Korean J. Math. **32** (2024), No. 3, pp. 407–424
https://dx.doi.org/10.11568/kjm.2024.32.3.407

INVARIANT AND SCREEN SEMI-INVARIANT LIGHTLIKE SUBMANIFOLDS OF A METALLIC SEMI-RIEMANNIAN MANIFOLD WITH A QUARTER SYMMETRIC NON-METRIC CONNECTION

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Abstract. The present work aims to introduce the geometry of invariant and screen semi-invariant lightlike submanifolds of a metallic semi-Riemannian manifold equipped with a quarter symmetric non-metric connection. The study establishes the characterization of integrability and parallelism of the distributions inherent in these submanifolds. Additionally, the conditions for distributions defining totally geodesic foliations on the invariant and screen semi-invariant lightlike submanifolds of metallic semi-Riemannian manifold are specified.

1. Introduction

The study of metallic Riemannian manifolds, which have emerged from the metallic numbers of the metallic means family, is effective in the realm of differential geometry. The metallic numbers exhibit important mathematical properties that constitute a bridge between mathematics and design. The "Metallic Means Family" introduced by [9] comprises of a range of means termed as the golden mean, silver mean, bronze mean, copper mean and others, which are defined in terms of values of metallic numbers. These means have been extensively explored for their mathematical properties and applications across various fields of research by [9], [10], [11], [12], [13], [14]. The polynomial structures on manifold were introduced by [19], [20] and a specific class known as metallic structure was introduced on a Riemannian manifold by [8].

The lightlike geometry of submanifolds, initiated by [15] is extremely relevant in different branches of mathematics equipped with degenerate metric and has led to the development of remarkable results in the fields where the non degenerate metric is not applicable. Since the tools used to investigate the geometry of submanifolds in a Riemannian manifold are not favourable in semi-Riemannian cases, so the lightlike (degenerate) geometry plays a pivotal role in the study of such structures. Within this framework, the invariant submanifolds for golden semi-Riemannnian manifolds

Received January 11, 2024. Revised April 8, 2024. Accepted June 26, 2024.
2010 Mathematics Subject Classification: 53C12, 53C22, 53C15, 53C40, 53C50.
Key words and phrases: metallic semi-Riemannian manifold, quarter symmetric non-metric connection, invariant lightlike submanifolds, screen semi-invariant lightlike submanifolds.
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have been investigated by [16] and for metallic semi-Riemannian manifolds with metric connection by [24]. One more interesting class of submanifolds, namely screen semi-invariant lightlike submanifold was established by [22] for the semi-Riemannian product manifold. [25] formulated the geometry of screen semi-invariant lightlike submanifolds for golden semi-Riemannian manifold which were further worked upon by many researchers.

A linear connection $\bar{\nabla}$ on a Riemannian manifold $(\bar{M}, \bar{g})$, introduced by [18], is said to be a quarter symmetric connection if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(X, Y) = \pi(Y)\phi(X) - \pi(X)\phi(Y),$$

where ϕ is a $(1,1)$-tensor field and π is a 1-form associated with a smooth unit vector ξ, called the characteristic vector field, by $\pi(X) = \bar{g}(X, \xi)$. If the linear connection $\bar{\nabla}$ is not a metric connection, then $\bar{\nabla}$ is called a quarter symmetric non-metric connection. Various researchers [2], [4], [23], [26], have advanced the geometric analysis of submanifolds for semi-Riemannian manifolds equipped with a quarter symmetric non-metric connection. In this context, the screen semi-invariant lightlike submanifolds for product semi-Riemannian manifold were introduced and studied extensively by [3].

The present research work propose to study the invariant and screen semi-invariant lightlike submanifolds of a metallic semi-Riemannian manifold endowed with a quarter symmetric non-metric connection. The integrability and parallelism of distributions have been analysed. The totally geodesic foliations on the invariant and screen semi-invariant submanifolds have also been explored. Examples elaborating the structure of the invariant and screen semi-invariant lightlike submanifolds have been presented.

2. Preliminaries

Let $(\bar{M}, \bar{g})$ be an $(m+n)$-dimensional semi-Riemannian manifold with semi-Riemannian metric $\bar{g}$ of constant index q such that $m, n \geq 1$, $1 \leq q \leq m + n - 1$.

Let (M, g) be a m-dimensional lightlike submanifold of $\bar{M}$. In this case, there exists a smooth distribution $RadTM$ on M of rank $r > 0$, known as radical distribution on M such that $RadTM_p = TM_p \cap TM_p^{\perp}, \forall\ p \in M$, where TM_p and $TM_p^{\perp}$ are degenerate orthogonal spaces but not complementary. Then, M is called an r-lightlike submanifold of $\bar{M}$.

Now, consider $S(TM)$, known as screen distribution, as a complementary distribution of radical distribution in TM i.e.,

$$TM = RadTM \perp S(TM),$$

and $S(TM^{\perp})$, called screen transversal vector bundle, as a complementary vector subbundle to $RadTM$ in $TM^{\perp}$ i.e.,

$$TM^{\perp} = RadTM \perp S(TM^{\perp}),$$

As $S(TM)$ is non degenerate vector subbundle of $T\bar{M}|_M$, we have

$$T\bar{M}|_M = S(TM) \perp S(TM)^{\perp},$$

where $S(TM)^{\perp}$ is the complementary orthogonal vector subbundle of $S(TM)$ in $T\bar{M}|_M$.

Let $tr(TM)$ and $ltr(TM)$ be complementary vector bundles to TM in $T\bar{M}|_M$ and to $RadTM$ in $S(TM^{\perp})^{\perp}$. Then we have

$$tr(TM) = ltr(TM) \perp S(TM^{\perp}),$$

$$T\bar{M}|_M = TM \oplus tr(TM)$$

$$= (RadTM \oplus ltr(TM)) \perp S(TM) \perp S(TM^{\perp}).$$

THEOREM 2.1. [15] *Let $(M, g, S(TM), S(TM^{\perp}))$ be an r-lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then there exists a complementary vector bundle $ltr(TM)$ called a lightlike transversal bundle of $Rad(TM)$ in $S(TM^{\perp})^{\perp}$ and basis of $\Gamma(ltr(TM)|_U)$ consisting of smooth sections $\{N_1, \cdots, N_r\}$ $S(TM^{\perp})^{\perp}|_U$ such that*

$$\bar{g}(N_i, \xi_j) = \delta_{ij}, \quad \bar{g}(N_i, N_j) = 0, \quad i, j = 0, 1, \cdots, r$$

where $\{\xi_1, \cdots, \xi_r\}$ is a lightlike basis of $\Gamma(RadTM)|_U$.

Let $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. We have, from the above mentioned theory, the Gauss and Weingarten formulae as:

$$\bar{\nabla}_U V = \nabla_U V + h(U, V) \quad \forall\, U, V \in \Gamma(TM)$$

and

$$\bar{\nabla}_U N = -A_N U + \nabla^t_U N \quad \forall\, U \in \Gamma(TM), N \in \Gamma(tr(TM))$$

where $\{\nabla_U V, -A_N U\}$ and $\{h(U, V), \nabla^t_U N\}$ belong to $\Gamma(TM)$ and $\Gamma(tr(TM))$ respectively. ∇ and ∇^t are linear connections on M and on the vector bundle $tr(TM)$.

Considering the projection morphisms L and S of $tr(TM)$ on $ltr(TM)$ and on $S(TM^{\perp})$, we have

$$\bar{\nabla}_U V = \nabla_U V + h^l(U, V) + h^s(U, V), \tag{1}$$

$$\bar{\nabla}_U N = -A_N U + \nabla^l_U N + D^s(U, N), \tag{2}$$

$$\bar{\nabla}_U W = -A_W U + \nabla^s_U W + D^l(U, W), \tag{3}$$

where $h^l(U, V) = Lh(U, V), h^s(U, V) = Sh(U, V), \{\nabla_U V, A_N U, A_W U\} \in \Gamma(TM)$, $\{\nabla^l_U N, D^l(U, W)\} \in \Gamma(ltr(TM))$ and $\{\nabla^s_U W, D^s(U, N)\} \in \Gamma(S(TM^{\perp}))$.Then considering $(1) - (3)$ and the fact that $\bar{\nabla}$ is a metric connection, the following holds:

$$\bar{g}(h^s(U, V), W) + \bar{g}(V, D^l(U, W)) = \bar{g}(A_W U, V),$$

$$\bar{g}(D^s(U, N), W) = \bar{g}(A_W U, N).$$

Let J be a projection of TM on $S(TM)$.Then we have

$$\nabla_U JV = \nabla^*_U JV + h^*(U, JV),$$

$$\nabla_U E = -A^*_E U + \nabla^{*t}_U E,$$

for any $U, V \in \Gamma(TM)$ and $E \in \Gamma(Rad(TM))$, where$\{\nabla^*_U JV, A^*_E U\}$ and $\{h^*(U, JV), \nabla^{*t}_U E\}$ belong to $\Gamma(S(TM))$ and $\Gamma(Rad(TM))$ respectively.

