0(X,\Delta,\phi,\lambda,q)$. Consequently $C_1^0(X,\Delta,\phi,\lambda,q)$ is a linear space □

THEOREM 2.2. *For a modulus function ϕ,*

$$C_1^0(X,\Delta,\phi,\lambda,q)\subset C_1^1(X,\Delta,\phi,\lambda,q)\subset C_1^\infty(X,\Delta,\phi,\lambda,q)$$

Proof. As first inclusion is trivial, so we proceed for second one.
Let $\xi \in C_1^1(X,\Delta,\phi,\lambda,q)$. Then

$$\begin{aligned}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right)\right]^{\lambda_m} &= \left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l + l\right)\right)\right]^{\lambda_m}\\ &\le \left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right)\right) + \phi(q(l))\right]^{\lambda_m}\\ &\le C\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right)\right)\right]^{\lambda_m} + C[\phi(q(l))]^{\lambda_m}.\end{aligned}$$

Let μ be positive integer such that $q(l) \le \mu$. Then

$$\begin{aligned}&\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right)\right]^{\lambda_m}\\ &\quad\le C\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right)\right)\right]^{\lambda_m} + \frac{C}{n}\sum_{m=1}^{n}\mu^H\left[\phi(1)\right]^{\lambda_m}\\ &\quad\le C\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right)\right)\right]^{\lambda_m} + \frac{C}{n}\mu^H \max\left[(\phi(1)^h, (\phi(1))^H\right]n.\end{aligned}$$

Hence $C_1^1(X,\Delta,\phi,\lambda,q) \subset C_1^\infty(X,\Delta,\phi,\lambda,q)$ and so the proof is complete.

The following examples state that the converse of Theorem 2.2 does not always hold.

EXAMPLE 2.3. Let $X = \mathbb{C}$ and $q(\xi) = |\xi|$, $\phi(\xi) = \xi$, $\lambda_m = 1\ \forall\ m \in \mathbb{N}$. Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right| = 0\ for\ \xi = (\xi_m) = (m)\ with\ l = -1.$$

But $\lim\limits_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right| = 1 \ne 0$.
Thus $(\xi_m) \notin C_1^0(X,\Delta,\phi,\lambda,q)$ but $(\xi_m) \in C_1^1(X,\Delta,\phi,\lambda,q)$.

EXAMPLE 2.4. [7] Consider $X = \mathbb{C}$, $\lambda_m = 1$, $\phi(\xi) = \xi$, $q(\xi) = |\xi|$ and the sequence (η_m) as $\eta_1 = 1$, $\eta_2 = 0$ and

$$\eta_m = \begin{cases} 1 & if\ \ 2^{i-1} < m \le 3(2^{i-2}),\\ 0 & \text{otherwise}\end{cases}\qquad i = 1,2,3,...$$

We can find a sequence (ξ_m) such that $\eta_m = \frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i$. Here $(\eta_m) \notin C_1$ and $(\eta_m) \in \ell_\infty$, i.e., $\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right) \notin C_1$ and $\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right) \in \ell_\infty$.
This implies $\lim\limits_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i)$ does not exist but $\left(\frac{1}{n}\sum_{m=1}^{n}(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i)\right) \in \ell_\infty$,
i.e., $\lim\limits_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i|$ does not exist but $\sup_n\frac{1}{n}\sum_{m=1}^{n}|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i| < \infty$
This gives $(\xi_m) \in C_1^\infty(X,\Delta,\phi,\lambda,q)$ but $(\xi_m) \notin C_1^1(X,\Delta,\phi,\lambda,q)$.

