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REGULARITY FOR SOLUTIONS OF FIRST ORDER EVOLUTION EQUATIONS OF VOLTERRA TYPE

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Abstract. In this paper we study the semilinear first order evolution problems of Volterra type with Lipschitz continuous nonlinearities. Using the variational formulation of problems due to Dautray and Lions [6], we have proved the fundamental results on existence, uniqueness and continuous dependence of solutions. Especially in the proof of the regularity we have used the double regularization method. Applications to nonlinear partial integro-differential equations are given.

1. Introduction

We consider the following nonlinear first order evolution problems of Volterra type in a Hilbert space H of the form

$$\begin{cases} \dfrac{dy}{dt} + A(t)y + \displaystyle\int_0^t K(t,s)y(s)ds = f(t,y) + g(t) \ \text{ in } (0,T), \\ y(0) = y_0, \end{cases} \tag{1}$$

where $A(t)$ and $K(t,s)$ are time varying differential operators, g is a forcing function, f is a nonlinear function and y_0 is an initial value.

In this paper, we mainly study the case where the integral kernel operator $K(t,s)$ in the above equation is a strong operator belonging to $\mathcal{L}(V,V')$. If the operator $K(t,s)$ is a weak operator belonging to $\mathcal{L}(V,H)$, other research results can be referred to, for example, Dautray and Lions [6, pp.651-654]. The results are almost similar to those for the parabolic equation. A specific physical example related to Eq.(1) with a strong integral kernel operator $K(t,s) \in \mathcal{L}(V,V')$ is as follows: The first model arise in the Coleman-Gurtin theory of heat propagation in real conductors [3], where the temperature evolution exhibits a *hyerbolic* character due to the conductor's inertia. Here, the classical Fourier law for the heat flux is replaced by a constitutive law based on the key assumption that the evolution of the heat flux is influenced by the past history of the temperature gradient, see [11, 12]. For this kind of the first

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order Volterra integro-differential evolution equations have been studied extensively by many authors. Among them, we refer to Chen and Grimmer [1, 2], Desch and Schappacher [7], Da Prato and Iannelli [4], and Nagel and Sinestrari [13], who adopted a semigroup treatment to clarify the existence, uniqueness, and asymptotic behavior of the solution. To achieve this, the authors imposed very strong assumptions on the operators of Eq.(1). However, as can be seen in Hwang and Nakagiri [8, Theorem 3.3], sometimes it is necessary to deal with more general Volterra-type equations using weaker conditions on the operators than previously considered. For this we should take the variational formulation of Eq.(1) to obtain the solutions under less conditions.

In particular, Dautray and Lions [6] gave basic results on uniqueness and existence under the variational formulation of the linear problem Eq.(1) with $f(t, y) = 0$. However, the regularity problem of the weak solution and the strong continuity problem of the approximate solution, which are necessary to handle the nonlinear case, are still unclear.

Therefore, in this paper, we study the nonlinear problem Eq.(1) and establish results on the existence, uniqueness, regularity, and continuity dependence of the solution according to the conditions of the nonlinear term f and the operators $A(t)$ and $K(t, s)$. Especially, in the proof of existence, it is not assumed the monotonicity of f with respect to y or the compact embedding of spaces. We only assume the Lipschitz continuity of f. Under Lipschitz continuity of the nonlinear term f, we prove that the approximate solution to Eq.(1) converges strongly to the weak solution to Eq.(1). The main idea of the proof comes from deducing strong convergence of the approximate solution presented in Dautray and Lions [6]. And for the strong memory case ($K(t, s) \in \mathcal{L}(V, V')$) the regularity is proved by the double regularization method given in [10, p.276]. This is main novelty of the paper.

The content of the paper is summarized as follows: In section 2 we present notations effective in this paper and assumptions for the terms of Eq.(1). In section 3 we prove the well-posedness of Eq.(1) including regularity of the solutions. In section 4 we present partial differential equations applicable to the results obtained in the previous section. Finally, we review the mathematical educational significance of various mathematical techniques implemented in this paper.

2. Notations and assumptions

First we explain the notations used in this chapter. Let H be a real pivot Hilbert space. By $(\cdot, \cdot)$ and $|\cdot|$ denote the inner product and the induced norm on H. Let V be a real separable Hilbert space. The symbols $(\cdot, \cdot)_V$ and $\|\cdot\|$ denote the inner product and the induced norm on V, respectively. The dual space of V is denoted by V' and $\langle \cdot, \cdot \rangle$ denotes the dual pairing between V' and V or V and V'. Assume that (V, H, V') is a Gelfand triple space with a notation,

$V \hookrightarrow H \equiv H' \hookrightarrow V'$, which means that the embedding ($V \subset H$) is continuous and V is dense in H. We denote by $\mathcal{D}'(0,T)$ the space of distributions on $(0,T)$. Let X be a Banach space and $p \in (1,\infty)$, $L^p(0,T;X)$ be the collection of all strongly measurable functions from $(0,T)$ into X for which pth power norm is integrable. $C([0,T];X)$ is the set of all continuous functions from $[0,T]$ into X.

We study the Cauchy problem described by the semilinear first order Volterra integro-differential equation Eq.(1). Firstly we impose the following conditions and assumptions on $A(t)$, $K(t,s)$ and $f(t,y)$ in Eq.(1).

(1) Assumptions on $A(t)$

Let $a(t;\phi,\varphi), t \in [0,T]$ be a bilinear form on $V \times V$. First we suppose the following conditions on $a(t;\phi,\varphi)$:

$$\begin{cases} \text{i)} & a(t;\phi,\psi) \text{ is measurable in } [0,T] \text{ for all } \phi,\psi \in V; \\ \text{ii)} & \text{there exists } c > 0 \text{ such that} \\ & |a(t;\phi,\varphi)| \le c\|\phi\|\|\varphi\| \text{ for all } \phi,\psi \in V \text{ and } t \in [0,T]; \\ \text{iii)} & \text{there exists } \alpha > 0 \text{ and } \lambda \in \mathbf{R} \text{ such that} \\ & a(t;\phi,\phi) \ge \alpha\|\phi\|^2 \text{ for all } \phi \in V \text{ and } t \in [0,T]. \end{cases} \tag{2.1}$$

Then we can define the operator $A(t) \in \mathcal{L}(V,V')$ for $t \in [0,T]$ deduced by the relation

$$a(t;\phi,\varphi) = \langle A(t)\phi,\varphi\rangle \text{ for all } \phi,\varphi \in V.$$

We also give the following stronger conditions on $a(t;\phi,\varphi)$:

$$\begin{cases} \text{i)} & a(t;\phi,\psi) = a(t;\psi,\phi) \text{ for all } \phi,\psi \in V \text{ and } t \in [0,T]; \\ \text{ii)} & \text{there exists } c_1 > 0 \text{ such that} \\ & |a'(t;\phi,\varphi)| \le c_1\|\phi\|\|\varphi\| \text{ for all } \phi,\psi \in V \text{ and } t \in [0,T], \end{cases} \tag{2.2}$$

where $' = \frac{d}{dt}$.

