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ON QUASI COVERED IDEALS AND QUASI BASES OF ORDERED SEMIGROUPS

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Abstract. This paper explores the concepts of quasi covered ideal, quasi base and the greatest quasi covered ideal within the context of an ordered semigroup, extending the study of algebraic structures to incorporate both the algebraic and order theoretic perspectives. An ordered semigroup provides a rich framework for investigating the interplay between algebraic and order structure. Also, we provide the conditions for the greatest ideal to be quasi covered ideal and develop the fundamental properties with implications of quasi covered ideal of an ordered semigroup. Moreover, we study the relationship between covered ideal with quasi covered ideal, greatest ideal with quasi covered ideal and the greatest quasi covered ideal with quasi base of an ordered semigroup.

1. INTRODUCTION AND PRELIMINARIES

N. Kehayopulu introduced the concepts of prime ideals and weakly prime ideals of an ordered semigroup [10, 11]. Fabrici [6, 7], described the notion of covered ideal in semigroup. This concept has gained significant influence in the study of algebraic structures such as semigroup [4, 5], ordered semigroup [2, 15, 16, 17, 8], ordered semihypergroup [1] and ternary semigroup [9].

When an algebraic structure $(\mathcal{S}, \cdot, \leq)$ satisfies $(\mathcal{S}, \cdot)$ is a semigroup along with a partial order relation $(\mathcal{S}, \leq)$. It is considered of an ordered semigroup and it satisfies $\alpha \leq \beta$ implies $\alpha y \leq \beta y, y\alpha \leq y\beta,\ \forall\ \alpha, \beta, y \in \mathcal{S}$ [2]. Let $\mathcal{N}_1(\neq \phi) \subset \mathcal{S}$ and $\mathcal{N}_2(\neq \phi) \subset \mathcal{S}$, then, $\mathcal{N}_1\mathcal{N}_2 = \{\alpha\beta \in \mathcal{S}, \alpha \in \mathcal{N}_1, \beta \in \mathcal{N}_2\}$, $(\mathcal{N}_1] = \{\alpha \in \mathcal{S}|$ for at least one $y \in \mathcal{N}_1, \alpha \leq y\}$ and $[\mathcal{N}_1) = \{\alpha \in \mathcal{S}|$ for at least one $y \in \mathcal{N}_1, y \leq \alpha\}$. $\mathcal{I}(\beta) = \{(\beta \cup \mathcal{S}\beta \cup \beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}]\colon\ \beta \in \mathcal{S}\}$ is an ideal generated by β [8].

Kehayopulu defined Green's relation $\mathcal{I} = \{(\alpha, \beta) \in \mathcal{S} \times \mathcal{S}|\mathcal{I}(\alpha) = \mathcal{I}(\beta)\}$ on $\mathcal{S}$, [12]. It is easy to prove that $\mathcal{I}$ is an equivalence relation on $\mathcal{S}$. For any $\alpha \in \mathcal{S}$, let $\mathcal{I}^\alpha$ be the $\mathcal{I}$-class containing α. The quasi ordering relation " $\preceq$ " on the set of $\mathcal{I}$-classes in $\mathcal{S}$ is defined as $\mathcal{I}^\alpha \preceq \mathcal{I}^\beta$ if and only if $\mathcal{I}(\alpha) \subseteq \mathcal{I}(\beta)$,