□

THEOREM 2.5. *$C_1^0(X,\Delta,\phi,\lambda,q)$ is a paranormed space, paranormed by*

$$g_\Delta(\xi)=\sup_n\left(\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right)\right]^{\lambda_m}\right)^{\dfrac{1}{P}}\quad \text{where } P=\max\{1,\sup_m\lambda_m\}.$$

Proof. (i) $g_\Delta(-\xi)=g_\Delta(\xi)$

(ii) For $\xi=\theta,\ g_\Delta(\xi)=0.$

(iii) Since $\lambda_m\le P$ and $P\ge 1$, so for each n, we get as an application of Minkowski inequality,

$$\begin{aligned}
g_\Delta(\xi+\eta)&=\left(\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta(\xi_i+\eta_i\right)\right)\right]^{\lambda_m}\right)^{\dfrac{1}{P}}\\
&\le\left(\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right)+\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\eta_i\right)\right)\right]^{\lambda_m}\right)^{\dfrac{1}{P}}\\
&\le\left(\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right)\right]^{\lambda_m}\right)^{\dfrac{1}{P}}+\left(\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\eta_i\right)\right)\right]^{\lambda_m}\right)^{\dfrac{1}{P}}\\
&\le g_\Delta(\xi)+g_\Delta(\eta).
\end{aligned}$$

(iv) In order to have continuity of multiplication, let us take any $\alpha\in\mathbb{C}$. Now $g_\Delta(\alpha\xi)\le\mu^{\frac{H}{P}}g_\Delta(x)$, where μ is a positive integer such that $|\alpha|\le\mu$. Hence $\alpha\to 0$, $\xi\to\theta$ imply $g_\Delta(\alpha\xi)\to 0$ and also $\xi\to\theta$, α fixed imply $g_\Delta(\alpha\xi)\to 0$. We now show that $\alpha\to 0$, ξ fixed imply $g_\Delta(\alpha\xi)\to 0$. Since $\xi=(\xi_m)$ is fixed in $C_1^0(X,\Delta,\phi,\lambda,q)$, so $\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\left[\phi(q(\frac{1}{m}\sum_{i=1}^{m}\alpha\Delta\xi_i))\right]^{\lambda_m}=0$. For $\varepsilon>0$, there exist $n_0\in\mathbb{N}$ such that

$$\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\alpha\Delta\xi_i\right)\right)\right]^{\lambda_m}<\varepsilon,\ \text{for all } n\ge n_0. \tag{1}$$

Now for $n\le n_0$, and using continuity of ϕ in $[0,\infty)$, we have

$$\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\alpha\Delta\xi_i\right)\right)\right]^{\lambda_m}<\varepsilon. \tag{2}$$

From (1) and (2), we have $g_\Delta(\alpha\xi)\to 0$ as $\alpha\to 0$.

Consequently, the proof is complete. □

THEOREM 2.6. *Let ϕ,ϕ_1 and ϕ_2 are modulus functions and $0<h=\inf_m\lambda_m\le\lambda_m\le\sup_m\lambda_m=H<\infty$. Then*

(i) $C_1^\Theta(X,\Delta,\phi_1,\lambda,q)\subseteq C_1^\Theta(X,\Delta,\phi\circ\phi_1,\lambda,q)$,

(ii) $C_1^\Theta(X,\Delta,\phi_1,\lambda,q)\cap C_1^\Theta(X,\Delta,\phi_2,\lambda,q)\subseteq C_1^\Theta(X,\Delta,\phi_1+\phi_2,\lambda,q)$.

Proof. It is sufficient to prove the the theorem for $\Theta=0$ as other cases may be prove on the similar ways.

(i) Let $(\xi_m) \in C_1^0(X, \Delta, \phi, \lambda, q)$. As ϕ is continuous at $t = 0$, so for $\varepsilon > 0 \; \exists \; \delta \; (0 < \delta < 1)$ such that $\phi(t) < \varepsilon$, whenever $0 \leq t < \delta$. Put $\eta_m = \phi_1(q(\frac{1}{m}\sum_{i=1}^{m} \Delta\xi_i))$. Then

$$\sum_{m=1}^{n}[\phi(\eta_m)]^{\lambda_m} = \sum_{\eta_m \leq \delta}[\phi(\eta_m)]^{\lambda_m} + \sum_{\eta_m > \delta}[\phi(\eta_m)]^{\lambda_m}.$$

