Honam Mathematical J. **46** (2024), No. 3, pp. 515–521
https://doi.org/10.5831/HMJ.2024.46.3.515

A REFINEMENT OF THE THIRD HANKEL DETERMINANT FOR CLOSE-TO-CONVEX FUNCTIONS

Laxmipriya Parida, Teodor Bulboacă, and Ashok Kumar Sahoo*

Abstract. In our paper, by using different inequalities regarding the coefficients of the normalized close-to-convex functions in the open unit disk, we found a smaller upper bound of the third Hankel determinant for the class of close-to-convex functions as compared with those obtained by Prajapat et. al. in 2015.

1. Introduction and Main Definitions

Let denote by $\mathcal{H}(\mathbb{D})$ the class of functions which are analytic in the open unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, and let $\mathcal{A}$ be the subclass of $\mathcal{H}(\mathbb{D})$ consisting of the functions f normalized with $f(0) = f'(0) - 1 = 0$, that is

$$f(z) = z + \sum_{k=2}^{\infty} a_k\, z^k,\ z \in \mathbb{D}. \tag{1}$$

Let $\mathcal{P}$ be the well-known class of *Carathéodory functions*, that is $P \in \mathcal{H}(\mathbb{D})$ with the power series expansion

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n\, z^n,\ z \in \mathbb{D}, \tag{2}$$

such that $\operatorname{Re} P(z) > 0$, $z \in \mathbb{D}$.

The *Hankel determinant* for a given function f of the form (1) is defined by

$$H_{q,k}(f) := \begin{vmatrix} a_k & a_{k+1} & \dots & a_{k+q-1} \\ a_{k+1} & a_{k+2} & \dots & a_{k+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{k+q-1} & a_{k+q} & \dots & a_{k+2q-2} \end{vmatrix} \quad \left(a_1 = 1,\ q, k \in \mathbb{N} := \{1, 2, 3, \dots\}\right),$$

Received March 22, 2024. Accepted April 25, 2024.
2020 Mathematics Subject Classification.Primary 30C45, Secondary 30C80..
Key words and phrases. analytic functions, univalent functions, Carathéodory functions, Hankel determinant, Fekete-Szegő functional, starlike functions, close-to-convex functions, Koebe function.
*Corresponding author

and the investigation of the Hankel determinants for various classes of analytic functions started in the 1960's. It was Pommerenke [9, 10] who first studied Hankel's determinant for the class $\mathcal{S}$ of the univalent functions given by (1), and he proved in [10] that for all the functions $f \in \mathcal{S}$ we have $|H_{q,k}(f)| < \mathrm{M}k^{-\left(\frac{1}{2}+\beta\right)q+\frac{3}{2}}$, where $k, q \in \mathbb{N}$, $q \geq 2$, $\beta > 1/4000$, and M depends only on q. Similar researches, but for different classes, were reported by Noor [7, 8].

Many recent papers have been devoted to the problem of finding the best upper bounds of $|H_{q,k}|$ for various subfamilies of $\mathcal{H}(\mathbb{D})$, and the majority of the results were obtained for $H_{2,2} = a_2a_4 - a_3^2$ which is called the *second Hankel determinant*.

For the particular values $q = 2$ and $n = 1$ the Hankel determinant reduces to $H_{2,1} = a_3 - a_2^2$. Fekete and Szegő made an early study for the estimate of well known *Fekete-Szegő functional* $\left|a_3 - \mu a_2^2\right|$, where μ is a real or a complex number. Also, there are many published papers that discuss the *third Hankel determinant* $H_{3,1}(f)$ (see, for example, [1, 11, 14]).

The function $f \in \mathcal{A}$ is said to be *close-to-convex* in $\mathbb{D}$, written $f \in \mathcal{K}$, if there exists a starlike function h in $\mathbb{D}$ normalized with the conditions $h(0) = h'(0) - 1 = 0$, such that

$$\operatorname{Re} \frac{zf'(z)}{h(z)} > 0,\ z \in \mathbb{D}. \tag{3}$$

The last known result about $|H_{3,1}(f)|$ for $f \in \mathcal{K}$ was obtained by Prajapat et al. in [11, Theorem 3], which proved that $|H_{3,1}(f)| \leq \dfrac{289}{12} = 24.08333\ldots$. In this paper we essentially improved the result for the above upper bound of third Hankel determinant $|H_{3,1}(f)|$ if $f \in \mathcal{K}$.