Using the above equations, we obtain

$$\bar{g}(h^l(U, JV), E) = g(A^*_E U, JV),$$

$$\bar{g}(h^*(U, JV), N) = g(A_N U, JV),$$

$$\bar{g}(h^l(U, E), E) = 0, \quad A^*_E E = 0.$$

In general, ∇ on M is not metric connection. Since $\bar{\nabla}$ is a metric connection, it follows from (1) that

$$(\nabla_U g)(V, Z) = \bar{g}(h^l(U, V), Z) + \bar{g}(h^l(U, Z), V),$$

for any $U, V, Z \in \Gamma(TM)$. Here ∇^* is a metric connection on $S(TM)$.

2.1. Metallic semi-Riemannian manifold. Some polynomial structures naturally arise as C^∞ tensor fields $\bar{J}$ of type $(1,1)$ which are roots of the algebraic equation

$$Q(\bar{J}) := \bar{J}^n + a_n \bar{J}^{n-1} + ... + a_2 \bar{J} + a_1 I_{\mathfrak{X}(\bar{M})} = 0,$$

where $I_{\mathfrak{X}(\bar{M})}$ is the identity map on the Lie algebra of vector fields on $\bar{M}$. In particular, if the structure polynomial is $Q(\bar{J}) = \bar{J}^2 - p\bar{J} - qI_{\mathfrak{X}(\bar{M})}$, with p and q positive integers, its solution $\bar{J}$ is called a metallic structure. For different values of p and q, the (p, q) metallic number introduced by [11] is the positive root of the quadratic equation $x^2 - px - q = 0$, namely $\sigma_{p,q} = \dfrac{p + \sqrt{p^2 + 4q}}{2}$ and is called the metallic mean.

A polynomial structure on a semi-Riemannian manifold $\bar{M}$ is known metallic if it is determined by $\bar{J}$ such that

$$\bar{J}^2 = p\bar{J} + qI, \tag{4}$$

If a semi-Riemannian metric $\bar{g}$ satisfies the equation

$$\bar{g}(U, \bar{J}V) = \bar{g}(\bar{J}U, V), \quad \forall \quad U, V \in \Gamma(T\bar{M}) \tag{5}$$

which yields

$$\bar{g}(\bar{J}U, \bar{J}V) = p\bar{g}(U, \bar{J}V) + q\bar{g}(U, V), \quad \forall \quad U, V \in \Gamma(T\bar{M}) \tag{6}$$

, then $\bar{g}$ is called $\bar{J}$-compatible.

Definition 2.2. [1] A semi-Riemannian manifold $(\bar{M}, \bar{g})$ equipped with $\bar{J}$ such that the semi-Riemannian metric $\bar{g}$ is $\bar{J}$-compatible, is called metallic semi-Riemannian manifold and $(\bar{g}, \bar{J})$ is called metallic structure on $\bar{M}$.

Let $(M, g, S(TM), S(TM^\perp))$ be a lightlike submanifold of a metallic semi-Riemannian manifold $(\bar{M}, \bar{g}, \bar{J})$. For each U tangent to M, $\bar{J}U$ can be written as follows:

$$\bar{J}U = fU + wU = fU + w_l U + w_s U, \tag{7}$$

where fU and wU are the tangential and the transversal parts of $\bar{J}U$; w_l and w_s are projections on $ltr(TM)$ and $S(TM^\perp)$ respectively. In addition, for any $V \in \Gamma(tr(TM))$, $\bar{J}V$ can be written as

$$\bar{J}V = BV + CV, \tag{8}$$

where BV and CV are the tangential and the transversal parts of $\bar{J}V$.

2.2. Quarter symmetric non-metric connection. As per [2], for a Levi-Civita connection $\bar{D}$ on the metallic semi-Riemannian manifold $\bar{M}$, by setting

$$\bar{D}_U V = \bar{\nabla}_U V + \pi(V)\bar{J}U, \tag{9}$$

for any $U, V \in \Gamma(T\bar{M})$, we see that $\bar{D}$ is linear connection on $\bar{M}$, where π is a 1-form on $\bar{M}$ with η as associated vector field such that

$$\pi(U) = \bar{g}(U, \eta),$$

Let the torsion tensor of $\bar{D}$ on $\bar{M}$ be denoted by $\bar{T}$.

$$\bar{T}^{\bar{D}}(U, V) = \pi(V)\bar{J}U - \pi(U)\bar{J}V,$$

$$(\bar{D}_U\bar{g})(V,Z) = -\pi(V)\bar{g}(\bar{J}U,Z) - \pi(Z)\bar{g}(\bar{J}U,V),$$

and

$$\bar{D}_U\bar{J}V = \bar{J}\bar{D}_UV - p\pi(V)\bar{J}U - q\pi(V)U + \pi(\bar{J}V)\bar{J}U, \tag{10}$$

for any $U,V,Z \in \Gamma(TM)$. Thus $\bar{D}$ is a quarter-symmetric non-metric connection on $\bar{M}$.

Consider a lightlike submanifold $(M,g,S(TM),S(TM^{\perp}))$ of the metallic semi-Riemannian manifold $(\bar{M},\bar{g})$ with quarter symmetric non-metric connection $\bar{D}$. Then the Gauss and Weingarten formulae with respect to $\bar{D}$ are given by

$$\bar{D}_UV = D_UV + \bar{h^l}(U,V) + \bar{h^s}(U,V), \tag{11}$$

$$\bar{D}_UN = -\bar{A}_NU + \bar{\nabla}^l_UN + \bar{D^s}(U,N), \tag{12}$$

$$\bar{D}_UW = -\bar{A}_WU + \bar{\nabla}^s_UW + \bar{D^l}(U,W), \tag{13}$$

for any $U,V \in \Gamma(TM)$, $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, where $\{D_UV, \bar{A}_NU, \bar{A}_WU\} \in \Gamma(TM)$ and $\bar{\nabla}^l$ and $\bar{\nabla}^s$ are linear connections on $ltr(TM)$ and $S(TM^{\perp})$ respectively. Both $\bar{A}_N$ and $\bar{A}_W$ are linear operators on $\Gamma(TM)$. From (9),(11),(12),(13), we obtain

$$D_UV = \nabla_UV + \pi(V)fU,$$

$$\bar{h^l}(U,V) = h^l(U,V) + \pi(V)w_lU, \tag{14}$$

$$\bar{h^s}(U,V) = h^s(U,V) + \pi(V)w_sU, \tag{15}$$

$$\begin{aligned}
\bar{A}_NU &= A_NU - \pi(N)fU,\\
\bar{\nabla}^l_UN &= \nabla^l_UN + \pi(N)w_lU,\\
\bar{D^s}(U,N) &= D^s(U,N) + \pi(N)w_sU,\\
\bar{A}_WU &= A_WU - \pi(W)fU,\\
\bar{\nabla}^s_UW &= \nabla^s_UW + \pi(W)w_sU,\\
\bar{D^l}(U,W) &= D^l(U,W) + \pi(W)w_lU,
\end{aligned}$$

From (11) we get,

$$(D_Ug)(V,Z) = g(h(U,V),Z) + g(h(U,Z),V) - \pi(V)g(fU,Z) - \pi(Z)g(fU,V),$$

On the other hand, the torsion tensor of the induced connection D is

$$T^D(U,V) = \pi(V)fU - \pi(U)fV,$$

$$\begin{aligned}\bar{g}(\bar{h}^s(U,V),W) + \bar{g}(V,\bar{D}^l(U,W)) = {}& \bar{g}(\bar{A}_WU,V) + \pi(W)\bar{g}(fU,V) +\\ & \pi(V)\bar{g}(w_sU,W) + \pi(W)\bar{g}(V,w_lU),\end{aligned} \tag{16}$$

$$\bar{g}(\bar{D}^s(U,N),W) = \bar{g}(\bar{A}_WU,N) + \pi(W)\bar{g}(fU,N) + \pi(N)\bar{g}(w_sU,W), \tag{17}$$