(2) Assumptions on $K(t,s)$

Let $k(t,s;\phi,\varphi)$ be a family of bilinear form over $V \times V$. We give the following conditions on $k(t,s;\phi,\varphi)$:

$$\begin{cases} \text{i)} & k(t;\phi,\varphi) \text{ is a bilinear form on } V \times V \\ & \text{and is a measurable in } [0,T]\times[0,T] \text{ for all } \phi \in V \text{ and } \varphi \in V; \\ \text{ii)} & \text{there exists } k_0 > 0 \text{ such that} \\ & |k(t,s;\phi,\varphi)| \le k_0\|\phi\|\|\varphi\| \text{ for all } \phi,\varphi \in V \\ & \text{and for all } (t,s) \in [0,T]\times[0,T]; \\ \text{iii)} & \text{the function } (t,s) \to k(t,s;\phi,\varphi) \text{ is differentiable for all } \phi,\varphi \in V \\ & \text{and there exists } k_1 > 0 \text{ with} \\ & \left|\dfrac{\partial k}{\partial t}(t,s;\phi,\varphi)\right| \le k_1\|\phi\|\|\varphi\|, \\ & \text{for all } \phi,\varphi \in V, \text{ and for all } (t,s) \in [0,T]\times[0,T]. \end{cases}$$

By the above condition the family $k(t,s;\phi,\varphi)$ defines a family of operators $K(t,s) \in \mathcal{L}(V,V')$

$$k(t,s;\phi,\varphi) = \langle K(t,s)\phi,\varphi\rangle. \tag{2.3}$$

(3) Assumptions of $f(t,y)$

For the nonlinear term $f(t,y)$ in Eq.(1) we give the following assumptions:

$$\begin{cases} \text{i)} & f:[0,T]\times V\to H \text{ and } t\to f(t,y) \\ & \text{is strongly measurable in } H \text{ for all } y\in V; \\ \text{ii)} & \text{there exists a } \beta\in L^2(0,T;\mathbf{R}^+) \text{ such that} \\ & |f(t,y)-f(t,z)|\le\beta(t)\|y-z\| \text{ a.e. } t\in[0,T] \text{ for } y,z\in V; \\ \text{iii)} & \text{there exists a } \gamma\in L^2(0,T;\mathbf{R}^+) \\ & \text{such that } |f(t,0)|\le\gamma(t) \text{ a.e. } t\in[0,T]; \end{cases} \tag{2.4}$$

Next, we introduce the solution Hilbert space $W(V,H)$ defined by

$$W(V,H)=\{\varphi|\varphi\in L^2(0,T;V),\varphi'\in L^2(0,T;H)\}$$

endowed with the norm

$$\|\varphi\|_{W(V,H)}=\left(\|\varphi\|^2_{L^2(0,T;V)}+\|\varphi'\|^2_{L^2(0,T;H)}\right)^{\frac{1}{2}}.$$

3. Existence, uniqueness and regularity of solutions

The purpose of this section is to prove the main results about the existence, uniqueness, and regularity of weak solutions to Eq.(1).

At first we present our notion of solutions for Eq.(1).

Definition 1. A function y is called *a weak solution for Eq.(1)* if

(i) $y\in W(V,H)$,

(ii) $\langle y'(\cdot),\phi\rangle+a(\cdot;y(\cdot),\phi)+\int^{\cdot} k(\cdot,s;y(s),\phi)ds=(f(\cdot,y(\cdot)),\phi)+\langle g(\cdot),\phi\rangle$
for all $\phi\in V$ in the sense of $\mathcal{D}'(0,T)$.

We will prove the energy equality necessary to clarify the regularity of weak solutions for the problem Eq.(1). To do so we need the following lemmas given in the Lions and Magenes [10].

Lemma 3.1. *Let X,Y be two Banach spaces, $X\subset Y$ with dense, and X being reflexive. Set*

$$C_s([0,T];Y)$$
$$=\{f\in L^\infty(0,T;Y)|\ \ \forall\phi\in Y',t\to\langle f(t),\phi\rangle_{Y,Y'} \text{ is continuous of } [0,T]\to R\}.$$

Then

$$L^\infty(0,T;X)\cap C_s([0,T];Y)\subset C_s([0,T];X).$$

Proof. See Lions and Magenes [10, Lemma 8.1]. □

Lemma 3.2. *Assume that $y\in L^\infty(0,T;V)$ is a weak solution of Eq.(1). Then we can assert (after possibly a modification on a set of measure zero) that*

$$y\in C_s([0,T];V). \tag{3.1}$$

Proof. We deduce that $W(V,H) \hookrightarrow C([0,T];H)$ (cf. Dautray and Lions [6, p.555]). Thus we have

$$y \in W(V,H) \cap L^\infty(0,T;V) \hookrightarrow C_s([0,T];H) \cap L^\infty(0,T;V) \subset C_s([0,T];V).$$

Here we use Lemma 3.1 for the case $X = V$ and $Y = H$.

This completes the proof. □

The result of scalar continuity of a weak solution y of Eq.(1) given in Lemma 3.2 can be improved to strong continuity of the solution y by making use of the following result.

Proposition 3.3. *Assume that a satisfies (2.1) and (2.2), k satisfies (2.3), $y_0 \in V$, and $F(\cdot)(= f(\cdot, y(\cdot)) + g(\cdot)) \in L^2(0,T;H)$. Let $y \in L^\infty(0,T;V)$ be the weak solution of Eq.(1). Then, for each $t \in [0,T]$ we have the following energy equality*

$$\begin{aligned} &a(t;y(t),y(t)) + 2\int_0^t |y'(s)|^2 ds \\ &= a(0;y_0,y_0) + \int_0^t a'(s;y(s),y(s))ds + 2\int_0^t (F(s),y'(s))ds \\ &-2\int_0^t k(t,s;y(s),y(t))ds + 2\int_0^t k(s,s;y(s),y(s))ds \\ &+2\int_0^t \int_0^s \frac{\partial k}{\partial s}(s,\sigma;y(\sigma),y(s))d\sigma ds. \end{aligned} \tag{3.2}$$

Remark 1. Starting from Eq.(1), we must multiply by y' which, via integration by parts, formally gives (3.2). This is only formal procedure because $y \in W(V,H)$, so $y' \in L^2(0,T;H)$ which is not the dual space of $L^2(0,T;V')$. Therefore, to justify (3.2) we need to approximate y' by functions valued in V which are verified by double regularization as shown below.

Proof of Proposition 3.3. By Lemma 3.2 and the uniform boundedness theorem argument as in Dautray and Lions [6, p. 577], it is clear that $y(t) \in V$ for each $t \in [0,T]$. Thus all functions in (3.2) are meaningful for all $t \in [0,T]$. We shall show the energy equality (3.2). Let $\delta > 0$ and $t_0 \in (0,T)$ be fixed. We introduce a continuous function

$$\mathcal{O}_\delta(t) = \mathcal{O}(t) = \begin{cases} \frac{t}{\delta} & \text{in } [0,\delta], \\ 1 & \text{in } [\delta, t_0-\delta], \\ \frac{1}{\delta}(t_0 - t) & \text{in } [t_0-\delta, t_0], \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

and a step function

$$\mathcal{O}_0(t) = \begin{cases} 1 & \text{in } [0,t_0], \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Let $\{\rho_n\}_{n=1}^{\infty}$ be a regularizing sequence of even functions such that $\int_{\mathbf{R_t}} \rho_n(t)dt = 1$ and supp$\rho_n \subset [-\frac{1}{n}, \frac{1}{n}]$. We shall extend $A(t), K(t,s)$ and $F(t)$ for all $t, s \in \mathbf{R}$, with the same properties on $[0,T]$. Especially we can suppose $K(t,s) = 0$ for $(t,s) \in (\mathbf{R} \setminus [0,T]) \times (\mathbf{R} \setminus [0,T])$. In the same way we shall assume that y is defined on $\mathbf{R}$, which is permissible by extension by reflection. For the simplicity we shall denote $[\ ,\ ]$ the scalar product in H or the anti duality V, V', and we shall denote $(\ ,\ )$ the antiduality $L^2(\mathbf{R_t}; V), L^2(\mathbf{R_t}; V')$ or the scalar product on $L^2(\mathbf{R_t}; H)$. We fix n and set $\rho_n = \rho$. Let $\rho * \rho$ be the convolution of ρ and ρ in $L^2(\mathbf{R}_t)$. At first, we have

$$\left\{ \begin{array}{l} (A'(t)(\rho * (\mathcal{O}_0 y), \rho * (\mathcal{O}_0 y)) + 2(\rho * (\mathcal{O}_0 A y), \rho * (\mathcal{O}_0 y')) \\ +2(A(\rho * (\mathcal{O}_0 y)) - \rho * (A\mathcal{O}_0 y), \rho' * (\mathcal{O}_0 y)) \\ +2[(\rho * \rho * (\mathcal{O}_0 A y))(0), y(0)] - 2[(\rho * \rho * (\mathcal{O}_0 A y))(t_0), y(t_0)] = 0. \end{array} \right. \tag{3.5}$$

The equality (3.5) is derived directly by the following equality

$$\int_{\mathbf{R_t}} \frac{d}{dt}[A(t)((\rho * (\mathcal{O}y)), \rho * (\mathcal{O}y)]dt = 0$$

and passage $\delta \to 0$ as in Lions and Magenes [10, p. 277].