2. Preliminary Results

The next lemmas contained in this section are necessary to prove our main results.

Lemma 2.1. [5, (3.5) and (3.9)], [6, (3.9) and (3.10)] *Let the function* P *given by* (2) *be a member of the class* $\mathcal{P}$. *Then,*

$$p_2 = \frac{1}{2}\left[p_1^2 + \left(4 - p_1^2\right)x\right],$$

and

$$p_3 = \frac{1}{4}\left[p_1^3 + 2\left(4 - p_1^2\right)p_1 x - \left(4 - p_1^2\right)p_1 x^2 + 2\left(4 - p_1^2\right)\left(1 - |x|^2\right)z\right]$$

for some complex numbers x, z *satisfying* $|x| \leq 1$ *and* $|z| \leq 1$.

The next result represents the well-known Carathéodory's lemma, and for complementary results see also [3].

Lemma 2.2. [2] *If $P \in \mathcal{P}$ and given by* (2), *then* $|p_k| \le 2$ *for all* $k \ge 1$, *and the result is best possible for the function* $P_*(z) := \dfrac{1+\rho z}{1-\rho z}$, *with* $\rho \in \mathbb{C}$, $|\rho| = 1$.

The following result deals with the sharp upper bounds of the coefficients of the close-to-convex functions.

Lemma 2.3. [13, Theorem 1] *If $f \in \mathcal{K}$ is given by* (1), *then* $|a_n| \le n$ *for all* $n \ge 2$. *The equality holds for all the values of n when f is a rotation of the* Koebe function, *that is* $K_\tau(z) = \dfrac{z}{\left(1+e^{i\tau}z\right)^2}$, $\tau \in \mathbb{R}$.

The next two results give us the upper bounds for some coefficient combinations for the functions of the class $\mathcal{S}^*$ of starlike normalized functions in $\mathbb{D}$.

Lemma 2.4. [12, Lemma 4] *If $h \in \mathcal{S}^*$ is given by* (6) *and* $\mu \in \mathbb{R}$, *then*

$$\left|b_3 - \mu b_2^2\right| \le \begin{cases} 1 + \left(\dfrac{1}{2} - \mu\right)|b_2|^2, & \text{if} \quad \mu \le \dfrac{3}{4}, \\ 1 + (\mu - 1)|b_2|^2, & \text{if} \quad \mu \ge \dfrac{3}{4}. \end{cases}$$

Lemma 2.5. [12, Lemma 5] *If $h \in \mathcal{S}^*$ is given by* (6), *then*

$$\left|b_4 - \frac{7}{9}b_2 b_3\right| \le H(|b_2|),$$

where

$$H(b) = \begin{cases} \dfrac{1}{3}\left(2 + \dfrac{7}{18}b^2 + \dfrac{25}{36}b^3\right), & \text{if} \quad b \in \left[0, \dfrac{6}{7}\right] \\ \dfrac{1}{9}\left(11b - 2b^3\right), & \text{if} \quad b \in \left[\dfrac{6}{7}, 2\right]. \end{cases} \tag{4}$$

The last two lemmas we will use in our proof are connected with a coefficient combination of the close-to-convex functions.

Lemma 2.6. [12, Theorem 1] *If $f \in \mathcal{K}$ is given by* (1), *then*

$$\left|a_2 a_4 - a_3^2\right| \le 1.242\ldots.$$

Lemma 2.7. [4, Corollary 3 for $\lambda = 1$] *If the function $f \in \mathcal{K}$ is given by* (1), *then*

$$|a_3 - a_2^2| \le 1.$$

The equality is attained for the function $\left(K_\pi(z^2)\right)^{1/2} = \dfrac{z}{1-z^2}$.

3. Main results

The next theorem will be used to obtain our main result.