PROPOSITION 2.3. *Let M be a lightlike submanifold of a metallic semi-Riemannian manifold $\bar{M}$ with a quarter symmetric non-metric connection $\bar{D}$. Then the induced connection D on the lightlike submanifold M is also a quarter symmetric non-metric connection.*

Let J be the projection of TM on $S(TM)$, then any $U \in \Gamma(TM)$, can be written as $U = JU + \Sigma_{i=1}^{r} \eta_i(U)\xi_i$,

$$\eta_i(U) = g(U, N_i), \tag{18}$$

where $\{\xi_i\}_{i=1}^{r}$ is a basis for $Rad(TM)$. Therefore,

$$D_U JV = D_U^* JV + \bar{h}^*(U, JV), \tag{19}$$

$$D_U \xi = -\bar{A}_\xi^* U + \bar{\nabla}_U^{*t} \xi, \tag{20}$$

For any $U, V \in \Gamma(TM)$, where $\{D_U^* JV, \bar{A}_\xi^* U\} \in \Gamma(S(TM))$ and $\{\bar{h}^*(U, JV), \bar{\nabla}_U^{*t} \xi\} \in \Gamma(RadTM)$. From (19 and (20), we obtain

$$D_U^* JV = \nabla_U^* JV + \pi(JV) JfU,$$

$$\bar{h}^*(U, JV) = h^*(U, JV) + \pi(JV)\Sigma_{i=1}^{r} \eta_i(fU)\xi_i, \tag{21}$$

and

$$\bar{A}_\xi^* U = A_\xi^* U - \pi(\xi) JfU, \tag{22}$$

$$\bar{\nabla}_U^{*t} \xi = \nabla_U^{*t} \xi + \pi(\xi)\eta(fU)\xi,$$

From (11),(14),(21) and (22),we have

$$g(\bar{h}^l(U, JV), \xi) = g(\bar{A}_\xi^* U, JV) + \pi(\xi) g(JfU, JV) + \pi(JV) g(w_l U, \xi),$$

$$g(\bar{h}^*(U, JV), N) = g(\bar{A}_N U, JV) + \pi(N) g(fU, JV) + \pi(JV)\eta(fU),$$

$$\bar{g}(\bar{h}^l(U, \xi), \xi) = \pi(\xi)\bar{g}(w_l U, \xi), \quad \bar{A}_\xi^* \xi = -\pi(\xi) f\xi.$$

3. Invariant lightlike submanifolds

This section presents an illustration that elucidates the configuration of an invariant lightlike submanifold of metallic semi-Riemannian manifold. Subsequently, it examines its geometry when the manifold is endowed with a quarter symmetric non-metric connection.

Definition 3.1. [24] Let M be a lightlike submanifold of a metallic semi-Riemannian manifold $\bar{M}$. If

$$\bar{J} Rad(TM) = Rad(TM), \quad \bar{J} S(TM) = S(TM), \tag{23}$$

then, M is said to be invariant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{M}$.

Proposition 3.2. [24] *Let M be an invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\bar{M}, \bar{J})$. Then, the distribution $ltr(TM)$ is invariant with respect to $\bar{J}$.*

Example 3.3. Consider a metallic semi-Riemannian manifold $\bar{M} = (R_1^5, \bar{g})$ of signature $(-,+,+,+,+)$ with respect to the basis $\{\partial y_1, \partial y_2, \partial y_3, \partial y_4, \partial y_5\}$, where the metallic structure $\bar{J}$ is defined by $\bar{J}(y_1, y_2, y_3, y_4, y_5) = (\omega y_1, (p-\omega) y_2, (p-\omega) y_3, \omega y_4, (p-\omega) y_5)$.

Let M be a submanifold of $(R_1^5, \bar{J}, \bar{g})$ given by

$$y_1 = 0, \quad y_2 = \frac{1}{2}(\sqrt{3} t_4 + t_2),$$

$$y_3 = \frac{1}{2}(-\sqrt{3}t_2 + t_4), \quad y_4 = t_1, \quad y_5 = t_4,$$

Then TM is spanned by $\{D_1, D_2, D_3\}$, where

$$D_1 = \frac{1}{2}(-\sqrt{3}\partial y_3 + \partial y_2),$$
$$D_2 = \partial y_4,$$
$$D_3 = \partial y_5 + \frac{1}{2}(\sqrt{3}\partial y_2 + \partial y_3),$$

Hence M is a 1-lightlike submanifold of R_1^5 with

$$Rad(TM) = Span\{D_3\} \quad and \quad S(TM) = Span\{D_1, D_2\},$$

Therefore

$$\bar{J}D_3 = (p-\omega)D_3 \in \Gamma(Rad(TM)), \quad \bar{J}D_1 = (p-\omega)D_1 \in \Gamma(S(TM)),$$
$$\bar{J}D_2 = \omega D_2 \in \Gamma(S(TM)),$$

which implies that $S(TM)$ and $Rad(TM)$ is invariant with respect to $\bar{J}$.

Also, for $N \in \Gamma(ltr(TM))$ and $W \in \Gamma(S(TM^{\perp}))$, where

$$N = \frac{1}{2}\{-\partial y_5 + \frac{1}{2}\partial y_3 + \frac{\sqrt{3}}{2}\partial y_2\}, \quad W = \partial y_1,$$

we derive that $\bar{J}N = (p-\omega)N \in \Gamma(ltr(TM))$ and $\bar{J}W = \omega W \in S(TM^{\perp})$. It follows that $ltr(TM)$ and $S(TM^{\perp})$ are invariant distributions with respect to $\bar{J}$. Thus, M is an invariant lightlike submanifold of $\bar{M}$.

We now discuss the integrability of radical distribution and screen distribution of invariant lightlike submanifold of a metallic semi-Riemannian manifold endowed with a quarter symmetric non-metric connection.

Definition 3.4. Let M be an invariant lightlike submanifold of metallic semi-Riemannian manifold $(\bar{M}, \bar{J}, \bar{g})$ with a quarter symmetric non-metric connection $\bar{D}$. Then, the distribution D is said to be integrable if and only if $[X, Y] \in \Gamma(D)$, for any $X, Y \in \Gamma(D)$.

Remark 3.5. The above definition implies that the distribution D is said to be integrable if and only if $g([X, Y], Z) = 0$, for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.

Theorem 3.6. *For an invariant lightlike submanifold M of a metallic semi-Riemannian manifold $\bar{M}$ with a quarter symmetric non-metric connection $\bar{D}$, the radical distribution is integrable if and only if*

$$\bar{A}^*_{\bar{J}U}V - p\bar{A}^*_U V = \bar{A}^*_{\bar{J}V}U - p\bar{A}^*_V U,$$

for $U, V \in \Gamma(Rad\, TM)$ and $Z \in \Gamma(S(TM))$.

Proof. Using the concept of Levi-Civita connection $\bar{\nabla}$ and quarter symmetric non-metric connection $\bar{D}$,

$g([U, V], Z) = \bar{g}(\bar{\nabla}_U V - \bar{\nabla}_V U, Z)$
$= \bar{g}(\bar{D}_U V - \pi(V)\bar{J}U - \bar{D}_V U + \pi(U)\bar{J}V, Z),$

Now, $Rad(TM)$ is integrable if and only if $g([U, V], Z) = 0$ for all $U, V \in \Gamma(Rad(TM))$ and $Z \in \Gamma(S(TM))$.

Using the integrability of $Rad(TM)$ along with equation (6), (10), (11), (20) and (23), we derive

$g([U,V],Z) = \frac{1}{q}[\bar{g}(\bar{D}_U\bar{J}V,\bar{J}Z) - p\bar{g}(\bar{D}_UV,\bar{J}Z) - \bar{g}(\bar{D}_V\bar{J}U,\bar{J}Z) + p\bar{g}(\bar{D}_VU,\bar{J}Z)]$
$= \frac{1}{q}[\bar{g}(-\bar{A}^*_{\bar{J}V}U + \bar{\nabla}^{*t}_U\bar{J}V,\bar{J}Z) - p\bar{g}(-\bar{A}^*_VU + \bar{\nabla}^{*t}_UV,\bar{J}Z) - \bar{g}(-\bar{A}^*_{\bar{J}U}V +$
$\bar{\nabla}^{*t}_V\bar{J}U,\bar{J}Z) + p\bar{g}(-\bar{A}^*_UV + \bar{\nabla}^{*t}_VU,\bar{J}Z)]$
$= \bar{g}(\bar{A}^*_{\bar{J}U}V - p\bar{A}^*_UV - \bar{A}^*_{\bar{J}V}U + p\bar{A}^*_VU,\bar{J}Z) = 0.$ □

Theorem 3.7. *If M be an invariant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{M}$ with a quarter symmetric non-metric connection $\bar{D}$, then the screen distribution is integrable if and only if*

$$\bar{h}^*(V,\bar{J}U) + p\bar{h}^*(U,V) = \bar{h}^*(U,\bar{J}V) + p\bar{h}^*(V,U),$$

for all $U,V \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$.