Next we shall prove

$$\left\{ \begin{array}{l} (\rho * (\mathcal{O}_0 K(\cdot,\cdot)y), \rho * (\mathcal{O}_0 y)) + (\rho * (\mathcal{O}_0 \int K'(\cdot,s)y(s)ds), \rho * (\mathcal{O}_0 y)) \\ +\ (\rho * (\mathcal{O}_0 \int K(\cdot,s)y(s)ds), \rho * (\mathcal{O}_0 y')) \\ -\ [(\mathcal{O}_0 \int K(\cdot,s)y(s)ds)(t_0), \rho * \rho * (\mathcal{O}_0 y)(t_0)] \\ -\ [\rho * \rho * (\mathcal{O}_0 \int K(\cdot,s)y(s)ds)(t_0), (\mathcal{O}_0 y)(t_0)] \\ +\ [\rho * \rho * (\mathcal{O}_0 \int K(\cdot,s)y(s)ds)(0), (\mathcal{O}_0 y)(0)] = 0, \end{array} \right. \tag{3.6}$$

where $\int$ means integration over $\mathbf{R_t}$. Starting from

$$\int_{\mathbf{R_t}} \frac{d}{dt}[\rho * (\mathcal{O} \int K(\cdot,s)z(s)ds), \rho * (\mathcal{O}z)]dt = 0,$$

we obtain that

$$\left\{ \begin{array}{l} (\rho * (\mathcal{O}' \int K(\cdot,s)y(s)ds), \rho * (\mathcal{O}y)) + (\rho * (\mathcal{O}K(\cdot,\cdot)y(\cdot)), \rho * (\mathcal{O}y)) \\ +\ (\rho * (\mathcal{O} \int K'(\cdot,s)y(s)ds), \rho * (\mathcal{O}y)) \\ +\ (\rho * (\mathcal{O} \int K(\cdot,s)y(s)ds), \rho * (\mathcal{O}'y)) \\ +\ (\rho * (\mathcal{O} \int K(\cdot,s)y(s)ds), \rho * (\mathcal{O}y')) = 0. \end{array} \right. \tag{3.7}$$

Now we let $\delta \to 0$ in (3.7). The first term of (3.7) may be written as

$$(\rho * (\mathcal{O}' \int K(\cdot,s)y(s)ds), \rho * ((\mathcal{O} - \mathcal{O}_0)y)) \\ +(\rho * (\mathcal{O}' \int K(\cdot,s)y(s)ds), \rho * (\mathcal{O}_0 y)). \tag{3.8}$$

Since

$$\begin{aligned}\|(\mathcal{O}-\mathcal{O}_0)y\|^2_{L^2(\mathbf{R}_t;V)} &= \int_0^\delta (\frac{s}{\delta})^2\|y(s)\|^2ds + \int_{t_0-\delta}^{t_0}(\frac{t_0-s}{\delta})^2\|y(s)\|^2ds \\ &\le 2\int_0^\delta(\frac{s}{\delta})^2 ds\|y\|^2_{L^\infty(0,t_0;V)} \\ &= \frac{2\delta}{3}\|y\|^2_{L^\infty(0,t_0;V)} \to 0 \qquad \text{as } \delta \to 0. \end{aligned} \tag{3.9}$$

Thus we see that

$$\rho * ((\mathcal{O}-\mathcal{O}_0)y) \to 0 \qquad \text{in} \quad L^\infty(\mathbf{R}_t;V). \tag{3.10}$$

Since $\int_{\mathbf{R_t}} |\mathcal{O}'|dt = 2$ and $\rho*(\mathcal{O}'\int K(\cdot,s)y(s)ds)$ is bounded in $L^1(\mathbf{R_t};V')$, then the first term of (3.8) goes to zero. Since ρ is even function we deduce that the second term in (3.8) is equal to

$$((\mathcal{O}'\int K(\cdot,s)y(s)ds),\rho*\rho*(\mathcal{O}_0y)),$$

which is also equal to

$$\begin{aligned} &\frac{1}{\delta}\int_0^\delta[\rho*\rho*(\mathcal{O}_0y)(t),\int_0^t K(t,s)y(s)ds]dt \\ -\;&\frac{1}{\delta}\int_{t_0-\delta}^{t_0}[\rho*\rho*(\mathcal{O}_0y)(t),\int_0^t K(t,s)y(s)ds]dt. \end{aligned} \tag{3.11}$$

Since the map

$$t \to [\rho*\rho*(\mathcal{O}_0y)(t),\int_0^t K(t,s)y(s)ds]$$

is continuous, so that (3.11) tends to

$$-[(\mathcal{O}_0\int K(\cdot,s)y(s)ds)(t_0),\rho*\rho*(\mathcal{O}_0y)(t_0)].$$

Here we note that

$$(\mathcal{O}_0\int K(\cdot,s)y(s)ds)(0)=0.$$

Similarly the fourth term in (3.7) tends to

$$[\rho*\rho*(\mathcal{O}_0\int K(\cdot,s)y(s)ds)(0),(\mathcal{O}_0y)(0)]-[\rho*\rho*(\mathcal{O}_0\int K(\cdot,s)y(s)ds)(t_0),(\mathcal{O}_0y)(t_0)].$$

Finally we add (3.5) and 2×(3.6), taking account of $A(t)y + \int_0^t K(t,s)y(s)ds = -y' + F(t)$, we obtain

$$\begin{cases} (A'(\rho * (\mathcal{O}_0 y)), \rho * (\mathcal{O}_0 y)) + 2((A(\rho * (\mathcal{O}_0 y)) - \rho * (A\mathcal{O}_0 y))', \rho * \mathcal{O}_0 y) \\ +2(\rho * (\mathcal{O}_0 K(\cdot,\cdot)y), \rho * (\mathcal{O}_0 y)) + 2(\rho * (\mathcal{O}_0 \int K'(\cdot,s)y(s)ds), \rho * (\mathcal{O}_0 y)) \\ +2(\rho * (\mathcal{O}_0 F(\cdot), \rho * (\mathcal{O}_0 y')) - 2(\rho * (\mathcal{O}_0 y'), \rho * (\mathcal{O}_0 y')) \\ -2[(\mathcal{O}_0 \int K(\cdot,s)y(s)ds)(t_0), \rho * \rho * (\mathcal{O}_0 y)(t_0)] \\ -2[\rho * \rho * (\mathcal{O}_0 \int K(\cdot,s)y(s)ds)(t_0), (\mathcal{O}_0 y)(t_0)] \\ +2[\rho * \rho * (\mathcal{O}_0 Ay)(0), y(0)] - 2[\rho * \rho * (\mathcal{O}_0 Ay)(t_0), y(t_0)] \\ +2[\rho * \rho * (\mathcal{O}_0 \int K(\cdot,s)y(s)ds)(0), (\mathcal{O}_0 y)(0)] = 0, \end{cases} \tag{3.12}$$

where $K'(\cdot,s) = \frac{dK}{dt}(\cdot,s)$. We set $\rho = \rho_n$ in (3.12), according to the vector Friedrichs' Lemma, the second term of (3.12) tends to 0 as $n \to \infty$. The first term tends to

$$(A'\mathcal{O}_0 y, \mathcal{O}_0 y), \tag{3.13}$$

and the third and fourth terms tend to

$$2(\mathcal{O}_0 K(\cdot,\cdot)y, \mathcal{O}_0 y) + 2(\mathcal{O}_0 \int K'(\cdot,s)y(s)ds, \mathcal{O}_0 y). \tag{3.14}$$

Moreover the fifth and sixth terms tend to

$$2((\mathcal{O}_0 F(\cdot)), (\mathcal{O}_0 y')) - 2((\mathcal{O}_0 y'), (\mathcal{O}_0 y')). \tag{3.15}$$

If we set $\gamma_n = \rho_n * \rho_n$, we can see that it is an even function that satisfies

$$\int_0^{t_0} \gamma_n(t)dt = \frac{1}{2}, \tag{3.16}$$

where n is sufficiently large. Thus, for sufficiently large n,

$$2[\gamma_n * (\mathcal{O}_0 Ay(t_0), y(t_0)] = 2\int_0^{t_0} \gamma_n(t)[Ay(t_0 - t), y(t_0)]dt.$$