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$\forall\ \alpha, \beta \in \mathcal{S}$. The $\mathcal{I}^{\alpha}$-class is maximal if there is no other $\mathcal{I}^{\beta}$-class such that $\mathcal{I}^{\alpha} \preceq \mathcal{I}^{\beta}$ and $\mathcal{I}^{\alpha}$ is the greatest $\mathcal{I}$-class of $\mathcal{S}$, if all other $\mathcal{I}$-classes are contained in $\mathcal{I}^{\alpha}$. $\mathcal{I}$ is maximal ideal of $\mathcal{S}$ if and only if $\mathcal{S} - \mathcal{I}$ is maximal $\mathcal{I}$-class [16]. $\mathcal{I}(\neq \phi) \subseteq \mathcal{S}$ is said to be a left (resp. right) ideal of $\mathcal{S}$, if it fulfil the following postulates: (i) $\mathcal{S}\mathcal{I} \subseteq \mathcal{I}$ (resp. $\mathcal{I}\mathcal{S} \subseteq \mathcal{I}$) (ii) $\mathcal{I} = (\mathcal{I}]$, i.e. for any $\alpha \in \mathcal{I}$ and $\beta \in \mathcal{S}$, $\beta \leq \alpha \Rightarrow \beta \in \mathcal{I}$. If $\mathcal{I}$ is both left and right ideal, then $\mathcal{I}$ is called an ideal or two-sided ideal of $\mathcal{S}$. An ideal $\mathcal{I}(\neq \phi) \subset \mathcal{S}$ is said to be maximal ideal of $\mathcal{S}$, if there exist any ideal T of $\mathcal{S}$ s.t. $\mathcal{I} \subseteq T \subseteq \mathcal{S}$, then either $\mathcal{I} = T$, or $T = \mathcal{S}$ [10]. If $\mathcal{S}$ does not contain any proper ideal, then it is called simple [13]. In [14], the below statements holds true (i) $\mathcal{I} \subseteq (\mathcal{I}]$, (ii) $\mathcal{I}_1 \subseteq \mathcal{I}_2$ implies that $(\mathcal{I}_1] \subseteq (\mathcal{I}_2]$, (iii) $(\mathcal{I}_1](\mathcal{I}_2] \subseteq (\mathcal{I}_1\mathcal{I}_2]$, (iv) $(\mathcal{I}_1 \cup \mathcal{I}_2] = (\mathcal{I}_1] \cup (\mathcal{I}_2]$, (v) $((\mathcal{I}_1]] = (\mathcal{I}_1]$.

Motivated while studying and analyzing the work related to covered ideals of an ordered semigroup. We got an idea to define the concepts of quasi covered ideal and generalized the results of covered ideal into quasi covered ideal of an ordered semigroup. In this research article, we have studied quasi covered ideal, the greatest quasi covered ideal and quasi base of an ordered semigroup. We go over their characteristics and also provide the connection between the greatest quasi covered ideal and quasi base of an ordered semigroup. Furthermore, we describe the conditions of covered ideal and greatest ideal to be quasi covered ideal of an ordered semigroup. Throughout the paper, we shall denote $\mathcal{S}$ as an ordered semigroup.

Definition 1.1. [3] *Let $\mathcal{L}$ be a proper left ideal of $\mathcal{S}$. Then $\mathcal{L}$ is said to be covered left ideal (shortly, $\mathcal{CL}$-ideal) of $\mathcal{S}$, if $\mathcal{L} \subseteq (\mathcal{S}(\mathcal{S} - \mathcal{L})]$. Accordingly, $\mathcal{R}$ is said to be covered right ideal ($\mathcal{CR}$-ideal), if $\mathcal{R} \subseteq ((\mathcal{S} - \mathcal{R})\mathcal{S}]$.*

Definition 1.2. [8] *Let $\mathcal{N}$ be a proper ideal of $\mathcal{S}$. Then $\mathcal{N}$ is said to be covered ideal (shortly, $\mathcal{C}$-ideal) of $\mathcal{S}$, if $\mathcal{N} \subseteq (\mathcal{S}(\mathcal{S} - \mathcal{N})\mathcal{S}]$.*

2. QUASI COVERED IDEALS

In this portion, we describe quasi covered ideals of $\mathcal{S}$ with examples and explores some of their properties. In simpler terms, quasi covered ideal of $\mathcal{S}$ is an ideal that is not only closed under the semigroup operation but also behaves nicely with respect to the order relation. Quasi covered ideal provides the way to study the relation between algebraic and order-theoretic properties within ordered semigroups, contributing a deeper understanding of their structure and behaviour. This explanation provides a general understanding of quasi covered ideal of an ordered semigroup.