Proof. $S(TM)$ is integrable if and only if $g([V,U],N) = 0$, for all $U,V \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$.

$0 = g([V,U],N) = \bar{g}(\bar{D}_VU,N) - \bar{g}(\bar{D}_UV,N),$

Using equations (6), (10), (11), (19) and (23), we obtain

$0 = \frac{1}{q}[\bar{g}(\bar{D}_V\bar{J}U,\bar{J}N) + p\pi(U)\bar{g}(\bar{J}V,\bar{J}N) + q\pi(U)\bar{g}(V,\bar{J}N) - \pi(\bar{J}U)\bar{g}(\bar{J}V,\bar{J}N) -$
$p\bar{g}(\bar{D}_VU,\bar{J}N)] - \frac{1}{q}[\bar{g}(\bar{D}_U\bar{J}V,\bar{J}N) + p\pi(V)\bar{g}(\bar{J}U,\bar{J}N) + q\pi(V)\bar{g}(U,\bar{J}N) -$
$\pi(\bar{J}V)\bar{g}(\bar{J}U,\bar{J}N) - p\bar{g}(\bar{D}_UV,\bar{J}N)]$
$= \frac{1}{q}\bar{g}(D^*_V\bar{J}U,\bar{J}N) + \frac{1}{q}\bar{g}(\bar{h}^*(V,\bar{J}U),\bar{J}N) - \frac{p}{q}\bar{g}(D^*_VU,\bar{J}N) - \frac{p}{q}\bar{g}(\bar{h}^*(V,U),\bar{J}N) -$
$\frac{1}{q}\bar{g}(D^*_U\bar{J}V,\bar{J}N) - \frac{1}{q}\bar{g}(\bar{h}^*(U,\bar{J}V),\bar{J}N) + \frac{p}{q}\bar{g}(D^*_UV,\bar{J}N) + \frac{p}{q}\bar{g}(\bar{h}^*(U,V),\bar{J}N).$

Using concept of invariant lightlike submanifold, we obtain

$$\bar{h}^*(V,\bar{J}U) + p\bar{h}^*(U,V) = \bar{h}^*(U,\bar{J}V) + p\bar{h}^*(V,U),$$

□

The conditions for radical distribution and screen distribution to define totally geodesic foliatons on an invariant lightlike submanifold M of metallic semi-Riemannian manifold $\bar{M}$ with a quarter symmetric non-metric connection are discussed here on.

Definition 3.8. Let $(M,g,S(TM),S(TM^{\perp}))$ be a invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\bar{M},\bar{g},\bar{J})$ with a quarter symmetric non-metric connection $\bar{D}$. Then, the distribution D defines a totally geodesic foliation on M if $\bar{D}_UV \in \Gamma(D)$, for any $U,V \in \Gamma(D)$.

Remark 3.9. From the above definition, the distribution D defines a totally geodesic foliation on M if and only if $g(\bar{D}_UV,N) = 0$, for any $U,V \in \Gamma(D)$ and $N \in \Gamma(D^{\perp})$.

Theorem 3.10. *Let M be an invariant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{M}$ with a quarter symmetric non-metric connection $\bar{D}$. Then, the radical distribution defines a totally geodesic foliation on M if and only if*

$$\bar{A}^*_{\bar{J}V}U = -p\bar{A}^*_VU,$$

for all $U,V \in \Gamma(Rad(TM))$ and $Z \in \Gamma(S(TM))$.

Proof. From the concept of quarter symmetric non-metric connection and the metallic structure on $\bar{M}$, we have

$g(D_UV,N) = \frac{1}{q}[\bar{g}(\bar{J}\bar{D}_UV,\bar{J}N) - p\bar{g}(\bar{J}\bar{D}_UV,N)],$

Using equations (9), (10), (11) and (20), we get
$g(D_U V, N) = \frac{1}{q}\bar{g}(\bar{D}_U \bar{J}V, \bar{J}N) + \frac{p}{q}\pi(V)\bar{g}(\bar{J}U, \bar{J}N) + \pi(V)\bar{g}(U, \bar{J}N) -$
$\frac{1}{q}\pi(\bar{J}V)\bar{g}(\bar{J}U, \bar{J}N) - \frac{p}{q}\bar{g}(\bar{D}_U V, \bar{J}N)$
$= \frac{1}{q}[\bar{g}(D_U \bar{J}V, \bar{J}N) - p\bar{g}(D_U V, \bar{J}N)]$
$= \frac{1}{q}\bar{g}(-\bar{A}^*_{\bar{J}V}U, \bar{J}N) + \frac{1}{q}\bar{g}(\bar{\nabla}^{*t}_U \bar{J}V, \bar{J}N) + \frac{p}{q}\bar{g}(\bar{A}^*_V U, \bar{J}N) - \frac{p}{q}\bar{g}(\bar{\nabla}^{*t}_U V, \bar{J}N) = 0.$
Since the radical distribution that defines a totally geodesic foliation, we have

$$\bar{A}^*_{\bar{J}V}U = -p\bar{A}^*_V U,$$

□

THEOREM 3.11. *Let M be an invariant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{M}$ with a quarter symmetric non-metric connection $\bar{D}$. Then the screen distribution defines a totally geodesic foliation on M if and only if*

$$\bar{h}^*(U, \bar{J}V) = p\bar{h}^*(U, V),$$

for any $U, V \in \Gamma(S(TM))$ and $N \in \Gamma(ltr(TM))$.

Proof. $S(TM)$ defines a totally geodesic foliation on M if and only if $D_U V \in \Gamma(S(TM))$.

Following (6), (9), (10), (11), (19) and (23), we have
$g(D_U V, N) = \frac{1}{q}[\bar{g}(\bar{D}_U \bar{J}V + p\pi(V)\bar{J}U + q\pi(V)U - \pi(\bar{J}V)\bar{J}U, \bar{J}N)] - \frac{p}{q}\bar{g}(\bar{D}_U V, \bar{J}N)$
$= \frac{1}{q}\bar{g}(D_U \bar{J}V, \bar{J}N) - \frac{p}{q}\bar{g}(D_U V, \bar{J}N) + \frac{1}{q}\bar{g}(\bar{h}^l(U, \bar{J}V), \bar{J}N) - \frac{p}{q}\bar{g}(\bar{h}^l(U, V), \bar{J}N) +$
$\frac{1}{q}\bar{g}(\bar{h}^s(U, \bar{J}V), \bar{J}N) - \frac{p}{q}\bar{g}(\bar{h}^s(U, V), \bar{J}N)$
$= \frac{1}{q}\bar{g}(D^*_U \bar{J}V, \bar{J}N) + \frac{1}{q}\bar{g}(\bar{h}^*(U, \bar{J}V), \bar{J}N) - \frac{p}{q}\bar{g}(D^*_U V, \bar{J}N) - \frac{p}{q}\bar{g}(\bar{h}^*(U, V), \bar{J}N)$
$= \frac{1}{q}[\bar{g}(\bar{h}^*(U, \bar{J}V) - p\bar{h}^*(U, V), \bar{J}N)] = 0.$
Thus, using the definition of an invariant lightlike submanifold, we obtain the required result. □

4. Screen semi-invariant lightlike submanifolds

In this section, the structure of screen semi-invariant submanifolds of a metallic semi-Riemannian manifold has been detailed out with an example.

DEFINITION 4.1. Let M be a lightlike submanifold of a metallic semi-Riemannian manifold $\bar{M}$.Then M is said to be a screen semi-invariant lightlike submanifold of $\bar{M}$ if the following conditions are satisfied:
(1) There exists a non-null distribution $B \subseteq S(TM)$ such that

(24) $$S(TM) = B \oplus B^{\perp},\ \bar{J}(B) = B,\ \bar{J}(B^{\perp}) \subseteq S(TM^{\perp}),\ B \cap B^{\perp} = \{0\},$$

where $B^{\perp}$ is orthogonal complementary to B in $S(TM)$.

(2) *Rad*(TM) is invariant with respect to $\bar{J}$ i.e. $\bar{J}(RadTM) = RadTM$.