Hence

$$\begin{aligned} 2[\gamma_n * (\mathcal{O}_0 Ay(t_0), y(t_0)] \quad - \quad & [Ay(t_0), y(t_0)] \\ = \quad & 2\int_0^{t_0} \gamma_n(t)[(Ay)(t_0 - t) - (Ay)(t_0), y(t_0)]dt \\ \to \quad & 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Therefore for the tenth term of (3.12) we deduce

$$2[\gamma_n * (\mathcal{O}_0 Ay)(t_0), y(t_0)] \to a(t_0; y(t_0), y(t_0)) \tag{3.17}$$

as $n \to \infty$. Similarly, as $n \to \infty$, we can also deduce for the remainder of (3.12) as follows:

$$\begin{aligned} & -2[(\mathcal{O}_0 \int K(\cdot,s)y(s)ds)(t_0), \gamma_n * (\mathcal{O}_0 y)(t_0)] \\ \to \quad & -\int_0^{t_0} k(t_0,s;y(s),y(t_0))ds; \end{aligned} \tag{3.18}$$

$$2[\gamma_n * (\mathcal{O}_0 \int K(\cdot,s)y(s)ds)(0), (\mathcal{O}_0 y)(0)] \to 0; \tag{3.19}$$

$$\begin{aligned} & -2[\gamma_n * (\mathcal{O}_0 \int K(\cdot,s)y(s)ds)(t_0), y(t_0)] \\ \to \quad & -\int_0^{t_0} k(t_0,s;y(s),y(t_0))ds; \end{aligned} \tag{3.20}$$

$$2[\gamma_n * (\mathcal{O}_0 Ay)(0), y(0)] \to a(0;y_0,y_0). \tag{3.21}$$

Combining from (3.12) to (3.21), we verify that the weak solution y of Eq.(1) satisfies the energy equality (3.2).

This completes the proof. □

We now pave the way for presenting the following main theorem.

Theorem 3.4. *Assume that a satisfies (2.1) and (2.2), k satisfies (2.3), $y_0 \in V$, $g \in L^2(0,T;H)$, and $f(t,y)$ satisfies (2.4). Then Eq.(1) admits a unique weak solution y. Moreover, the weak solution y belongs to $C([0,T];V)$.*

Proof. We divide the proof into five steps.

Step 1. *Approximate solutions.*

We use the Faedo-Galerkin approximation method. Since V is separable, there exists a complete orthonormal system $\{w_m\}_{m=1}^\infty$ in H such that $\{w_m\}_{m=1}^\infty$ is free and total in V. Let $\mathbf{N}$ be the set of natural numbers. For each $m \in \mathbf{N}$ we define an approximate solution of the equation (1) by

$$y_m(t) = \sum_{j=1}^m g_{jm}(t) w_j,$$

where $y_m(t)$ satisfies

$$\begin{cases} (y_m'(t), w_j) + a(t;y_m(t),w_j) + \displaystyle\int_0^t k(t,s;y_m(s),w_j)ds \\ \qquad\qquad = (f(t,y_m(t)),w_j) + \langle g(t),w_j\rangle, \quad t \in [0,T], \quad 1 \le j \le m, \\ y_m(0) = y_{0m} = \displaystyle\sum_{j=1}^m (y_0,w_j)w_j. \end{cases} \tag{3.22}$$

We see easily that

$$V_m = \text{Span}\,\{w_1,\cdots,w_m\} \ni y_{0m} \to y_0 \text{ in } V \text{ as } m \to \infty. \tag{3.23}$$

By standard manipulation the equations (3.22) induce the following system of first order Volterra integro-differential equations for $g_{jm}(t)$.

$$\frac{d}{dt}\vec{g}_m + \tilde{A}(t)\vec{g}_m + \int_0^t \tilde{K}(t,s)\vec{g}_m(s)ds = \vec{f}_m(t,\vec{g}_m) + \vec{G}_m$$

with initial values $g_{jm}(0) = (y_0, w_j), j = 1, \cdots, m,$. where

$$\begin{aligned}
&\tilde{A}(t) = \left(a(t; w_i, w_j) :_{j=1,\cdots,m}^{i=1,\cdots,m}\right), \quad \tilde{K}(t,s) = \left(k(t,s; w_i, w_j) :_{j=1,\cdots,m}^{i=1,\cdots,m}\right) \\
&\vec{f}_m(t,\vec{g}_m) = [(f(t, \Sigma_{j=1}^m g_{jm} w_j), w_1), \cdots, (f(t, \Sigma_{j=1}^m g_{jm} w_j), w_m)]^t, \\
&\vec{G}_m = [(g(t), w_1), \cdots, (g(t), w_m)]^t.
\end{aligned}$$

$[\cdots]^t$ denotes the transpose of $[\cdots]$. By (2.4) it is verified that the system admits a unique solution $g_{jm}(t), j = 1, \cdots, m$ over $[0,T]$. Hence we can construct the approximate solution $y_m(t)$ of (3.22).

Step 2. *A priori estimates.*
We shall derive a priori estimates of $y_m(t)$. We multiply both sides of the equation (3.22) by $g'_{jm}(t)$ and sum over j to have

$$\begin{aligned}
&(y'_m(t), y'_m(t)) + a(t; y_m(t), y'_m(t)) + \int_0^t k(t,s; y_m(s), y'_m(t))ds \\
= \;& (f(t, y_m(t)), y'_m(t)) + (g(t), y'_m(t)). \qquad (3.24)
\end{aligned}$$

Using

$$a(t; y_m(t), y'_m(t)) = \frac{1}{2}\frac{d}{dt}a(t; y_m(t), y_m(t)) - \frac{1}{2}a'(t; y_m(t), y_m(t)), \qquad (3.25)$$

$$\begin{aligned}
\int_0^t k(t,s; y_m(s), y'_m(t))ds = \frac{d}{dt}\int_0^t k(t,s; y_m(s), y_m(t))ds \\
-k(t,t; y_m(t), y_m(t)) - \int_0^t \frac{\partial k}{\partial t}(t,s; y_m(s), y_m(t))ds, \qquad (3.26)
\end{aligned}$$

(3.24) can be written as

$$\begin{aligned}
&|y'_m(t)|^2 + \frac{1}{2}\frac{d}{dt}[a(t; y_m(t), y_m(t)) + 2\int_0^t k(t,s; y_m(s), y_m(t))ds] \\
= \;& \frac{1}{2}a'(t; y_m(t), y_m(t)) + (f(t, y_m(t)) + g(t), y'_m(t)) \\
&+ k(t,t; y_m(t), y_m(t)) + \int_0^t \frac{\partial k}{\partial t}(t,s; y_m(s), y_m(t))ds. \qquad (3.27)
\end{aligned}$$

Let us integrate (3.27) on $[0,t]$ and estimate it to obtain a priori estimates of $\{y_m\}$. Let $\epsilon > 0$ be an arbitrary real number. From ii) and iii) of (2.4) we obtain

$$\begin{aligned}
& 2\Big|\int_0^t (f(s,y_m(s))+g(s),y_m'(s))ds\Big| \\
= \; & 2\Big|\int_0^t (f(s,y_m(s))-f(s,0)+f(s,0)+g(s),y_m'(s))ds\Big| \\
\le \; & 2\int_0^t \beta(s)\|y_m(s)\||y_m'(s)|ds + 2\int_0^t (\gamma(s)+|g(s)|)|y_m'(s)|ds \\
\le \; & \frac{1}{\epsilon}(\|\gamma\|^2_{L^2(0,T;\mathbf{R}^+)}+\|g\|^2_{L^2(0,T;H)}) \\
& + \frac{1}{\epsilon}\int_0^t \beta(s)^2\|y_m(s)\|^2 ds + 3\epsilon\int_0^t |y_m'(s)|^2 ds. \qquad (3.28)
\end{aligned}$$

And from (2.3) we have

$$\begin{aligned}
\Big|\int_0^t k(t,s;y_m(s),y_m(t))ds\Big| \;&\le\; k_0\|y_m(t)\|\int_0^t \|y_m(s)\|ds \\
&\le\; \epsilon\|y_m(t)\|^2 + c(\epsilon)\int_0^t \|y_m(s)\|^2 ds \qquad (3.29)
\end{aligned}$$

for some $c(\epsilon)>0$. We can deduce from (2.3):

$$\begin{aligned}
\Big|\int_0^t\int_0^s \frac{\partial k}{\partial s}(s,\sigma;y_m(\sigma),y_m(s))d\sigma ds\Big| \;&\le\; k_1\Big(\int_0^t \|y_m(s)\|ds\Big)^2; \qquad (3.30) \\
\Big|\int_0^t k(s,s;y_m(s),y_m(s))ds\Big| \;&\le\; k_0\int_0^t \|y_m(s)\|^2 ds. \qquad (3.31)
\end{aligned}$$