Using the continuity of ϕ,

$$\sum_{\eta_m \leq \delta}[\phi(\eta_m)]^{\lambda_m} < n \; \max(\varepsilon^h, \varepsilon^H). \tag{3}$$

As $\eta_m > \delta$ and $0 < \delta < 1$, so $\eta_m < \dfrac{\eta_m}{\delta} \leq 1 + \left[\dfrac{\eta_m}{\delta}\right]$. Now

$$\begin{aligned}\phi(\eta_m) &\leq \phi\left(1 + \left[\frac{\eta_m}{\delta}\right]\right)\\ &\leq \left(1 + \left[\frac{\eta_m}{\delta}\right]\right)\phi(1)\\ &\leq 2\frac{\eta_m}{\delta}\phi(1).\end{aligned}$$

Hence

$$\frac{1}{n}\sum_{\eta_m > \delta}[\phi(\eta_m)]^{\lambda_m} \leq \max\left\{1, \left(\frac{2\phi(1)}{\delta}\right)^H\right\}.\frac{1}{n}\sum_{m=1}^{n}(\eta_m)^{\lambda_m} \tag{4}$$

By (3),(4) we have $C_1^0(X, \Delta, \phi, \lambda, q) \subseteq C_1^0(X, \Delta, \phi \circ \phi_1, \lambda, q)$.

(ii) Let $(\xi_m) \in C_1^0(X, \Delta, \phi_1, \lambda, q) \cap C_1^0(X, \Delta, \phi_2, \lambda, q)$. Now

$$\begin{aligned}&\frac{1}{n}\sum_{m=1}^{n}\left[(\phi_1+\phi_2)\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right)\right]^{\lambda_m}\\ &\quad= \frac{1}{n}\sum_{m=1}^{n}\left[\phi_1\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right) + \phi_2\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right)\right]^{\lambda_m}\\ &\quad\leq C.\frac{1}{n}\sum_{m=1}^{n}\left[\phi_1\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right)\right]^{\lambda_m} + C.\frac{1}{n}\sum_{m=1}^{n}\left[\phi_2\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right)\right)\right]^{\lambda_m}\\ &\quad\longrightarrow 0 \; as \; n \to \infty.\end{aligned}$$

Hence $(\xi_m) \in C_1^0(X, \Delta, \phi_1 + \phi_2, \lambda, q)$.
Thus $C_1^0(X, \Delta, \phi_1, \lambda, q) \cap C_1^0(X, \Delta, \phi_2, \lambda, q) \subseteq C_1^0(X, \Delta, \phi_1 + \phi_2, \lambda, q)$.

□

Corollary 2.7. *For modulus function ϕ, $C_1^\Theta(X, \Delta, \lambda, q) \subseteq C_1^\Theta(X, \Delta, \phi, \lambda, q)$.*

Proof. Taking $\phi_1(\xi) = \xi$, in the part(i) of above theorem, we get the result. □

Remark 2.8. For semi-norms q_1, q_2 on X, we have $C_1^\Theta(X, \Delta, \phi, \lambda, q_1) \cap C_1^\Theta(X, \Delta, \phi, \nu, q_2) \neq \emptyset$, where $\lambda = (\lambda_m)$ and $\nu = (\nu_m)$ are sequences of positive real numbers, as $\{\theta\} \subseteq C_1^\Theta(X, \Delta, \phi, \lambda, q_1) \cap C_1^\Theta(X, \Delta, \phi, \nu, q_2)$

Theorem 2.9. *For q_1, q_2 are semi-norms on X,*

$$C_1^\Theta(X, \Delta, \phi, \lambda, q_1) \cap C_1^\Theta(X, \Delta, \phi, \lambda, q_2) \subseteq C_1^\Theta(X, \Delta, \phi, \lambda, q_1 + q_2).$$