Theorem 3.1. *If the function $f \in \mathcal{K}$ is given by* (1), *then*

$$|a_2a_3 - a_4| \le 2.33333333333333348\ldots. \tag{5}$$

Proof. Suppose that $f \in \mathcal{K}$ has the form (1) and satisfy the condition (3), that is

$$\frac{zf'(z)}{h(z)} = P(z),\ z \in \mathbb{D}, \quad \text{with} \quad h \in \mathcal{S}^*,\ P \in \mathcal{P}.$$

If h be given by

$$h(z) = z + \sum_{k=2}^{\infty} b_k\, z^k,\ z \in \mathbb{D}, \tag{6}$$

and P has the form (2), we get

$$z + \sum_{n=2}^{\infty} n a_n z^n = \left(z + \sum_{n=2}^{\infty} b_n z^n\right)\left(1 + \sum_{n=1}^{\infty} p_n z^n\right),\ z \in \mathbb{D},$$

which yields

$$a_2 = \frac{1}{2}(b_2 + p_1), \tag{7}$$

$$a_3 = \frac{1}{3}(b_3 + b_2p_1 + p_2), \tag{8}$$

$$a_4 = \frac{1}{4}(b_4 + b_3p_1 + b_2p_2 + p_3). \tag{9}$$

Since $f \in \mathcal{K}$, by using (7), (8) and (9) we get

$$\begin{aligned} a_2a_3 - a_4 = &-\frac{1}{4}\left(b_4 - \frac{7}{9}b_2b_3\right) - \frac{1}{36}b_2b_3 - \frac{1}{12}\left(b_3 - 2b_2^2\right)p_1 \\ &+\frac{1}{12}\left(2p_1 - b_2\right)p_2 + \frac{1}{6}\left(b_2p_1^2\right) - \frac{1}{4}p_3, \end{aligned}$$

and from Lemma 2.1 it follows

$$\begin{aligned} a_2a_3 - a_4 = &-\frac{1}{4}\left(b_4 - \frac{7}{9}b_2b_3\right) - \frac{1}{36}b_2b_3 - \frac{1}{12}\left(b_3 - 2b_2^2\right)p_1 \\ +\frac{1}{48}p_1^3 + \frac{4-p_1^2}{24}x(2p - b_2 - 3p) &+ \frac{1}{8}b_2p_1^2 + \frac{1}{16}\left(4 - p_1^2\right)p_1x^2 \\ &-\frac{1}{8}\left(4 - p_1^2\right)\left(1 - |x|^2\right)z, \end{aligned} \tag{10}$$

where $|x| \le 1$ and $|z| \le 1$.

Denoting $|p_1| =: p \in [0,2]$, $|x| =: \rho \in [0,1]$, and using the triangle's inequality in (10) we get

$$|a_2a_3 - a_4| \le \frac{1}{4}\left|b_4 - \frac{7}{9}b_2b_3\right| + \frac{1}{36}|b_2|\,|b_3| + \frac{1}{12}\left|b_3 - 2b_2^2\right|\,|p| + \frac{1}{48}|p|^3 + \frac{4-p^2}{24}\rho\,|p+b_2|$$
$$+\frac{1}{8}|b_2|\,|p|^2 + \frac{1}{16}\left(4-p^2\right)|p|\,\rho^2 + \frac{1}{8}\left(4-p^2\right)\left(1-\rho^2\right).$$

Using Lemma 2.3, Lemma 2.4, and Lemma 2.5 with the substitution $|b_2| =: b \in [0,2]$ we obtain

$$|a_2a_3 - a_4| \le \frac{1}{4}H(b) + \frac{1}{12}b + \frac{1}{12}\left(1+b^2\right)p + \frac{1}{48}p^3 + \frac{4-p^2}{24}\rho\,(p+b)$$
$$+\frac{1}{8}b\,p^2 + \frac{1}{16}\left(4-p^2\right)p\,\rho^2 + \frac{1}{8}\left(4-p^2\right)\left(1-\rho^2\right) =: F(p,b,\rho), \tag{11}$$

where

$$F(p,b,\rho) = A + B\rho + C\rho^2,\ p,b \in [0,2],\ \rho \in [0,1], \tag{12}$$

with

$$C = \frac{1}{16}\left(4-p^2\right)(p-2),$$
$$B = \frac{1}{24}(p+b)\left(4-p^2\right),$$
$$A = \frac{H(b)}{4} + \frac{b}{12} + \frac{b^2p}{12} + \frac{p}{12} + \frac{p^3}{48} + \frac{bp^2}{8} - \frac{p^2}{8} + \frac{1}{2},$$

and $H(b)$ given by (4).