Definition 2.1. *Any proper ideal $\mathcal{N}$ of $\mathcal{S}$ satisfies the condition*

$$\mathcal{N} \subseteq ((\mathcal{S} - \mathcal{N})\mathcal{S} \cup \mathcal{S}(\mathcal{S} - \mathcal{N}) \cup \mathcal{S}(\mathcal{S} - \mathcal{N})\mathcal{S}]$$

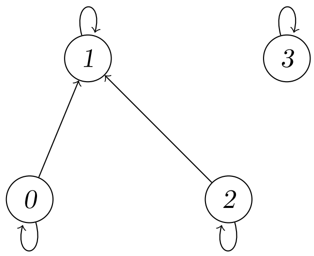
is called quasi covered ideal (shortly, $\mathcal{QC}$-ideal) of $\mathcal{S}$.

Example 2.2. *Let $\mathcal{S} = \{0, 1, 2, 3\}$ and $(\mathcal{S}, \cdot, \leq)$ be an ordered semigroup with the binary operation $'\cdot'$ and the poset $' \leq'$ defined as follows:*

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	1	0	3

$$\leq = \{(0,0),(1,1),(2,2),(3,3),(0,1),(2,1)\}$$

Also, we define the poset $' \leq'$ with the help of figure as follows:



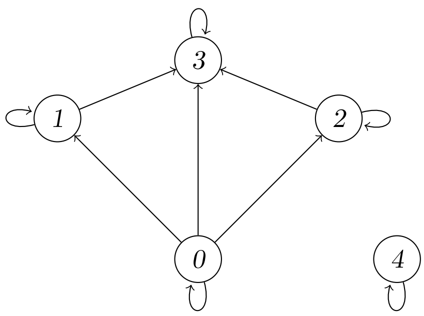
It is easy to verify that $\{0\}$, $\{0,1\}$, $\{0,2\}$ and $\{0,1,2\}$ are $\mathcal{QC}$-ideals of $\mathcal{S}$.

Example 2.3. *Let $\mathcal{S} = \{0, 1, 2, 3, 4\}$ and $(\mathcal{S}, \cdot, \leq)$ be an ordered semigroup with the binary operation $'\cdot'$ and the poset $' \leq'$ defined as follows:*

$\cdot$	0	1	2	3	4
0	0	1	1	1	1
1	1	1	1	1	1
2	1	1	1	4	4
3	1	1	1	1	4
4	4	4	4	4	4

$$\leq = \{(0,0),(1,1),(2,2),(3,3),(4,4),(0,1),(0,2),(0,3),(1,3),(2,3)\}$$

Also, we define the poset $' \leq'$ with the help of figure as follows:



Then $\{0, 1, 4\}$ *is an* $\mathcal{QC}$*-ideal of* $\mathcal{S}$.

Example 2.4. *Let* $\mathcal{S} = \{0, 1, 2, 3, 4, 5, ...\}$ *and* $(\mathcal{S}, \cdot, \leq)$ *be an ordered semigroup with the binary operation* $'\cdot'$ *and the poset* $'\leq'$ *defined as* $p_1 \cdot p_2 =$ *sum of* $\{p_1, p_2\}$, $\forall\ p_1, p_2 \in \mathcal{S}$.

$$\leq = \begin{cases} (0,0),(0,1)(0,2),(0,3),(0,4),(0,5),... \\ (1,1),(1,2),(1,3),(1,4),(1,5),... \\ (2,2),(2,3),(2,4),(2,5),... \\ (3,3),(3,4),(3,5),(3,6),... \\ (4,4),(4,5),(4,6),(4,7),... \\ .... \\ .... \end{cases}$$

Then $\mathcal{S}_p = \{p + I, \forall\ p, I \in N\}$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$.