Then,

(25) $$\bar{J}(ltr(TM)) = ltr(TM),$$

and

$$TM = B^{'} \oplus B^{\perp}, \quad B^{'} = B \perp Rad(TM), \tag{26}$$

It follows that $B^{'}$ is also invariant with respect to $\bar{J}$. Thus, the orthogonal complement to $\bar{J}(B^{\perp})$ in $S(TM^{\perp})$ is indicated by B_o and we obtain

$$tr(TM) = ltr(TM) \perp \bar{J}(B^{\perp}) \perp B_o. \tag{27}$$

PROPOSITION 4.2. *Let M be a screen semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\bar{M}, \bar{g}, \bar{J})$. Then, M is an invariant lightlike submanifold of $\bar{M}$ if and only if $B^{\perp} = \{0\}$.*

Proof. Since M is a invariant lightlike submanifold of $\bar{M}$, therefore $\bar{J}(TM) = (TM)$. Hence $B^{\perp} = \{0\}$. Similarly, the converse holds. □

PROPOSITION 4.3. *If a screen semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{M}$ is a isotropic or totally lightlike , then it is an invariant lightlike submanifold of $\bar{M}$.*

LEMMA 4.4. *Let M be a screen semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{M}$. Then*
$f^2U = pfU + qU - BwU, \quad pwU - CwU = wfU,$
$fBN = pBN - BCN, \quad C^2N = pCN + qN - wBN,$
$g(fU, V) - g(U, fV) = g(U, wV) - g(wU, V),$
$g(fU, fV) = pg(fU, V) + qg(U, V) + pg(wU, V) - g(fU, wV) - g(wU, fV) - g(wU, wV),$
for any $U, V \in \Gamma(TM)$ and $N \in \Gamma(tr(TM))$.

Proof. $\bar{J}U = fU + wU$ for $U \in \Gamma(TM)$,
Applying $\bar{J}$ on both sides and using (4), we derive
$pfU + pwU + qU = f^2U + wfU + BwU + CwU$,
Taking tangential and transversal parts of the above equation,
$f^2U = pfU + qU - BwU$ and $pwU - CwU = wfU$,
On applying of $\bar{J}$ on (8) , we obtain

$$\bar{J}^2N = \bar{J}BN + \bar{J}CN$$

Using equation (4) and and then comparing the tangential and transversal parts on both sides, we obtain
$fBN = pBN - BCN$ and $C^2N = pCN + qN - wBN$,
Using equations (4), (5) and (6), we get
$g(fU, V) - g(U, fV) = g(U, wV) - g(wU, V)$,
$g(fU, fV) = pg(fU, V) + qg(U, V) + pg(wU, V) - g(fU, wV) - g(wU, fV) - g(wU, wV)$.
□

THEOREM 4.5. *Let M be a screen semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{M}$. Then, f is a metallic structure on $B^{'}$.*

Proof. For a screen semi-invariant lightlike submanifold, we have $wU = 0$ for any $U \in \Gamma(B^{'})$. Then, it follows from the lemma (4.4) that f become a metallic structure on $B^{'}$. □

EXAMPLE 4.6. Consider an 9-dimensional semi-Euclidean space $(\bar{M} = \mathbb{R}^9_2, \bar{g})$ with signature $(-,-,+,+,+,+,+,+,+)$. Let $(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9)$ be the standard coordinate system of $\bar{M}$.Then by setting

$$\bar{J}(z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8, z_9) = ((\sigma z_1, (p-\sigma)z_2, \sigma z_3, \sigma z_4, \sigma z_5, \sigma z_6, \sigma z_7, \sigma z_8, (p-\sigma)z_9),$$

we have $\bar{J}^2 = p\bar{J} + qI$ which shows that $\bar{J}$ is a metallic structure on $\bar{M}$.

Consider a submanifold M of $\mathbb{R}^9_2$ given by the equations

$$z_1 = -y_2 + y_1, \quad z_2 = \frac{\sigma}{\sqrt{q}} y_4,$$
$$z_3 = y_3 + y_5, \quad z_4 = y_2 + y_3,$$
$$z_5 = y_4, \quad z_6 = y_3 - y_5,$$
$$z_7 = y_2 + y_1, \quad z_8 = -y_2 + y_3, \quad z_9 = 0,$$

Then, TM is spanned by $\{D_1, D_2, D_3, D_4, D_5\}$, where

$$D_1 = -\partial z_1 + \partial z_4 + \partial z_7 - \partial z_8, \quad D_2 = \partial z_3 + \partial z_4 + \partial z_6 + \partial z_8,$$
$$D_3 = \partial z_1 + \partial z_7, \quad D_4 = \partial z_3 - \partial z_6, \quad D_5 = \frac{\sigma}{\sqrt{q}} \partial z_2 + \partial z_5,$$

Also, M is a 2-lightlike submanifold with $Rad(TM)$=Span $\{D_1, D_2\}$.

Further, $S(TM)$ and $S(TM^\perp)$ are spanned by $\{D_3, D_4, D_5\}$ and $\{W\}$ respectively, where

$$W = -\sqrt{q}\partial z_2 + \sigma \partial z_5,$$

The lightlike transversal vector bundle $ltr(TM)$ is spanned by

$$N_1 = \frac{1}{4}(-\partial z_1 - \partial z_4 + \partial z_7 + \partial z_8),$$
$$N_2 = \frac{1}{4}(\partial z_3 - \partial z_4 + \partial z_6 - \partial z_8),$$

Hence $B = Span\{D_3, D_4\}$, $B^\perp = Span\{D_5\}$, $B_o = \{0\}$, and $B' = Span\{D_1, D_2, D_3, D_4\}$. Moreover, $\bar{J}(Rad(TM)) = Rad(TM)$, $\bar{J}(B) = B$, $\bar{J}(ltr(TM)) = ltr(TM)$ and $\bar{J}(B^\perp) = S(TM^\perp)$. Therefore, M becomes a screen semi-invariant lightlike submanifold of the metallic semi-Riemannian manifold $\bar{M}$.

We now analyze the geometric characteristics of the screen semi-invariant lightlike submanifold for the metallic semi-Riemannian manifold admitting a quarter symmetric non-metric connection.

LEMMA 4.7. *Let $(M, g, S(TM), S(TM^\perp))$ be a screen semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\bar{M}, \bar{g}, \bar{J})$ with a quarter symmetric non-metric connection $\bar{D}$. Then*

$$g(\bar{h}(U,V), \xi) = g(\bar{A}^*_\xi U, V) + \pi(\xi) g(JfU, V),$$

for any $\xi \in \Gamma(Rad(TM))$, $U \in \Gamma(B)$, $V \in \Gamma(B')$.

Proof. For any $\xi \in \Gamma(Rad(TM))$, $U \in \Gamma(B)$ and $V \in \Gamma(B')$,

$$g(\bar{h}(U,V), \xi) = g(h(U,V) + \pi(V) wU, \xi) = g(A^*_\xi U, V)$$
$$= g(\bar{A}^*_\xi U, V) + \pi(\xi) g(JfU, V).$$

□

Thereafter, we discuss the integrability of the distributions $Rad(TM)$, B, $B^{\perp}$ and $B^{'}$ of a screen semi-invariant submanifold of a metallic semi-Riemannian manifold $(\bar{M}, \bar{g}, \bar{J})$ with a quarter symmetric non-metric connection $\bar{D}$.

DEFINITION 4.8. Let $(M, g, S(TM), S(TM^{\perp}))$ be a screen semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\bar{M}, \bar{g}, \bar{J})$. Then, the distribution B is integrable if and only if $[U, V] \in \Gamma(B)$, for any $U, V \in \Gamma(B)$.

This leads to the following remark:

REMARK 4.9. The distribution B is said to be integrable if and only if $g([X, Y], Z) = 0$, for any $X, Y \in \Gamma(B)$ and $Z \in \Gamma(B^{\perp})$.

THEOREM 4.10. *Let M be a screen semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{M}$ with a quarter symmetric non-metric connection $\bar{D}$.Then, the radical distribution is integrable if and only if*
(i) $\bar{A}^{*}_{\bar{J}E}E^{'} + p\bar{A}^{*}_{E^{'}}E = \bar{A}^{*}_{\bar{J}E^{'}}E + p\bar{A}^{*}_{E}E^{'}$ for all $E, E^{'} \in \Gamma(Rad(TM))U \in \Gamma(B)$.
(ii) $\bar{h}^{s}(E, \bar{J}E^{'}) = \bar{h}^{s}(E^{'}, \bar{J}E)$, for all $E, E^{'} \in \Gamma(Rad(TM)), Z \in \Gamma(B^{\perp})$.