Therefore by using (2.1), (2.2), (2.3) and (3.28)-(3.31), we can obtain the following inequality

$$\begin{aligned}
& \int_0^t |y_m'(s)|^2 ds + \alpha\|y_m(t)\|^2 \\
\le \; & a(0;y_{0m},y_{0m}) + \epsilon\|y_m(t)\|^2 \\
& + (c_2+k_0+c(\epsilon))\int_0^t \|y_m(s)\|^2 ds + k_1\Big(\int_0^t \|y_m(s)\|ds\Big)^2 \\
& + k_1\|y_m(t)\|\int_0^t \|y_m(s)\|ds + \frac{1}{\epsilon}(\|\gamma\|^2_{L^2(0,T;\mathbf{R}^+)}+\|g\|^2_{L^2(0,T;H)}) \\
& + \frac{1}{\epsilon}\int_0^t \beta(s)^2\|y_m(s)\|^2 ds + 3\epsilon\int_0^t |y_m'(s)|^2 ds, \qquad (3.32)
\end{aligned}$$

furthermore we deduce from (3.32)

$$\begin{aligned}
&\int_0^t |y_m'(s)|^2 ds + \alpha \|y_m(t)\|^2 \\
\leq\ & a(0; y_{0m}, y_{0m}) + 2\epsilon \|y_m(t)\|^2 + 3\epsilon \int_0^t |y_m'(s)|^2 ds \\
&+(c_2 + k_0 + c(\epsilon) + k_1 c'(\epsilon) + k_1 T) \int_0^t \|y_m(s)\|^2 ds \\
&+\frac{1}{\epsilon}(\|\gamma\|^2_{L^2(0,T;R^+)} + \|g\|^2_{L^2(0,T;H)}) + \frac{1}{\epsilon}\int_0^t \beta(s)^2 \|y_m(s)\|^2 ds. \quad (3.33)
\end{aligned}$$

If we put $\beta_1(s) := (c_2 + k_0 + c(\epsilon) + k_1 c'(\epsilon) + k_1 T + \frac{1}{\epsilon}\beta(s)^2) \in L^1(0, T; \mathbf{R}^+)$, then we arrive at

$$\begin{aligned}
&(1-3\epsilon)\int_0^t |y_m'(s)|^2 ds + (\alpha - 2\epsilon)\|y_m(t)\|^2 \\
\leq\ & C(y_0, \gamma, g) + \int_0^t \beta_1(s)\|y_m(s)\|^2 ds, \quad (3.34)
\end{aligned}$$

where C is constant and we choose $\epsilon < \min\{\frac{1}{6},\ \frac{\alpha}{4}\}$. Thus it follows by Bellman-Gronwall's inequality that

$$\begin{aligned}
&\|y_m(t)\|^2 + \int_0^t |y_m'(s)|^2 ds \\
\leq\ & C'(y_0, \gamma, g) \exp\Big(\int_0^T \frac{\beta_1(s)}{\min\{1-3\epsilon, \alpha - 2\epsilon\}} ds\Big). \quad (3.35)
\end{aligned}$$

Step 3. *Passage to the limit.*

From (3.35) we can deduce that y_m and y_m' remain in a bounded sets of $L^\infty(0, T; V)$ and $L^2(0, T; H)$ respectively. Hence we can extract a subsequence $\{y_{m_k}\}$ of $\{y_m\}$ and find $z \in L^\infty(0, T; V)$, $z' \in L^2(0, T; H)$ such that

$$\begin{aligned}
&\text{i)}\quad y_{m_k} \to z \ \text{ weakly-star in } \ L^\infty(0, T; V) \\
&\qquad\qquad \text{and weakly in } L^2(0, T; V), \quad (3.36) \\
&\text{ii)}\quad y_{m_k}' \to z' \ \text{ weakly in } L^2(0, T; H), \quad (3.37) \\
&\text{iii)}\quad A(\cdot)y_{m_k} \to A(\cdot)z \ \text{ weakly in } \ L^2(0, T; V'), \quad (3.38) \\
&\text{iv)}\quad K(t, \cdot)y_{m_k} \to K(t, \cdot)z \ \text{ weakly in } \ L^2(0, t; V'), \quad (3.39)
\end{aligned}$$

and by ii) and iii) of (2.4) we can induce the following inequality

$$\|f(\cdot, y_m(\cdot))\|^2_{L^2(0,T;H)} \leq 2\|y_m\|^2_{L^\infty(0,T;V)}\|\beta\|^2_{L^2(0,T)} + 2\|\gamma\|^2_{L^2(0,T)}.$$

Therefore, we deduce that there exists a $F(\cdot) \in L^2(0, T; H)$ such that

$$f(\cdot, y_{m_k}) \to F(t) \ \text{ weakly in } \ L^2(0, T; H), \quad (3.40)$$

as $k \to \infty$.

Let $\phi \in \mathcal{D}(0,T)$ and $v \in V$ be fixed. Since $\{w_m\}_{m=1}^{\infty}$ is free and total in V, there exists $\{v_m\}_{m\in\mathbf{N}}, v_m \in V_m = \text{Span }\{w_1, \cdots, w_m\}$ for all m such that $v_m \to v$ strongly in V. We introduce

$$\begin{cases} \psi_m &= \phi \otimes v_m \\ \psi &= \phi \otimes v, \end{cases} \tag{3.41}$$

and we particularly have

$$\psi_m \to \psi \text{ strongly in } \; L^2(0,T;V). \tag{3.42}$$

From (3.22) we have

$$\begin{aligned} &\int_0^T (y_m'(t), \psi_m(t))dt + \int_0^T a(t; y_m(t), \psi_m(t))dt \\ &+ \int_0^T \int_0^t k(t,s; y_m(s), \psi_m(t))dsdt \\ = &\int_0^T (f(t, y_m(t)), \psi_m(t))dt + \int_0^T \langle g(t), \psi_m(t)\rangle dt. \end{aligned} \tag{3.43}$$

By letting $m = m_k \to \infty$ in (3.43) and using (3.36)-(3.40) we deduce

$$\begin{aligned} \int_0^T (z'(t), \psi(t))dt + \int_0^T a(t; z(t), \psi(t))dt + \int_0^T \int_0^t k(t,s; z(s), \psi(t))dsdt \\ = \int_0^T (F(t), \psi(t))dt + \int_0^T \langle g(t), \psi(t)\rangle dt. \end{aligned} \tag{3.44}$$

Hence

$$\begin{aligned} \int_0^T (z'(t), v)\phi(t)dt + \int_0^T \Big[a(t; z(t), v) + \int_0^t k(t,s; z(s), v)ds\Big]\phi(t)dt \\ = \int_0^T (F(t), v)\phi(t)dt + \int_0^T \langle g(t), v\rangle \phi(t)dt. \end{aligned}$$

This shows that z satisfies for all $v \in V$

$$(z'(\cdot), v) + a(\cdot; z(\cdot), v) + \int_0^{\cdot} k(\cdot; z(s), v)ds = (F(\cdot), v) + \langle g(\cdot), v\rangle \tag{3.45}$$

in the sense of $\mathcal{D}'(0,T)$. Obviously, (3.36) and (3.37) imply $z \in W(V,H)$.

Let $\phi \in C^1[0,T]$ with $\phi(0) \neq 0$ and $\phi(T) = 0$. Then we have

$$\int_0^T (y_m'(t), \psi_m(t))dt = -(y_{0m}, \psi_m(0)) - \int_0^T (y_m(t), \psi_m'(t))dt, \tag{3.46}$$

from which by the passage of $m = m_k \to \infty$,

$$\int_0^T (z'(t), \psi(t))dt = -(y_0, \psi(0)) - \int_0^T (z(t), \psi'(t))dt. \tag{3.47}$$

Moreover, we can see that

$$\int_0^T (z'(t), \psi(t))dt = -(z(0), \psi(0)) - \int_0^T (z(t), \psi'(t))dt. \tag{3.48}$$

Thus by the comparison of (3.47) with (3.48), we deduce

$$(z(0), v) = (y_0, v) \quad \text{for all } v \in V, \tag{3.49}$$

which implies $z(0) = y_0$. This shows that z is a weak solution of Eq.(1) with $f(t, z) = F(t)$.