Example 2.5. *Let* $\mathcal{S} = \{0, 1, 2, 3, 4, 5, ...\}$ *and* $(\mathcal{S}, \cdot, \leq)$ *be an ordered semigroup with the binary operation* $'\cdot'$ *and the poset* $'\leq'$ *defined as* $p_1 \cdot p_2 = min\{p_1, p_2\}$, $\forall\ p_1, p_2 \in \mathcal{S}$.

$$\leq = \begin{cases} (0,0),(0,1)(0,2),(0,3),(0,4),(0,5),... \\ (1,1),(1,2),(1,3),(1,4),(1,5),... \\ (2,2),(2,3),(2,4),(2,5),... \\ (3,3),(3,4),(3,5),(3,6),... \\ (4,4),(4,5),(4,6),(4,7),... \\ .... \\ .... \end{cases}$$

Then $\{0, 1, 2, 3, 4, 5, ...n\}$, $\forall\ n \in N$ *is an* $\mathcal{QC}$*-ideal of* $\mathcal{S}$.

Theorem 2.6. *Every* $\mathcal{C}$*-ideal is an* $\mathcal{QC}$*-ideal of* $\mathcal{S}$. *But the converse need not be true.*

Proof. Consider $\mathcal{N}$ is an $\mathcal{C}$-ideal of $\mathcal{S}$. Then, we have $\mathcal{N} \subseteq (\mathcal{S}(\mathcal{S} - \mathcal{N})\mathcal{S}]$. It implies

$$\mathcal{N} \subseteq (\mathcal{S}(\mathcal{S} - \mathcal{N}) \cup (\mathcal{S} - \mathcal{N})\mathcal{S} \cup \mathcal{S}(\mathcal{S} - \mathcal{N})\mathcal{S}].$$

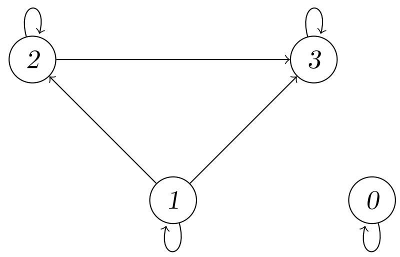
Hence, $\mathcal{N}$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. Converse need not be true, which is shown by below example. ☐

Example 2.7. *Let* $\mathcal{S} = \{0, 1, 2, 3\}$ *and* $(\mathcal{S}, \cdot, \leq)$ *be an ordered semigroup with the binary operation* $'\cdot'$ *and the poset* $'\leq'$ *defined as follows:*

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	0	1	1

$$\leq = \{(0,0),(1,1),(2,2),(3,3),(1,2),(2,3),(1,3)\}$$

Also, we define the poset $' \leq '$ *with the help of figure as follows:*



Let an ideal $\mathcal{N} = \{0,1\}$ *of* $\mathcal{S}$ *such that* $(\mathcal{S}(\mathcal{S}-\mathcal{N})\mathcal{S}] = \{0\}$ *and* $(\mathcal{S}(\mathcal{S}-\mathcal{N}) \cup (\mathcal{S}-\mathcal{N})\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N})\mathcal{S}] = \{0,1\}$. *Therefore* $\mathcal{N}$ *is an* $\mathcal{QC}$*-ideal but it is not* $\mathcal{C}$*-ideal of* $\mathcal{S}$.

Theorem 2.8. *If* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *are two ideals with* $(\mathcal{N}_1 \cap \mathcal{N}_2) \neq \phi$ *and at least one is an* $\mathcal{QC}$*-ideal of* $\mathcal{S}$. *Then* $(\mathcal{N}_1 \cap \mathcal{N}_2)$ *is an* $\mathcal{QC}$*-ideal of* $\mathcal{S}$.