Proof. It is sufficient to prove the result for $\Theta = 0$, as other cases may be proved on similar line.
Let $(\xi_m) \in C_1^0(X, \Delta, \phi, \lambda, q_1) \cap C_1^0(X, \Delta, \phi, \lambda, q_2)$. Then $(\xi_m) \in C_1^0(X, \Delta, \phi, \lambda, q_1)$ and $(\xi_m) \in C_1^0(X, \Delta, \phi, \lambda, q_2)$. This implies
$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \left[\phi(q_1(\frac{1}{m}\sum_{i=1}^{m} \Delta\xi_i))\right]^{\lambda_m} = 0$ and $\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \left[\phi(q_2(\frac{1}{m}\sum_{i=1}^{m} \Delta\xi_i))\right]^{\lambda_m} = 0$.
Now

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \left[\phi\left((q_1+q_2)\left(\frac{1}{m}\sum_{i=1}^{m} \Delta\xi_i\right)\right)\right]^{\lambda_m} \\
&\quad \le \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \left[\phi\left(q_1\left(\frac{1}{m}\sum_{i=1}^{m} \Delta\xi_i\right)\right) + \phi\left(q_2\left(\frac{1}{m}\sum_{i=1}^{m} \Delta\xi_i\right)\right)\right]^{\lambda_m} \\
&\quad \le C\frac{1}{n} \sum_{m=1}^{n} \left[\phi\left(q_1\left(\frac{1}{m}\sum_{i=1}^{m} \Delta\xi_i\right)\right)\right]^{\lambda_m} + C\frac{1}{n} \sum_{m=1}^{n} \left[\phi\left(q_2\left(\frac{1}{m}\sum_{i=1}^{m} \Delta\xi_i\right)\right)\right]^{\lambda_m} \\
&\quad \longrightarrow 0 \ as \ n \to \infty.
\end{aligned}$$

Hence $\xi = (\xi_m) \in C_1^\Theta(X, \Delta, \phi, \lambda, q_1 + q_2)$. Consequently, the proof is complete. □

THEOREM 2.10. *For q_2 rough than q_1, $C_1^\Theta(X, \Delta, \phi, \lambda, q_1) \subseteq C_1^\Theta(X, \Delta, \phi, \lambda, q_2)$.*

Proof. As q_2 is rough than q_1, so there exists a positive integer μ such that $q_2(x) \le \mu q_1(x)$. We prove the result for $\Theta = 1$, as other cases may be proved on similar ways.
Let $\xi = (\xi_m) \in C_1^1(X, \Delta, \phi, \lambda, q_1)$. Now

$$\begin{aligned}
\frac{1}{n} \sum_{m=1}^{n} \left[\phi\left(q_2\left(\frac{1}{m}\sum_{i=1}^{m} \Delta\xi_i - l\right)\right)\right]^{\lambda_m} &\le \frac{1}{n} \sum_{m=1}^{n} \left[\phi\left(\mu q_1\left(\frac{1}{m}\sum_{i=1}^{m} \Delta\xi_i - l\right)\right)\right]^{\lambda_m} \\
&\le \frac{1}{n}\mu^H \sum_{m=1}^{n} \left[\phi\left(q_1\left(\frac{1}{m}\sum_{i=1}^{m} \Delta\xi_i - l\right)\right)\right]^{\lambda_m} \\
&\longrightarrow 0 \ as \ n \to \infty.
\end{aligned}$$

Thus $\xi \in C_1^1(X, \Delta, \phi, \lambda, q_2)$. Thus $C_1^1(X, \Delta, \phi, \lambda, q_1) \subseteq C_1^1(X, \Delta, \phi, \lambda, q_2)$. □

THEOREM 2.11. *If q_1 and q_2 are equivalent semi-norms then*

$$C_1^1(X, \Delta, \phi, \lambda, q_1) = C_1^1(X, \Delta, \phi, \lambda, q_2).$$

Proof. If $q_1 \equiv q_2$, then for every $u \in X$ that satisfies $q_2(u) > 0$, there exists positive integers T_1 and T_2 such that $T_1 \le \dfrac{q_1(u)}{q_2(u)} \le T_2$. Let $\xi = (\xi_i) \in C_1^1(X, \Delta, \phi, \lambda, q_1)$.