Step 4. *Strong convergence of approximate solutions*
The main difficulty is showing $F(\cdot) = f(\cdot, z(\cdot))$. We shall show $y_m \to z$ strongly in $L^2(0, T; V)$. Integrating (3.24) on $[0, t]$, we obtain

$$\begin{aligned} & a(t; y_m(t), y_m(t)) + 2\int_0^t |y_m'(s)|^2 ds \\ = \; & a(0; y_{m0}, y_{m0}) + \int_0^t a'(s; y_m(s), y_m(s))ds - 2\int_0^t k(t, s; y_m(s), y_m(t))ds \\ & +2\int_0^t k(s, s; y_m(s), y_m(s))ds + 2\int_0^t (f(s, y_m(s)) + g(s), y_m'(s))ds \\ & +2\int_0^t \int_0^s \frac{\partial k}{\partial s}(s, \sigma; y_m(\sigma), y_m(s))d\sigma ds. \end{aligned} \tag{3.50}$$

By Proposition 3.3, we see that the weak solution z of Eq.(1) with $f(t, z) = F(t)$ satisfies the following energy equality

$$\begin{aligned} & a(t; z(t), z(t)) + 2\int_0^t |z'(s)|^2 ds \\ = \; & a(0; y_0, y_0) + \int_0^t a'(s; z(s), z(s))ds - 2\int_0^t k(t, s; z(s), z(t))ds \\ & + 2\int_0^t k(s, s; z(s), z(s))ds + 2\int_0^t (F(s) + g(s), z'(s))ds \\ & + 2\int_0^t \int_0^s \frac{\partial k}{\partial s}(s, \sigma; z(\sigma), z(s))d\sigma ds. \end{aligned} \tag{3.51}$$

Adding (3.50) to (3.51), we have

$$\begin{aligned}
& a(t; y_m(t) - z(t), y_m(t) - z(t)) + 2\int_0^t |y'_m(s) - z'(s)|^2 ds \\
= \; & \sum_{i=1}^{7} \Phi_m^i(t) + a(0; y_{m0} - y_0, y_{m0} - y_0) \\
& + \int_0^t a'(s; y_m(s) - z(s), y_m(s) - z(s)) ds \\
& - 2\int_0^t k(t, s; y_m(s) - z(s), y_m(t) - z(t)) ds \\
& +2\int_0^t k(s, s; y_m(s) - z(s), y_m(s) - z(s)) ds \\
& + 2\int_0^t (f(s, y_m) - f(s, z), y'_m(s) - z'(s)) ds \\
& + 2\int_0^t \int_0^s \frac{\partial k}{\partial s}(s, \sigma; y_m(\sigma) - z(\sigma), y_m(s) - z(s)) d\sigma ds, \qquad (3.52)
\end{aligned}$$

where

$$\begin{aligned}
\Phi_m^1 &= 2a(0; y_{0m}, y_0), \\
\Phi_m^2 &= -2a(t; y_m, z) - 4\int_0^t (y'_m(s), z'(s)) ds, \\
\Phi_m^3 &= 2\int_0^t a'(s; y_m, z) ds, \\
\Phi_m^4 &= -2\int_0^t k(t, s; y_m(s), z(t)) ds - 2\int_0^t k(t, s; z(s), y_m(t)) ds, \\
\Phi_m^5 &= 2\int_0^t k(s, s; y_m(s), z(s)) ds + 2\int_0^t k(s, s; z(s), y_m(s)) ds, \\
\Phi_m^6 &= 2\int_0^t \int_0^s (\frac{\partial k}{\partial s}(s, \sigma; y_m(\sigma), z(s)) + \frac{\partial k}{\partial s}(s, \sigma; z(\sigma), y_m(s))) d\sigma ds, \\
\Phi_m^7 &= 2\int_0^t (f(s, y_m(s)), z'(s)) + (F(s), y'_m(s))) ds \\
& \quad +2\int_0^t (f(s, z(s)) - F(s), y'_m(s) - z'(s)) ds \\
& \quad +2\int_0^t (g, z') ds + 2\int_0^t (g, y'_m) ds.
\end{aligned}$$

For simplicity we set

$$\Phi_m(t) = \sum_{i=1}^{7} \Phi_m^i(t).$$

By (3.35) and (3.36), we deduce for any fixed $t \in [0, T]$ it follows that

$$y_m(t) \;\to\; \xi_t \quad \text{weakly in} \quad V \quad \text{as } m \to \infty. \tag{3.53}$$

By similar reasoning to that done to clarify (3.49) (cf. Dautray and Lions [6, p. 579]), we verify that

$$\xi_t = z(t). \tag{3.54}$$

Thus together with (3.37) we have that

$$\Phi_m^2(t) \;\to\; -2a(t; z(t), z(t)) - 4\int_0^t |z'(s)|^2 ds \tag{3.55}$$

as $m \to \infty$. It is clear from (3.23) that

$$\Phi_m^1 \;\to\; 2a(0; y_0, y_0) \quad \text{as} \quad m \to \infty. \tag{3.56}$$

From (3.36)-(3.39) it follows immediately that

$$\begin{aligned}
\Phi_m^3(t) \quad &\to \quad 2\int_0^t a'(s; z, z)ds, &(3.57)\\
\Phi_m^4(t) \quad &\to \quad -4\int_0^t k(t, s; z(s), z(t))ds, &(3.58)\\
\Phi_m^5(t) \quad &\to \quad 4\int_0^t k(s, s; z(s), z(s))ds, &(3.59)\\
\Phi_m^6(t) \quad &\to \quad 4\int_0^t \int_0^s \frac{\partial k}{\partial t}(s, \sigma; z(\sigma), z(s))d\sigma ds, &(3.60)\\
\Phi_m^7(t) \quad &\to \quad 4\int_0^t (F(s) + g(s), z'(s))ds &(3.61)
\end{aligned}$$

as $m \to \infty$. Hence by (3.55)-(3.61) and the energy equality (3.2) for z, we have

$$\Phi_m(t) \;\to\; 0 \qquad \text{as} \;\; m \;\to\; \infty. \tag{3.62}$$

Then, by routine estimation of each term of 3.52), we have

$$\begin{aligned}
&(\alpha - \epsilon)\|y_m(t) - z(t)\|^2 + (2 - \epsilon)\int_0^t |y_m'(s) - z'(s)|^2 ds\\
&\le \Phi_m(t) + c_0\|y_{0m} - y_0\|^2 + \int_0^t h(s)\|y_m(s) - z(s)\|^2 ds, \qquad (3.63)
\end{aligned}$$

where $h(s) = c_1 + \frac{T}{\epsilon}4k_0^2 + 2k_0 + \frac{2}{\epsilon}\beta(s)^2 + 2k_1 T \;\in\; L^1(0, T; \mathbf{R}^+)$. We divide (3.63) by $\alpha - \epsilon > 0$ and if we set

$$\begin{aligned}
M_m(t) &= \|y_m(t) - z(t)\|^2, &(3.64)\\
\Psi_m(t) &= \frac{1}{\alpha - \epsilon}\Phi_m(t) + \frac{c_0}{\alpha - \epsilon}\|y_{m0} - y_0\|^2, &(3.65)\\
h_1(s) &= \frac{1}{\alpha - \epsilon}h(s), &(3.66)
\end{aligned}$$

then we can have

$$M_m(t) \le \Psi_m(t) + \int_0^t h_1(s) M_m(s) ds. \tag{3.67}$$

By the Gronwall's lemma, we can induce the following inequality

$$M_m(t) \le \Psi_m(t) + \exp(\|h_1\|_{L^1(0,T;\mathbf{R}^+)}) \int_0^t \Psi_m(s) h_1(s) ds. \tag{3.68}$$

Thanks to (3.62) and (3.23) we have

$$|\Psi_m(t)| \to 0 \ \text{ when } m \ \to \ \infty, \tag{3.69}$$

and we can easily deduce that

$$|\Psi_m(t)| < C \quad \text{ for all } \ t \in [0, T], \tag{3.70}$$

where C is a constant. Thus by (3.68)-(3.70) and the Lebesgue dominated convergence theorem, we can obtain

$$\limsup_{m\to\infty} M_m(t) = 0 \ \text{ for all } \ t \in [0, T]. \tag{3.71}$$

Therefore we can know for all $t \in [0, T]$ that

$$y_m(t) \ \to \ z(t) \ \text{ strongly in } V \quad \text{as } \ m \to \infty. \tag{3.72}$$

Moreover with the condition ii) of (2.4) and (3.72) it is readily followed that

$$F(\cdot) = f(\cdot, z(\cdot)). \tag{3.73}$$

Therefore we have proved the existence of a weak solutions and strong convergent of them.