Proof. Consider that $\mathcal{N}_1$ and $\mathcal{N}_2$ are two ideals and considering $\mathcal{N}_1$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. As we know their intersection is also an ideal of $\mathcal{S}$. Therefore $(\mathcal{N}_1 \cap \mathcal{N}_2) \subseteq \mathcal{N}_1 \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{N}_1) \cup (\mathcal{S}-\mathcal{N}_1)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}_1)\mathcal{S}]$. Since $\mathcal{N}_1$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. It implies that, $(\mathcal{N}_1 \cap \mathcal{N}_2) \subseteq (\mathcal{S}(\mathcal{S}-(\mathcal{N}_1 \cap \mathcal{N}_2)) \cup (\mathcal{S}-(\mathcal{N}_1 \cap \mathcal{N}_2))\mathcal{S} \cup \mathcal{S}(\mathcal{S}-(\mathcal{N}_1 \cap \mathcal{N}_2)\mathcal{S}]$. Hence, $(\mathcal{N}_1 \cap \mathcal{N}_2)$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. □

Theorem 2.9. *If* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *are two* $\mathcal{QC}$*-ideals of* $\mathcal{S}$. *Then* $(\mathcal{N}_1 \cap \mathcal{N}_2)$ *is an* $\mathcal{QC}$*-ideal of* $\mathcal{S}$.

Proof. Similar proof as the proof of Theorem 2.8 . □

Corollary 2.10. *If* $\{\mathcal{M}_\lambda : \lambda \in \mathcal{N}\}$ *is the family of* $\mathcal{QC}$*-ideals, then* $\bigcap_{\lambda \in \mathcal{N}} \mathcal{M}_\lambda$ *is an* $\mathcal{QC}$*-ideal of* $\mathcal{S}$.

Corollary 2.11. *Let* $\mathcal{N}_1$ *be an* $\mathcal{QC}$*-ideal of* $\mathcal{S}$ *and* $\mathcal{S}_1$ *is sub-semigroup of* $\mathcal{S}$. *Then* $(\mathcal{N}_1 \cap \mathcal{S}_1)$ *is an* $\mathcal{QC}$*-ideal of* $\mathcal{S}$.

Theorem 2.12. *Let* $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *be two ideals of* $\mathcal{S}$ *such that* $\mathcal{N}_1 \subseteq \mathcal{N}_2$. *If* $\mathcal{N}_2$ *is an* $\mathcal{QC}$*-ideal of* $\mathcal{S}$, *then* $\mathcal{N}_1$ *is also an* $\mathcal{QC}$*-ideal of* $\mathcal{S}$.

Proof. Let $\mathcal{N}_2$ be an $\mathcal{QC}$-ideal of $\mathcal{S}$ such that $\mathcal{N}_1 \subseteq \mathcal{N}_2$. Therefore $(\mathcal{S}-\mathcal{N}_1) \supseteq (\mathcal{S}-\mathcal{N}_2)$. It implies $\mathcal{N}_2 \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{N}_1) \cup (\mathcal{S}-\mathcal{N}_1)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}_1)\mathcal{S}]$. Thus, we have $\mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{N}_1) \cup (\mathcal{S}-\mathcal{N}_1)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}_1)\mathcal{S}]$. Hence, $\mathcal{N}_1$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. □

Theorem 2.13. *If $\mathcal{N}_1$ and $\mathcal{N}_2$ are $\mathcal{CL}$-ideal and $\mathcal{CR}$-ideal of $\mathcal{S}$ such that $(\mathcal{N}_1 \cap \mathcal{N}_2)$ is an ideal of $\mathcal{S}$. Then their intersection is an $\mathcal{QC}$-ideal of $\mathcal{S}$.*