Consider

$$\begin{aligned}
&\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q_2\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i-l\right)\right)\right]^{\lambda_m}\\
&\le \frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(\frac{1}{T_1}q_1\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i-l\right)\right)\right]^{\lambda_m}\\
&\le \frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(\left(1+\left[\frac{1}{T_1}\right]\right)q_1\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i-l\right)\right)\right]^{\lambda_m}\\
&\le \max\left(1,\left(1+\left[\frac{1}{T_1}\right]\right)^H\right)\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q_1\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i-l\right)\right)\right]^{\lambda_m}\\
&\longrightarrow 0 \; as \; n\to\infty.
\end{aligned}$$

So $C_1^1(X,\Delta,\phi,\lambda,q_1)\subseteq C_1^1(X,\Delta,\phi,\lambda,q_2)$.
Now let $\xi\in C_1^1(X,\Delta,\phi,\lambda,q_2)$. Consider

$$\begin{aligned}
&\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q_1\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i-l\right)\right)\right]^{\lambda_m}\\
&\le \frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(T_2q_2\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i-l\right)\right)\right]^{\lambda_m}\\
&\le \max\left(1,(1+[T_2])^H\right)\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q_2\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i-l\right)\right)\right]^{\lambda_m}\\
&\longrightarrow 0 \; as \; n\to\infty.
\end{aligned}$$

Hence $C_1^1(X,\Delta,\phi,\lambda,q_2)\subseteq C_1^1(X,\Delta,\phi,\lambda,q_1)$.
Thus $C_1^1(X,\Delta,\phi,\lambda,q_1)=C_1^1(X,\Delta,\phi,\lambda,q_2)$. □

DEFINITION 2.12. A sequence (ξ_m) is termed as $C_1(\Delta,q)-$statistically convergent to $l\in X$ if for given $\varepsilon>0$,

$$\lim_{n\to\infty}\frac{1}{n}\text{card}\left(\left\{m\le n: q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i-l\right)\ge\varepsilon\right\}\right)=0,$$

and we write $\xi_m\to l(C_1(\Delta,q))$. By $SC_1(\Delta,q)$, we refer the class of all $C_1(\Delta,q)-$statistically convergent sequences.

THEOREM 2.13. *For given modulus function* ϕ, $C_1^1(X,\Delta,\phi,\lambda,q)\subset SC_1(\Delta,q)$.

Proof. Let $\xi=(\xi_i)\in C_1^1(X,\Delta,\phi,\lambda,q)$. Then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\left[\phi(q(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i-l))\right]^{\lambda_m}=0,\text{ for some } l\in X.$$

Let $\eta_m = q(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l)$. Now for $\varepsilon > 0$.

$$\begin{aligned}\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right)\right)\right]^{\lambda_m} &= \frac{1}{n}\sum_{m=1}^{n}[\phi(\eta_m)]^{\lambda_m}\\ &= \frac{1}{n}\sum_{\substack{1\le m\le n\\ \eta_m\ge\epsilon}}[\phi(\eta_m)]^{\lambda_m} + \frac{1}{n}\sum_{\substack{1\le m\le n\\ \eta_m<\varepsilon}}[\phi(\eta_m)]^{\lambda_m}\\ &\ge \frac{1}{n}\sum_{\substack{1\le m\le n\\ \eta_m\ge\epsilon}}[\phi(\eta_m)]^{\lambda_m}\\ &\ge \frac{1}{n}\sum_{\substack{1\le m\le n\\ \eta_m\ge\epsilon}}[\phi(\varepsilon)]^{\lambda_m}\\ &\ge \min\left((\phi(\varepsilon))^H, (\phi(\varepsilon))^h\right).\frac{1}{n}\mathrm{card}(\{m\le n : \eta_m\ge\varepsilon\}).\end{aligned}$$