According to the inequality (11), to find the upper bound for $|a_2a_3 - a_4|$ we should determine the value

$$M := \max\left\{F(p,b,\rho) : (p,b) \in [0,2]\times[0,2], \rho \in [0,1]\right\},$$

and from Lemma 2.5, according to (12), a simple computation shows that

$$F(p,b,\rho) = \begin{cases} H_1(p,b,\rho) := \dfrac{\left(4-p^2\right)(p-2)\,\rho^2}{16} + \dfrac{\left(4-p^2\right)(p+b)\,\rho}{24} + \dfrac{2}{3} + \dfrac{7b^2}{216} + \dfrac{25b^3}{432} \\ \qquad + \dfrac{b}{12} + \dfrac{b^2p}{12} + \dfrac{p}{12} + \dfrac{p^3}{48} + \dfrac{bp^2}{8} - \dfrac{p^2}{8}, \quad \text{if} \quad 0 \le b \le \dfrac{6}{7}, \\ H_2(p,b,\rho) := \dfrac{\left(4-p^2\right)(p-2)\,\rho^2}{16} + \dfrac{\left(4-p^2\right)(p+b)\,\rho}{24} - \dfrac{b^3}{18} + \dfrac{7b}{18} + \dfrac{b^2p}{12} \\ \qquad + \dfrac{p}{12} + \dfrac{p^3}{48} + \dfrac{bp^2}{8} - \dfrac{p^2}{8} + \dfrac{1}{2}, \quad \text{if} \quad \dfrac{6}{7} \le b \le 2. \end{cases}$$

Using the MAPLE™ computer software following codes

```
with(Optimization)
Maximize(H1, {0 <= b, 0 <= p, 0 <= y, b <= 6/7, p <= 2, y <= 1});
Maximize(H2, {0 <= p, 0 <= y, b <= 2, p <= 2, y <= 1, 6/7 <= b});
```

where H1 $:= H_1(p, b, \rho)$ and H2 $:= H_2(p, b, \rho)$, we obtain

$$\begin{aligned}\max\left\{H_1(p,b,\rho): p\in[0,2], \rho\in[0,1], 0\le b\le \frac{6}{7}\right\} &= H_1\left(1.81101117886148, \frac{6}{7}, 1\right)\\ &= 1.19688382931563142\ldots,\\ \max\left\{H_2(p,b,\rho): p\in[0,2], \rho\in[0,1], \frac{6}{7}\le b\le 2\right\} &= H_2(2, 2, 0.915977063761207)\\ &= 2.33333333333333348\ldots,\end{aligned}$$

hence $M = 2.33333333333333348\ldots$ and the assertion (5) is reached. □

Remark 3.2. *1. In* [11] *the authors estimated that* $|a_2a_3 - a_4| \le 3$*, while Theorem 3.1 improve this result.*

2. Since we didn't proved that our result is the best possible, to find the smallest upper bound of $|a_2a_3 - a_4|$ *for* $f \in \mathcal{K}$ *given by* (1) *remains an interesting open problem.*

The next theorem contains our main result where we determined the upper bound for $|H_{3,1}(f)|$ if $f \in \mathcal{K}$.

Theorem 3.3. *If the function* $f \in \mathcal{K}$ *is given by* (1)*, then*

$$|H_{3,1}(f)| \le 18.058\ldots. \tag{13}$$

Proof. Let $f \in \mathcal{K}$ be of the form (1). Since

$$H_{3,1}(f) = a_3\left(a_2a_4 - a_3^2\right) - a_4\left(a_4 - a_3a_2\right) + a_5\left(a_3 - a_2^2\right),$$

and from the triangle's inequality we get

$$|H_{3,1}(f)| \le |a_3|\left|a_2a_4 - a_3^2\right| + |a_4|\left|a_4 - a_3a_2\right| + |a_5|\left|a_3 - a_2^2\right|. \tag{14}$$