Proof. $Rad(TM)$is integrable if and only if

$$g([E, E^{'}], Z) = 0, \quad g([E, E^{'}], U) = 0, \tag{28}$$

for any $E, E^{'} \in \Gamma(Rad(TM)),\ Z \in \Gamma(B^{\perp}),\ U \in \Gamma(B)$.
From equations (6),(9),(10),(11),(20) and (28), we get

$$\begin{aligned}
0 &= \tfrac{1}{q}[\bar{g}(\bar{D}_{E}\bar{J}E^{'} + p\pi(E^{'})\bar{J}E + q\pi(E^{'})E - \pi(\bar{J}E^{'})\bar{J}E, \bar{J}U) - p\bar{g}(\bar{D}_{E}E^{'}, \bar{J}U)] - \\
&\quad \tfrac{1}{q}[\bar{g}(\bar{D}_{E^{'}}\bar{J}E + p\pi(E)\bar{J}E^{'} + q\pi(E)E^{'} - \pi(\bar{J}E)\bar{J}E^{'}, \bar{J}U) - p\bar{g}(\bar{D}_{E^{'}}E, \bar{J}U)] \\
&= \tfrac{1}{q}[\bar{g}(-\bar{A}^{*}_{\bar{J}E^{'}}E, \bar{J}U) + \bar{g}(\bar{\nabla}^{*t}_{E}\bar{J}E^{'}, \bar{J}U) - p\bar{g}(-\bar{A}^{*}_{E^{'}}E, \bar{J}U) - p\bar{g}(\bar{\nabla}^{*t}_{E}E^{'}, \bar{J}U)] - \\
&\quad \tfrac{1}{q}[\bar{g}(-\bar{A}^{*}_{\bar{J}E}E^{'}, \bar{J}U) + \bar{g}(\bar{\nabla}^{*t}_{E^{'}}\bar{J}E, \bar{J}U) - p\bar{g}(-\bar{A}^{*}_{E}E^{'}, \bar{J}U) - p\bar{g}(\bar{\nabla}^{*t}_{E^{'}}E, \bar{J}U)] \\
&= \tfrac{1}{q}\bar{g}(\bar{A}^{*}_{\bar{J}E}E^{'} + p\bar{A}^{*}_{E^{'}}E - \bar{A}^{*}_{\bar{J}E^{'}}E - p\bar{A}^{*}_{E}E^{'}, \bar{J}U),
\end{aligned}$$

which leads to (i).

$$g([E, E^{'}], Z) = \bar{g}(\bar{D}_{E}E^{'} - \pi(E^{'})\bar{J}E - \bar{D}_{E^{'}}E + \pi(E)\bar{J}E^{'}, Z),$$

Using equations, (9), (10) and (11), we have

$$\begin{aligned}
g([E, E^{'}], Z) &= \tfrac{1}{q}\bar{g}(D_{E}\bar{J}E^{'}, \bar{J}Z) + \tfrac{1}{q}\bar{g}(\bar{h}^{l}(E, \bar{J}E^{'}), \bar{J}Z) + \tfrac{1}{q}\bar{g}(\bar{h}^{s}(E, \bar{J}E^{'}), \bar{J}Z) - \\
&\quad \tfrac{p}{q}\bar{g}(D_{E}E^{'}, \bar{J}Z) - \tfrac{p}{q}\bar{g}(\bar{h}^{l}(E, E^{'}), \bar{J}Z) - \tfrac{p}{q}\bar{g}(\bar{h}^{s}(E, E^{'}), \bar{J}Z) - \tfrac{1}{q}\bar{g}(D_{E^{'}}\bar{J}E, \bar{J}Z) - \\
&\quad \tfrac{1}{q}\bar{g}(\bar{h}^{l}(E^{'}, \bar{J}E), \bar{J}Z) - \tfrac{1}{q}\bar{g}(\bar{h}^{s}(E^{'}, \bar{J}E), \bar{J}Z) + \tfrac{p}{q}\bar{g}(D_{E^{'}}E, \bar{J}Z) + \\
&\quad \tfrac{p}{q}\bar{g}(\bar{h}^{l}(E^{'}, E), \bar{J}Z) + \tfrac{p}{q}\bar{g}(\bar{h}^{s}(E^{'}, E), \bar{J}Z),
\end{aligned}$$

Further, equations (24), (15) and (28) imply

$$g([E, E^{'}], Z) = \tfrac{1}{q}[\bar{g}(\bar{h}^{s}(E, \bar{J}E^{'}) - p\pi(E^{'})w_{s}E - \bar{h}^{s}(E^{'}, \bar{J}E) + p\pi(E)w_{s}E^{'}, \bar{J}Z)] = 0,$$

From the concept of screen semi-invariant lightlike submanifold, $w_{s}E = 0, \quad w_{s}E^{'} = 0$,
$\bar{h}^{s}(E, \bar{J}E^{'}) = \bar{h}^{s}(E^{'}, \bar{J}E)$ which proves (ii). □

THEOREM 4.11. *Let M be a screen semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{M}$ with a quarter symmetric non-metric connection $\bar{D}$.Then, the necessary and sufficient condition for B to be integrable is that*
(i) $\bar{h}^{s}(U, \bar{J}V) = \bar{h}^{s}(V, \bar{J}U)$ for all $U, V \in \Gamma(B), Z \in \Gamma(B^{\perp})$.
(ii) $\bar{h}^{*}(U, \bar{J}V) + p\bar{h}^{*}(V, U) = \bar{h}^{*}(V, \bar{J}U) + p\bar{h}^{*}(U, V)$,
for all $U, V \in \Gamma(B), N \in \Gamma(ltr(TM))$.

Proof. B is integrable if and only if

$$g([U,V],Z)=0, \quad g([U,V],N)=0, \tag{29}$$

Using the concept of a quarter symmetric non-metric connection, we derive

$g([U,V],Z)=\bar{g}(\bar{D}_U V-\pi(V)\bar{J}U-\bar{D}_V U+\pi(U)\bar{J}V,Z)=\bar{g}(\bar{D}_U V,Z)-\bar{g}(\bar{D}_V U,Z),$

From equations (6) and (10), we have

$g([U,V],Z)=\frac{1}{q}\bar{g}(\bar{J}\bar{D}_U V,\bar{J}Z)-\frac{p}{q}\bar{g}(\bar{D}_U V,\bar{J}Z)-\frac{1}{q}\bar{g}(\bar{J}\bar{D}_V U,\bar{J}Z)+\frac{p}{q}\bar{g}(\bar{D}_V U,\bar{J}Z),$

Equations (11), (15) and (29) imply

$\frac{1}{q}[\bar{g}(\bar{h}^s(U,\bar{J}V)-p\pi(V)w_sU-\bar{h}^s(V,\bar{J}U)+p\pi(U)w_sV,\bar{J}Z)]=0,$

Thus, from (24) and (7), we obtain the result (i) .

Also, $g([U,V],N)=\frac{1}{q}[\bar{g}(\bar{D}_U\bar{J}V+p\pi(V)\bar{J}U+q\pi(V)U-\pi(\bar{J}V)\bar{J}U,\bar{J}N)-$
$p\bar{g}(\bar{D}_U V,\bar{J}N)]-\frac{1}{q}[\bar{g}(\bar{D}_V\bar{J}U+p\pi(U)\bar{J}V+q\pi(U)V-\pi(\bar{J}U)\bar{J}V,\bar{J}N)-$
$p\bar{g}(\bar{D}_V U,\bar{J}N)],$

Since M is a screen semi-invariant submanifold of $\bar{M}$, therefore, on using equations (11), (19), (21) and (29), we have

$g([U,V],N)=\frac{1}{q}[\bar{g}(\bar{h}^*(U,\bar{J}V)+ph^*(V,U)+\pi(U)\eta(fV)\xi-\bar{h}^*(V,\bar{J}U)-$
$\pi(V)\eta(fU)\xi-ph^*(U,V),\bar{J}N)]=0.$

Then, the equations (24) and (18) proves the result (ii). □

THEOREM 4.12. *The distribution B' of a screen semi-invariant lightlike submanifold M of a metallic semi-Riemannian manifold $\bar{M}$ equipped with a quarter symmetric non-metric connection is integrable if and only if*

$$\bar{h}^s(U,\bar{J}V)=\bar{h}^s(V,\bar{J}U),$$

for any $U,V\in\Gamma(B)$, $Z\in\Gamma(B^\perp)$.