Hence $\xi \in SC_1(\Delta, q)$. Thus $C_1^1(X, \Delta, \phi, \lambda, q) \subset SC_1(\Delta, q)$. □

THEOREM 2.14. *For bounded modulus function ϕ, $SC_1(\Delta, q) \subseteq C_1^1(X, \Delta, \phi, \lambda, q)$.*

Proof. Let $\varepsilon > 0$ be given. Since ϕ is bounded, so there exists an integer μ such that $\phi(\xi) < \mu$, for all $\xi \ge 0$. Then (as in Theorem 2.13) we have,

$$\begin{aligned}\frac{1}{n}\sum_{m=1}^{n}\left[\phi\left(q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right)\right)\right]^{\lambda_m} &= \frac{1}{n}\sum_{m=1}^{n}[\phi(\eta_m)]^{\lambda_m}\\ &= \frac{1}{n}\sum_{\substack{1\le m\le n\\ \eta_m\ge\epsilon}}[\phi(\eta_m)]^{\lambda_m} + \frac{1}{n}\sum_{\substack{1\le m\le n\\ \eta_m<\varepsilon}}[\phi(\eta_m)]^{\lambda_m}\\ &= \frac{1}{n}\sum_{\substack{1\le m\le n\\ \eta_m\ge\epsilon}}[\mu]^{\lambda_m} + \frac{1}{n}\sum_{\substack{1\le m\le n\\ \eta_m<\varepsilon}}\max\left((\phi(\varepsilon))^h, (\phi(\varepsilon))^H\right)\\ &\le \mu^H.\frac{1}{n}\mathrm{card}(\{m\le n : \eta_m\ge\varepsilon\})\\ &\qquad + \max\left((\phi(\varepsilon))^h, (\phi(\varepsilon))^H\right).\end{aligned}$$

Hence $\xi \in C_1^1(X, \Delta, \phi, \lambda, q)$. Thus $SC_1(\Delta, q) \subseteq C_1^1(X, \Delta, \phi, \lambda, q)$. □

THEOREM 2.15. *$SC_1(\Delta, q) = C_1^1(X, \Delta, \phi, \lambda, q)$ iff ϕ is bounded.*

Proof. Let ϕ be a bounded modulus function. Then by Theorem 2.13 and Theorem 2.14,

$$SC_1(\Delta, q) = C_1^1(X, \Delta, \phi, \lambda, q).$$

Conversely, suppose that $SC_1(\Delta, q) = C_1^1(X, \Delta, \phi, \lambda, q)$ and let if possible ϕ is unbounded. Then there exists a sequence (z_n), $z_n > 0$ with $\phi(z_n) = n^2$, $n = 1, 2, 3, ...$ If we choose $\xi = (\xi_i)$ such that

$$\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i = \begin{cases} z_n & \text{if } m = n^2\\ 0 & \text{otherwise,}\end{cases}$$

then

$$\frac{1}{n}\text{card}\left(\left\{m \le n : q\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right) \ge \varepsilon\right\}\right) \le \frac{\sqrt{n}}{n} \longrightarrow 0 \; as \; n \to \infty.$$

Thus $\xi_m \to 0 \, SC_1(\Delta, q)$, but $\xi \notin C_1^1(\Delta, \phi, q)$ for $X = \mathbb{C}$ and $q(\xi) = |\xi|$

Indeed

$$\left(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right) = (z_1, 0, 0, z_2, 0, 0, 0, 0, z_3, ...).$$

Let $s_n = \frac{1}{n}\sum_{m=1}^{n}\phi\left(\left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right|\right)$. Then $\langle s_n\rangle$ has a subsequence $\langle s_{n^2}\rangle$ where

$$\begin{aligned} s_{n^2} &= \frac{1}{n^2}(1^2 + 2^2 + 3^2 + ... + n^2) \\ &= \frac{n(n+1)(2n+1)}{6n^2} \\ &\to \infty \; as \; n \to \infty. \end{aligned}$$

Therefore (s_n) is not convergent, i.e., $\frac{1}{n}\sum_{m=1}^{n}\phi\left(\left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i\right|\right)$ is not convergent as $n \to \infty$.