Step 5. *Regularity and uniqueness of weak solutions*
By Proposition 3.3 we verify that the weak solution z of Eq.(1) satisfies the energy equality (3.51) in which $F(\cdot) = f(\cdot, z(\cdot))$ and observe that all terms of (3.51) except $a(t; z(t), z(t))$ are integral terms with absolutely integrable integrands. This implies the map $t \ \to \ a(t; z(t), z(t))$ is continuous on $[0, T]$. Therefore using the scalar continuity of z, i.e. $z \in C_s([0, T]; V)$, and referring to the proof as in Lions and Magenes [10, pp. 278 - 279], we deduce that

$$z \in C([0, T]; V).$$

Finally the uniqueness of the weak solutions follows directly from the equality (3.51). Indeed, let y_1 and y_2 be the weak solutions of Eq.(1) and $y := y_1 - y_2$.

Then by Proposition 3.3 we deduce y satisfies

$$\begin{aligned}
&a(t;y(t),y(t)) + 2\int_0^t |y'(s)|^2 ds \\
= &\int_0^t a'(s;y(s),y(s))ds - 2\int_0^t k(t,s;y(s),y(t))ds \\
&+ 2\int_0^t k(s,s;y(s),y(s))ds + 2\int_0^t (f(s,y_1(s)) - f(s,y_2(s)), y'(s))ds \\
&+ 2\int_0^t \int_0^s \frac{\partial k}{\partial s}(s,\sigma;y(\sigma),y(s))d\sigma ds. \qquad (3.74)
\end{aligned}$$

If we estimate (3.74) as we did before, we can arrive at

$$\|y(t)\|^2 + \int_0^t |y'(s)|^2 ds \le C_1 \int_0^t \sigma(s)\|y(s)\|^2 ds, \qquad (3.75)$$

where C_1 is a constant and $\sigma \in L^1(0,T;\mathbf{R}^+)$. Finally, applying the Gronwall's inequality to (3.75), we have $y = 0$.

This concludes the proof. □

4. Applications to partial differential equations

Example 1. We set $V = H_0^1(\Omega)$ with the inner product and norm are defined by $(\phi,\varphi)_{H_0^1(\Omega)} = \int_\Omega \nabla\phi(x)\cdot\nabla\varphi(x)dx$, $\|\phi\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla\phi(x)|^2 dx$, respectively, and we put $H = L^2(\Omega)$. For each $t \in [0,T]$, we consider a family of bilinear forms on $V \times V$ by

$$a(t;\phi,\psi) = \sum_{i,j=1}^n \int_\Omega a_{ij}(t,x)\frac{\partial\phi(x)}{\partial x_i}\frac{\partial\psi(x)}{\partial x_j}dx, \qquad \forall\phi,\psi \in V, \quad (4.1)$$

$$k(t;\phi,\psi) = \sum_{i=1}^n \int_\Omega b_{ij}(t,x)\frac{\partial\phi(x)}{\partial x_i}\frac{\partial\psi(x)}{\partial x_j}dx, \qquad \forall\phi,\psi \in V, \quad (4.2)$$

where

$$\begin{cases}
a_{ij} = a_{ji}, & \text{for all } i,j = 1,2,...,n, \\
a_{ij}, b_{ij} \in C^1([0,T];L^\infty(\Omega)), & \text{for all } i,j = 1,2,...,n, \\
\displaystyle\sum_{i,j=1}^n a_{ij}(t,x)\xi_i\xi_j \ge \alpha\sum_{i=1}^n \xi_i^2, & \exists\alpha > 0,\ \forall\xi_i \in \mathbf{R}.
\end{cases} \qquad (4.3)$$

(4.3) implies that bilinear families $a(t;\phi,\psi)$ satisfies (2.1), (2.2) and $k(t;\phi,\psi)$ satisfies (2.3). Now we define the differential operators associated with (4.1) and

(4.2) by

$$A(t,x,\frac{\partial}{\partial x}) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}(t,x)\frac{\partial}{\partial x_j}),$$

$$K(t,x,\frac{\partial}{\partial x}) = -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(b_{ij}(t,x)\frac{\partial}{\partial x_j}), \qquad \forall (t,x) \in Q. \quad (4.4)$$

Let us define a scalar function $f : [0,T] \times \Omega \times \mathbf{R} \times \mathbf{R}^n \to \mathbf{R}$ satisfying the following conditions:

(i) $f(\cdot,\cdot,\xi,\eta)$ is measurable on $[0,T]\times\Omega$ for each $\xi \in \mathbf{R}$ and $\eta \in \mathbf{R}^n$,

(ii) there is a $\beta \in L^2(0,T;\mathbf{R}^+)$ such that for each $t \in [0,T]$ and $x \in \Omega$

$$|f(t,x,\xi,\eta,) - f(t,x,\xi',\eta')| \le \beta(t)(|\xi-\xi'|+|\eta-\eta'|), \quad \forall \xi,\xi' \in \mathbf{R},\ \forall \eta,\eta' \in \mathbf{R}^n,$$

(iii) there is a $\gamma \in L^2(0,T;\mathbf{R}^+)$ such that $|f(t,x,0,0)| \le \gamma(t), \quad \forall (t,x) \in Q.$

We define a function $\hat{f} : [0,T] \times H \times V \to H$ by

$$\hat{f}(t,y,\nabla y)(x) = f(t,x,y(x),\nabla y(x)), \qquad \forall x \in \Omega.$$

Then we have

$$\begin{aligned} |\hat{f}(t,y_1,\nabla y_1) - \hat{f}(t,y_2,\nabla y_2)|^2 &= \int_\Omega (\hat{f}(t,y_1,\nabla y_1) - \hat{f}(t,y_2,\nabla y_2))^2 dx \\ &\le 2\beta^2(t) \int_\Omega (|y_1(x) - y_2(x)|^2 + |\nabla y_1(x) - \nabla y_2(x)|^2) dx \\ &\le (c^2+1)2\beta^2(t) \int_\Omega |\nabla y_1(x) - \nabla y_2(x)|^2 dx, \end{aligned}$$

where c is a constant by Poincaré's inequality. Therefore the condition ii) of (2.4) is satisfied. Furthermore we assume that $g \in L^2(0,T;H)$ and $y_0 \in V$. Then by the Theorem 3.4, there exists a unique weak solution $y \in W(V,H)$ and y satisfies the Dirichlet problem

$$\begin{cases} \dfrac{\partial y}{\partial t} + A(t,x,\dfrac{\partial}{\partial x})y + \displaystyle\int_0^t K(t-s,x,\dfrac{\partial}{\partial x})y(s)ds = f(t,x,y,\nabla y) + g(t,x) \ \text{ in } \ Q, \\ y = 0 \ \text{ on } \ \Sigma, \\ y(0,x) = y_0(x) \quad \text{ in } \ \Omega. \end{cases} \quad (4.5)$$

If we take $V = H^1(\Omega)$. Then there exists a unique weak solution $y \in W(V,H) \cap C([0,T];V)$ satisfying

$$\begin{cases} \dfrac{\partial y}{\partial t} + A(t,x,\dfrac{\partial}{\partial x})y + \displaystyle\int_0^t K(t-s,x,\dfrac{\partial}{\partial x})y(s)ds = f(t,x,y,\nabla y) + g(t,x) \ \text{ in } \ Q, \\ \dfrac{\partial y}{\partial \nu_{A(t)}} + \displaystyle\int_0^t \dfrac{\partial y(s)}{\partial \nu_{K(t-s)}} ds = 0 \ \text{ on } \ \Sigma, \\ y(0,x) = y_0(x) \quad \text{ in } \ \Omega, \end{cases} \quad (4.6)$$

where

$$\frac{\partial y}{\partial \nu_{A(t)}}+\int_0^t \frac{\partial y(s)}{\partial \nu_{K(t-s)}}ds=-\sum_{i=1}^n[\sum_{j=1}^n a_{ij}(t,x)\frac{\partial y}{\partial x_j}+\int_0^t b_{ij}(t-s,x)\frac{\partial y(s)}{\partial x_j}ds]\cos(\mathrm{n}_i).$$