Using Lemma 2.3 and the inequalities of Lemma 2.6, Lemma 2.7 and (5), from (14) it follows that

$$|H_{3,1}(f)| \le 3\cdot 1.242\ldots + 4\cdot 2.333\ldots + 5\cdot 1 = 18.058\ldots.$$

□

Remark 3.4. *1. The bound given by* (13) *is an improvement of Theorem 3 from* [11]*, where it was proved that* $|H_{3,1}(f)| \le \dfrac{289}{12} = 24.08333333\ldots.$

2. We used in the proof of this theorem the inequality (5) *that possibly doesn't gives the best upper bound for* $|a_2a_3 - a_4|$ *whenever* $f \in \mathcal{K}$ *is given by* (1)*. Thus, we didn't proved that the bound of* (13) *is the best possible, and this remains a challenging open question.*

References

[1] D. Bansal, S. Maharana, and J. K. Prajapat, *Third order Hankel determinant for certain univalent functions*, J. Korean Math. Soc. **52** (2015), no. 6, 1139–1148.

[2] C. Carathéodory, *Über den Variabilitätsbereich der Koeffizienten von Potenzreihen, die gegebene Werte nicht annehmen,* Math. Ann. **64** (1907), no. 1, 95–115.

[3] C. Carathéodory, *Über den Variabilitätsbereich der Fourier'schen Konstanten von Positiven Harmonischen Funktionen.* Rend. Circ. Mat. Palermo **32** (1911), 193–217.

[4] W. Koepf, *On the Fekete-Szegő problem for close-to-convex functions*, Proc. Amer. Math. Soc. **101** (1987), no. 1, 89–95.

[5] R. J. Libera and E. J. Zlotkiewicz, *Early coefficients of the inverse of a regular convex function*, Proc. Amer. Math. Soc. **85** (1982), no. 2, 225–230.

[6] R. J. Libera and E. J. Zlotkiewicz, *Coeffcient bounds for the inverse of a function with derivative in $\mathcal{P}$*, Proc. Amer. Math. Soc. **87** (1983), no. 2, 251–257.

[7] K. I. Noor, *On the Hankel determinant problem for strongly closed-to-convex functions,* J. Nat. Geom. **11** (1997), no. 1, 29–34.

[8] K. I. Noor, *On certain analytic functions related with strongly close-to-convex functions,* Appl. Math. Comput. **197** (2008), no. 1, 149–157.

[9] Ch. Pommerenke, *On the coefficients and Hankel determinants of univalent functions*, J. Lond. Math. Soc. **41** (1966), no. 1, 111–122.

[10] Ch. Pommerenke, *On the Hankel determinants of univalent functions*, Mathematika **14** (1967), no. 1, 108–112.

[11] J. K. Prajapat, D. Pansal, A. Singh, and A. K. Mishra, *Boundson third Hankel determinant for close-to-convex functions*, Acta Univ. Sapientiae Math. **7** (2015), no. 2, 210–219.

[12] D. Răducanu and P. Zaprawa, *Second Hankel determinant for closed-to-convex functions*, §C. R. Math. Acad. Sci. Paris, Ser. I **355** (2017), 1063–1071.

[13] M. O. Reade, *On close-to-convex univalent functions*, Michigan Math. J. **3** (1955/1956), no. 1, 59–62.

[14] P. Zaprawa, *Third Hankel determinant for subclasses of univalent functions*, Mediterr. J. Math. **14** (2017), no .19 Article ID: 19.

Laxmipriya Parida
Basic Science and Humanities, School of Engineering and Technology,
Gandhi Institute of Engineering and Technology University,
Gunupur, Rayagada-765022, Odisha, India.
E-mail: laxmipriya.parida94@gmail.com

Teodor Bulboacă
Faculty of Mathematics and Computer Science, Babeş-Bolyai University,
400084 Cluj-Napoca, Romania.
E-mail: bulboaca@math.ubbcluj.ro

Ashok Kumar Sahoo
Department of Mathematics, Central University,
Koraput, Odisha 763004, India.
E-mail: aksahoo@cuo.ac.in, ashokuumt@gmail.com