Thus a contradiction to $SC_1(\Delta, q) = C_1^1(X, \Delta, \phi, \lambda, q)$. Hence ϕ is bounded. □

Remark 2.16. If we take $X = \mathbb{C}$, $\lambda_m = 1$ for $m \in \mathbb{N}$ and $\phi(\xi) = \xi$ with $q(\xi) = |\xi|$, then we shall write $C_1^1(X, \Delta, \phi, \lambda, q)$ as $C_1^1(\Delta)$, i.e.,

$$C_1^1(\Delta) = \left\{\xi = (\xi_m) : \lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right| = 0 \text{ for some } l \in \mathbb{C}\right\}$$

and for the space $SC_1(\Delta, q)$, we write $SC_1(\Delta)$, i.e.,

$$SC_1(\Delta) = \left\{(\xi_m) \in s(X) : \lim_{n\to\infty}\frac{1}{n}\text{card}\left(\left\{m \le n : \left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right| \ge \epsilon\right\}\right) = 0 \text{ for some } l \in \mathbb{C}\right\}$$

Theorem 2.17. *(i) If $(\xi_m) \in C_1^1(\Delta)$ then $(\xi_m) \in SC_1(\Delta)$.*

(ii) If $(\xi_m) \in \ell_\infty(\Delta)$ and $(\xi_m) \in SC_1(\Delta)$ then $(\xi_m) \in C_1^1(\Delta)$.

(iii) $SC_1(\Delta) \cap \ell_\infty(\Delta) = C_1^1(\Delta) \cap \ell_\infty(\Delta)$.

Proof. (i) Let $\varepsilon > 0$ and $(\xi_m) \in C_1^1(\Delta)$. Then $\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right| = 0$ for some $l \in X$. Now

$$\begin{aligned} \sum_{m=1}^{n}\left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right| &\ge \sum_{\substack{m\le n \\ \left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right| \ge \epsilon}}\left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right| \\ &\ge \varepsilon.\text{card}\left(\left\{m \le n : \left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right| \ge \varepsilon\right\}\right). \end{aligned}$$

Therefore $\lim_{n\to\infty}\frac{1}{n}\text{card}\left(\left\{m \le n : \left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right| \ge \varepsilon\right\}\right) = 0$ that is $(\xi_m) \in SC_1(\Delta)$.

(ii) Let $(\xi_m) \in \ell_\infty(\Delta)$ and $(\xi_m) \in SC_1(\Delta)$. Then $(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i)$ is statistically convergent and so $(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i)$ is convergent (because $(\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i) \in \ell_\infty$).

Say $\left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right| \leq M \;\forall\; m$. We have

$$\frac{1}{n}\sum_{m=1}^{n}\left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right| = \frac{1}{n}\sum_{\substack{m\leq n \\ \left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right|\geq\epsilon}}\left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right| + \frac{1}{n}\sum_{\substack{m\leq n \\ \left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right|<\varepsilon}}\left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right|$$

$$\leq \frac{1}{n}M.\text{card}\left(\left\{m\leq n : \left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right| \geq \varepsilon\right\}\right) + \frac{1}{n}\varepsilon.n$$

and hence $\lim_{n\to\infty}\frac{1}{n}\sum_{m=1}^{n}\left|\frac{1}{m}\sum_{i=1}^{m}\Delta\xi_i - l\right| = 0$. Thus $(\xi_m) \in C_1^1(\Delta)$.

(iii) By (i) and (ii), it is clear that $SC_1(\Delta)\cap\ell_\infty(\Delta) = C_1^1(\Delta)\cap\ell_\infty(\Delta)$.

□

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