Proof. B' is integrable if and only if

$$g([U,V],Z)=0, \tag{30}$$

Using the compatibility of metallic structure with quarter symmetric non-metric connection, we get

$g([U,V],Z)=\bar{g}(\bar{\nabla}_U V-\bar{\nabla}_V U,Z)=\bar{g}(\bar{D}_U V,Z)-\bar{g}(\bar{D}_V U,Z)$
$=\frac{1}{q}[\bar{g}(\bar{D}_U\bar{J}V+p\pi(V)\bar{J}U+q\pi(V)U-\pi(\bar{J}V)\bar{J}U,\bar{J}Z)]-\frac{p}{q}\bar{g}(\bar{D}_U V,\bar{J}Z)-$
$\frac{1}{q}[\bar{g}(\bar{D}_V\bar{J}U+p\pi(U)\bar{J}V+q\pi(U)V-\pi(\bar{J}U)\bar{J}V,\bar{J}Z)]+\frac{p}{q}\bar{g}(\bar{D}_V U,\bar{J}Z),$

Equations (11), (15) and (30) imply

$g([U,V],Z)=\frac{1}{q}\bar{g}(\bar{h}^s(U,\bar{J}V),\bar{J}Z)-\frac{p}{q}\bar{g}(\pi(V)w_sU,\bar{J}Z)-\frac{1}{q}\bar{g}(\bar{h}^s(V,\bar{J}U),\bar{J}Z)+$
$\frac{p}{q}\bar{g}(\pi(U)w_sV,\bar{J}Z)=0.$

Therefore, from the concept of screen semi-invariant lightlike submanifold M, we have

$$\bar{h}^s(U,\bar{J}V)=\bar{h}^s(V,\bar{J}U),$$

□

Further, we establish some necessary and sufficient conditions for parallelism of an invariant distribution B' with respect to a quarter symmetric non-metric connection $\bar{D}$ and induced metric connection ∇.

DEFINITION 4.13. Let $(\bar{M},\bar{J},\bar{g})$ be a metallic semi-Riemannian manifold and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$ with respect to $\bar{g}$. Then the distribution D is parallel with respect to $\bar{\nabla}$ if and only if $\bar{\nabla}_U V=0$, for all $U,V\in\Gamma(D)$.

Remark 4.14. In case of quarter symmetric non-metric connection, the distribution D is parallel with respect to $\bar{D}$ if and only if $\bar{D}_U V = 0$, for all $U, V \in \Gamma(D)$.

Theorem 4.15. *For a screen semi-invariant lightlike submanifold M of a metallic semi-Riemannian manifold $\bar{M}$ with a quarter symmetric non-metric connection $\bar{D}$, the screen distribution is parallel if and only if*
$-\bar{A}_{\bar{J}Z}U + p\pi(Z)\bar{J}U = \pi(\bar{J}Z)\bar{J}U + p\bar{h}^*(U, Z)$, *for any* $U \in \Gamma(S(TM)), Z \in \Gamma(B^{\perp})$.

Proof. $S(TM)$ is parallel with respect to $\bar{D}$ if and only if

$$\bar{g}(\bar{D}_U Z, N) = 0, \tag{31}$$

for any $U \in \Gamma(S(TM)), Z \in \Gamma(B^{\perp}), N \in \Gamma(ltr(TM))$.
From the hypothesis of metallic semi-Riemannian manifold and a quarter symmetric non-metric connection, we derive

$$\begin{aligned}\bar{g}(\bar{D}_U Z, N) &= \tfrac{1}{q}[\bar{g}(\bar{D}_U \bar{J}Z + p\pi(Z)\bar{J}U - \pi(\bar{J}Z)\bar{J}U - p\bar{D}_U Z, \bar{J}N)]\\ &= \tfrac{1}{q}[\bar{g}(-\bar{A}_{\bar{J}Z}U, \bar{J}N) + \bar{g}(\bar{\nabla}^s_U \bar{J}Z, \bar{J}N) + \bar{g}(D^l(U, \bar{J}Z), \bar{J}N) - p\bar{g}(D_U Z, \bar{J}N) -\\ &\quad p\bar{g}(\bar{h}^l(U, Z), \bar{J}N) - p\bar{g}(\bar{h}^s(U, Z), \bar{J}N)] + p\pi(Z)\bar{g}(\bar{J}U, \bar{J}N) - \pi(\bar{J}Z)\bar{g}(\bar{J}U, \bar{J}N)\\ &= \tfrac{1}{q}[\bar{g}(-\bar{A}_{\bar{J}Z}U, \bar{J}N) - p\bar{g}(D^*_U Z + \bar{h}^*(U, Z), \bar{J}N) + p\pi(Z)\bar{g}(\bar{J}U, \bar{J}N) -\\ &\quad \pi(\bar{J}Z)\bar{g}(\bar{J}U, \bar{J}N)].\end{aligned}$$

Therefore the result is obtained by using (6), (10), (11), (13) and (31). □

Proposition 4.16. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a screen semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $(\bar{M}, \bar{g}, \bar{J})$. Then the following assertion holds:*
The distribution B' is parallel with respect to induced connection ∇ if and only if $h^s(U, \bar{J}V) = 0$, $U, V \in \Gamma(B')$.

Proof. From the Gauss-Weingarten formulae,

$$g(\nabla_U \bar{J}V, Z) = g(h^s(U, V), \bar{J}Z),$$

for $U, V \in \Gamma(B')$ and $Z \in \Gamma(S(TM^{\perp}))$.
Therefore, the assertion follows that the distribution B' is parallel with respect to induced connection ∇ i.e $\nabla_U \bar{J}V \in \Gamma(B')$. □

Theorem 4.17. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a screen semi-invariant submanifold of a metallic semi-Riemannian manifold $(\bar{M}, \bar{g}, \bar{J})$. The distribution B' is parallel with respect to quarter symmetric non-metric connection D if and only if B' is parallel with respect to induced connection ∇.*

Proof. For any $U, V \in \Gamma(B'), Z \in \Gamma(B^{\perp})$ $\quad w_s U = 0$, Therefore, $\bar{h}^s(U, \bar{J}V) = h^s(U, \bar{J}V)$. Thus the result follows using proposition (4.16). □

We have the following corollary from proposition (4.16).

Corollary 4.18. *Let $(M, g, S(TM), S(TM^{\perp}))$ be a screen semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{M}$. Then, the following assertions are equivalent:*
(a) The distribution B' is parallel with respect to quarter symmetric non-metric connection D,
(b) $\bar{h}^s(U, \bar{J}V) = 0, \forall\, U, V \in \Gamma(B')$,
(c) B' is parallel with respect to induced connection ∇,
(d) $h^s(U, \bar{J}V) = 0, \forall\, U, V \in \Gamma(B')$.

REMARK 4.19. Since $\bar{h}$ is not symmetric, therefore $\bar{h}(Y,X)$ may not be zero, if $\bar{h}(X,Y)=0$.

Next, we present the conditions of B, $B^{\perp}$ and $B^{'}$ to define totally geodesic foliatons on screen semi-invariant lightlike submanifolds M of metallic semi-Riemannian manifold $\bar{M}$ with a quarter symmetric non-metric connection.

DEFINITION 4.20. If $\bar{D}_U V \in \Gamma(B)$, for any $U,V \in \Gamma(B)$ in a screen semi-invariant lightlike submanifold $(M,g,S(TM),S(TM^{\perp}))$ of a metallic semi-Riemannian manifold $(\bar{M},\bar{g},\bar{J})$ with a quarter symmetric non-metric connection $\bar{D}$, then B defines a totally geodesic foliation on M.

REMARK 4.21. The distribution B defines a totally geodesic foliation on M if and only if $g(\bar{D}_U V, N)=0$, for any $U,V\in\Gamma(B)$ and $N\in\Gamma(B^{\perp})$.

THEOREM 4.22. *Suppose M be a screen semi-invariant lightlike submanifold of a metallic semi-Riemannian manifold $\bar{M}$ with a quarter symmetric non metric connection. Then, the distribution B defines a totally geodesic foliation on $S(TM)$ if and only if*

$$\bar{J}\bar{h}^s(U,V) = p\pi(V)\bar{J}U + q\pi(V)U - \pi(\bar{J}V)\bar{J}U,$$

for all $U\in\Gamma(TM),\ V\in\Gamma(B),\ Z\in\Gamma(B^{\perp})$.