Example 2. Let $H=L^2(\Omega)$ and let us define a Hilbert space

$$H(\Delta;\Omega)=\{\phi\in L^2(\Omega)\mid \ \Delta\phi\in L^2(\Omega)\} \tag{4.7}$$

with the inner product $(\cdot,\cdot)$ define by

$$(\phi,\psi)=\int_\Omega \phi(x)\psi(x)dx+\int_\Omega \Delta\phi(x)\Delta\psi(x)dx.$$

Let V be a Hilbert space which satisfies $H_0^2(\Omega)\subset V\subset H(\Delta;\Omega)$ with the inner product of $H(\Delta;\Omega)$ and the corresponding norm $\|\phi\|:=|\phi|+|\Delta\phi|$.

In this case we consider the fourth order equation which involve nonlinear functions associated with gradient and Laplacian terms.
Let V be a Hilbert space which satisfies $H_0^2(\Omega)\subset V\subset H(\Delta;\Omega)$ with the inner product of $H(\Delta;\Omega)$. We consider the bilinear forms

$$a(t;\phi,\psi)=\int_\Omega a(t,x)\Delta\phi(x)\Delta\psi(x)dx,\quad \forall\phi,\psi\in V\subset H(\Delta;\Omega), \tag{4.8}$$

$$k(t;\phi,\psi)=\int_\Omega k(t)\Delta\phi(x)\Delta\psi(x)dx,\quad \forall\phi,\psi\in V\subset H(\Delta;\Omega), \tag{4.9}$$

where $k(\cdot)\in C^1([0,T])$ and $a\in C^1([0,T];L^\infty(\Omega))$, furthermore $a(t,x)\ge\alpha>0$ for all $(t,x)\in Q$. Then it is easy to verify that a and k satisfies (2.1), (2.2) and (2.3), respectively. We consider the function $f:[0,T]\times\Omega\times\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}\to\mathbf{R}$ satisfying the following conditions:

(i) $f(\cdot,\cdot,y,z,\xi)$ is measurable on Q for each $(y,z,\xi)\in\mathbf{R}\times\mathbf{R}^n\times\mathbf{R}$,

(ii) there is a $\beta\in L^2(0,T;\mathbf{R}^+)$ such that
$$|f(t,x,y,z,\xi)-f(t,x,y',z',\xi')|\le\beta(t)(|y-y'|+|z-z'|+|\xi-\xi'|),$$
$$\forall(t,x)\in Q,\quad \forall y,y',\xi,\xi'\in\mathbf{R},\quad \forall z,z'\in\mathbf{R}^n,$$

(iii) there is a $\gamma\in L^2(0,T;\mathbf{R}^+)$ such that
$$|f(t,x,0,0,0)|\le\gamma(t),\quad \forall(t,x)\in Q.$$

We remark that the norm $\|\cdot\|$ on V is equivalent to the norm of $H^2(\Omega)$, i.e., there exists $c_1>0$ such that

$$\|\phi\|_{H^2(\Omega)}\le c_1\|\phi\|,\quad \forall\phi\in H^2(\Omega).$$

For the above f, we define the nonlinear function $\bar{f} : [0,T] \times V \to H$ by $\bar{f}(t,y)(x) = f(t,x,y(x),\nabla y(x),\Delta y(x)),$ *a.e.* $x \in \Omega$. Then by (ii),

$$\begin{aligned} & |\bar{f}(t,y_1) - \bar{f}(t,y_2)|^2 \\ = \ & \int_\Omega (f(t,x,y_1,\nabla y_1,\Delta y_1) - f(t,x,y_2,\nabla y_2,\Delta y_2))^2 dx \\ \le \ & 3\beta^2(t) \int_\Omega (|y_1 - y_2|^2 + |\nabla y_1 - \nabla y_2|^2 + |\Delta y_1 - \Delta y_2|^2) dx \\ \le \ & 3\beta^2(t) \|y_1 - y_2\|^2_{H^2(\Omega)} \le 3c_1\beta^2(t)\|y_1 - y_2\|^2. \end{aligned}$$

From this we can show that the nonlinear term $\bar{f}$ satisfies ii) of (2.4). Hence the result of Theorem 3.4 can be applied. Hence we can verify that there exists a unique weak solution $y \in W(V,H) \cap C([0,T];V)$ of

$$\begin{cases} \dfrac{\partial y}{\partial t} + \Delta(a(t,x)\Delta y) + \displaystyle\int_0^t k(t-s)\Delta^2 y(s)ds = f(t,x,y,\nabla y,\Delta y) + g(t,x) \ \text{ in } Q, \\ y(0,x) = y_0(x) \ \text{ in } \Omega, \end{cases} \tag{4.10}$$

where $y_0 \in V$ and $g \in L^2(Q)$.

As in Case 3, we consider the various cases of the space V according to different boundary conditions.

Case 1. $V = H^2_0(\Omega)$ **(Dirichlet boundary condition)**
There exists a unique weak solution $y \in W(V,H) \cap C([0,T];V)$ of (4.10).

Case 2. $V = \{\phi \in H(\Delta;\Omega) \mid \phi = 0$ **on** $\Gamma\}$ **(Mixed boundary condition)**
There is a unique weak solution $y \in W(V,H) \cap C([0,T];V)$ of

$$\begin{cases} \dfrac{\partial y}{\partial t} + \Delta(a(t,x)\Delta y) + \displaystyle\int_0^t k(t-s)\Delta^2 y(s)ds = f(t,x,y,\nabla y,\Delta y) + g(t,x) \ \text{ in } Q \\ y(t,x) = 0, \ \Delta y(t,x) = 0 \qquad \text{on } \Sigma \\ y(0,x) = y_0(x) \qquad \text{in } \Omega. \end{cases} \tag{4.11}$$

Case 3. $V = \{\phi \in H(\Delta;\Omega) \mid \dfrac{\partial}{\partial \mathrm{n}}\phi = 0$ **on** $\Gamma\}$ **(Mixed boundary condition)**
There exists a unique weak solution $y \in W(V,H) \cap C([0,T];V)$ of

$$\begin{cases} \dfrac{\partial y}{\partial t} + \Delta(a(t,x)\Delta y) + \displaystyle\int_0^t k(t-s)\Delta^2 y(s)ds = f(t,x,y,\nabla y,\Delta y) + g(t,x) \ \text{ in } Q \\ \dfrac{\partial}{\partial \mathrm{n}}(a\Delta y) + k * \dfrac{\partial}{\partial \mathrm{n}}\Delta y = 0, \ \dfrac{\partial}{\partial \mathrm{n}} y(t,x) = 0 \ \text{ on } \Sigma \\ y(0,x) = y_0(x) \text{ in } \Omega, \end{cases} \tag{4.12}$$

where $*$ is a convolution.

5. Conclusive remarks

Through the paper, we understand mathematical concepts, principles, and laws related to various physical phenomena, recognize the value of mathematics, and think and communicate mathematically to solve various problems. It also develops a mathematical attitude toward mathematical thinking and develops creative problem-solving skills. The mathematics educational significance of this paper can be summarized as follows.

- (i) It was an opportunity to understand the structure of mathematical problems related to physical phenomena and to actively and confidently solve various problems by utilizing related mathematics.
- (ii) It served as an opportunity to communicate mathematical thinking and strategies related to phenomena and to recognize the convenience of mathematical expressions.
- (iii) It was an opportunity to explore the connection between mathematical concepts, principles, and laws, and to recognize the usefulness of mathematics by applying it to real life.
- (iv) It was an opportunity to mathematically analyze the model of the phenomenon and infer reasonable results based on that information.

The above matters are evaluated as important research in terms of mathematics education in that they are consistent with the goals pursued by current mathematics education.

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