Proof. Using the concept of quarter symmetric non-metric connection $\bar{D}$,
$\bar{g}(\bar{D}_U V,\bar{J}Z) = \bar{g}(\bar{D}_U\bar{J}V,Z)+p\pi(V)\bar{g}(\bar{J}U,Z)+q\pi(V)\bar{g}(U,Z)-\pi(\bar{J}V)\bar{g}(\bar{J}U,Z)$
$=\bar{g}(D^*_U\bar{J}V,Z)+\bar{g}(\bar{h}^*(U,\bar{J}V),Z)+p\pi(V)\bar{g}(\bar{J}U,Z)+q\pi(V)\bar{g}(U,Z)-\pi(\bar{J}V)\bar{g}(\bar{J}U,Z)$,
B defines totally geodesic foliation on $S(TM)$ if and only if $g(D^*_U\bar{J}V,Z)=0$,
Therefore, $\bar{g}(\bar{D}_U V,\bar{J}Z)=\bar{g}(D_U V,\bar{J}Z)+\bar{g}(\bar{h}^l(U,V),\bar{J}Z)+\bar{g}(\bar{h}^s(U,V),\bar{J}Z)$
$=\bar{g}(\bar{J}\bar{h}^s(U,V),Z)$,
Hence, $\bar{g}(\bar{J}\bar{h}^s(U,V),Z)=p\pi(V)\bar{g}(\bar{J}U,Z)+q\pi(V)\bar{g}(U,Z)-\pi(\bar{J}V)\bar{g}(\bar{J}U,Z)$. □

THEOREM 4.23. *Let M be a screen semi-invariant lightlike submanifold of $(\bar{M},\bar{g},\bar{J})$ with a quarter symmetric non-metric connection.Then $B^{'}$ defines a totally geodesic foliation on M if and only if*
$\bar{g}(\bar{J}E,\bar{D}^l(U,\bar{J}Z))=\bar{g}(\bar{A}_{\bar{J}Z}U,\bar{J}E)+\pi(\bar{J}Z)\bar{g}(\bar{J}U,\bar{J}E)$,
$\bar{g}(\bar{A}_{\bar{J}Z}U,\bar{J}F)=-\pi(\bar{J}Z)\bar{g}(\bar{J}U,\bar{J}F)$,
for any $E\in\Gamma(Rad(TM)),\ U\in\Gamma(B^{'}),\ Z\in\Gamma(B^{\perp}),\ F\in\Gamma(B)$.

Proof. $B^{'}$ defines a totally geodesic foliation on M if and only if

$$g(D_U V,Z)=0,\quad g(D_U\bar{J}V,Z)=0\quad\forall\quad U,V\in\Gamma(B^{'}). \tag{32}$$

Following (6), (10) and (11), we have
$g(D_U V,Z)=\frac{1}{q}[\bar{g}(\bar{D}_U\bar{J}V,\bar{J}Z)+p\pi(V)\bar{g}(\bar{J}U,\bar{J}Z)+q\pi(V)\bar{g}(U,\bar{J}Z)-$
$\pi(\bar{J}V)\bar{g}(\bar{J}U,\bar{J}Z)-p\bar{g}(\bar{D}_U\bar{J}V,Z)-p^2\pi(V)\bar{g}(\bar{J}U,Z)-$
$pq\pi(V)\bar{g}(U,Z)+p\pi(\bar{J}V)\bar{g}(\bar{J}U,Z)]$
$=\frac{1}{q}[\bar{g}(D_U\bar{J}V,\bar{J}Z)+\bar{g}(\bar{h}^l(U,\bar{J}V),\bar{J}Z)+\bar{g}(\bar{h}^s(U,\bar{J}V),\bar{J}Z)-$
$p\bar{g}(D_U\bar{J}V,Z)-p\bar{g}(\bar{h}^l(U,\bar{J}V),Z)-p\bar{g}(\bar{h}^s(U,\bar{J}V),Z)]$,
Using (32), we get
$\bar{g}(\bar{h}^s(U,\bar{J}V),\bar{J}Z)=0$,
Using (26) and (7), $w_sU=0\quad w_lU=0$,
$\bar{g}(\bar{J}V,\bar{D}^l(U,\bar{J}Z))=\bar{g}(\bar{A}_{\bar{J}Z}U,\bar{J}V)+\pi(\bar{J}Z)\bar{g}(\bar{J}U,\bar{J}V)$,
$Case-1$: Taking $V=E\in\Gamma(Rad(TM))$,

$\bar{g}(\bar{J}E, \bar{D}^l(U, \bar{J}Z)) = \bar{g}(\bar{A}_{\bar{J}Z}U, \bar{J}E) + \pi(\bar{J}Z)\bar{g}(\bar{J}U, \bar{J}E)$,
$Case-2$: Taking $V = F \in \Gamma(B)$,$\bar{g}(\bar{J}F, \bar{D}^l(U, \bar{J}Z)) = 0$,
$\bar{g}(\bar{A}_{\bar{J}Z}U, \bar{J}F) = -\pi(\bar{J}Z)\bar{g}(\bar{J}U, \bar{J}F)$. □

THEOREM 4.24. *Let M be a screen semi-invariant lightlike submanifold of $(\bar{M}, \bar{g}, \bar{J})$ with a quarter symmetric non-metric connection $\bar{D}$. Then, $B^{\perp}$ defines a totally geodesic foliation on M if and only if*
$\bar{g}(\bar{h}^s(U, \bar{J}Z), \bar{J}V) = \pi(\bar{J}Z)\bar{g}(\bar{J}U, \bar{J}V)$,
$\bar{g}(\bar{D}^s(U, \bar{J}N), \bar{J}V) = \pi(\bar{J}N)\bar{g}(\bar{J}U, \bar{J}V)$,
for any $U, V \in \Gamma(B^{\perp}),\ Z \in \Gamma(B),\ N \in \Gamma(ltr(TM))$.

Proof. Following the concept of screen semi-invariant lightlike submanifold with equations (6), (10) and (13), we obtain

$$g(D_UV, \bar{J}Z) = \bar{g}(\bar{D}_U\bar{J}V, Z) + p\pi(V)\bar{g}(\bar{J}U, Z) + q\pi(V)\bar{g}(U, Z) - \pi(\bar{J}V)\bar{g}(\bar{J}U, Z)$$
$$= \bar{g}(-\bar{A}_{\bar{J}V}U, Z) + \bar{g}(\bar{\nabla}^s_U\bar{J}V, Z) + \bar{g}(\bar{D}^l(U, \bar{J}V), Z),$$
$$\bar{g}(-\bar{A}_{\bar{J}V}U, Z) = 0,$$
$$g(D_UV, \bar{J}N) = \bar{g}(\bar{D}_U\bar{J}V, N) + p\pi(V)\bar{g}(\bar{J}U, N) + q\pi(V)\bar{g}(U, N) - \pi(\bar{J}V)\bar{g}(\bar{J}U, N)$$
$$= \bar{g}(-\bar{A}_{\bar{J}V}U + \bar{\nabla}^s_U\bar{J}V + \bar{D}^l(U, \bar{J}V), N) = \bar{g}(-\bar{A}_{\bar{J}V}U, N) = 0,$$

Using (16) and (17), we have

$$\bar{g}(\bar{h}^s(U, \bar{J}Z), \bar{J}V) = \bar{g}(\bar{A}_{\bar{J}V}U, \bar{J}Z) + \pi(\bar{J}Z)\bar{g}(\bar{J}U, \bar{J}V),$$
$$\bar{g}(\bar{D}^s(U, \bar{J}N), \bar{J}V) = \bar{g}(\bar{A}_{\bar{J}V}U, \bar{J}N) + \pi(\bar{J}N)\bar{g}(\bar{J}U, \bar{J}V).$$ □

5. Relevance of the study

This research work has investigated the lightlike geometry of invariant and screen semi-invariant lightlike submanifolds in the metallic semi-Riemannian manifold equipped with a quarter symmetric non-metric connection.

The metallic Riemannian manifolds and the various classes of metallic means namely golden mean, silver mean and the bronze mean based on the values of the metallic numbers , are instrumental as the basis of proportion to design sculptures. These manifolds are of great use to physicists to analyze the behaviour of nonlinear dynamic systems in the transition from periodicity to semi-periodicity. The study undertaken suggests that there is a potential for further investigation into the characteristics of invariant and screen semi invariant submanifolds within a metallic semi Riemannian manifold endowed with a quarter symmetric non-metric connection. The geometry of these submanifolds can be examined for various classes of metallic semi-Riemannian manifold equipped with distinct connections inherent to their structure. Also, the metallic semi-Riemannian manifolds, being intrinsically related to the theoretical explanation of behavior in quantum physics, can motivate the geometers to delve into the applications of this noteworthy area of study.

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