Proof. Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be $\mathcal{CL}$-ideal and $\mathcal{CR}$-ideal of $\mathcal{S}$. Then, we have $\mathcal{N}_1 \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{N}_1)], \mathcal{N}_2 \subseteq ((\mathcal{S}-\mathcal{N}_2)\mathcal{S}]$ and $(\mathcal{N}_1 \cap \mathcal{N}_2)$ is an ideal of $\mathcal{S}$. Thus, we have, $\mathcal{N}_1 \cap \mathcal{N}_2 \subseteq \mathcal{N}_1 \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{N}_1)] \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{N}_1) \cup \mathcal{S}(\mathcal{S}-\mathcal{N}_1)\mathcal{S}]$. Also, we have $\mathcal{N}_1 \cap \mathcal{N}_2 \subseteq \mathcal{N}_2 \subseteq ((\mathcal{S}-\mathcal{N}_2)\mathcal{S}] \subseteq ((\mathcal{S}-\mathcal{N}_2)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}_2)\mathcal{S}]$. Therefore $\mathcal{N}_1 \cap \mathcal{N}_2 \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{N}_1) \cup \mathcal{S}(\mathcal{S}-\mathcal{N}_1)\mathcal{S} \cup (\mathcal{S}-\mathcal{N}_2)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}_2)\mathcal{S}] \subseteq (\mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1 \cap \mathcal{N}_2)\big) \cup \mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1 \cap \mathcal{N}_2)\big)\mathcal{S} \cup \big(\mathcal{S}-(\mathcal{N}_1 \cap \mathcal{N}_2)\big)\mathcal{S} \cup \mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1 \cap \mathcal{N}_2)\big)\mathcal{S}]$, it implies that $\mathcal{N}_1 \cap \mathcal{N}_2 \subseteq (\mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1 \cap \mathcal{N}_2)\big) \cup \mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1 \cap \mathcal{N}_2)\big)\mathcal{S} \cup \big(\mathcal{S}-(\mathcal{N}_1 \cap \mathcal{N}_2)\big)\mathcal{S}]$. Hence, $(\mathcal{N}_1 \cap \mathcal{N}_2)$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$ □

Theorem 2.14. *If $\mathcal{N}_1$ and $\mathcal{N}_2$ are two proper ideals of $\mathcal{S}$ such that $\mathcal{N}_1 \neq \mathcal{N}_2$ and $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{S}$. Then neither $\mathcal{N}_1$ nor $\mathcal{N}_2$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$.*

Proof. Let $\mathcal{N}_1 \cup \mathcal{N}_2 = \mathcal{S}$, thus $\mathcal{S}-\mathcal{N}_1 \subset \mathcal{N}_2$, $\mathcal{S}-\mathcal{N}_2 \subset \mathcal{N}_1$. If possible one of them, say $\mathcal{N}_1$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. Then $\mathcal{N}_1 \subseteq ((\mathcal{S}-\mathcal{N}_1)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}_1) \cup \mathcal{S}(\mathcal{S}-\mathcal{N}_1)\mathcal{S}]$, which implies $\mathcal{N}_1 \subset (\mathcal{S}\mathcal{N}_2 \cup \mathcal{N}_2\mathcal{S} \cup \mathcal{S}\mathcal{N}_2\mathcal{S}] \subseteq \mathcal{N}_2$. i.e. $\mathcal{N}_1 \subseteq \mathcal{N}_2$, which is a contradiction. Similarly, if $\mathcal{N}_2$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$, then we can show that $\mathcal{N}_2 \subseteq \mathcal{N}_1$, which is again a contradiction. Hence, neither $\mathcal{N}_1$ nor $\mathcal{N}_2$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. □

The result of Theorem 2.14 is the following corollary.

Corollary 2.15. *If there are more than one maximal ideals in $\mathcal{S}$. Then, none of them is an $\mathcal{QC}$-ideal of $\mathcal{S}$.*

Theorem 2.16. *Suppose $\mathcal{N}_1$ and $\mathcal{N}_2$ are two $\mathcal{QC}$-ideals of $\mathcal{S}$ and $(\mathcal{N}_1 \cap \mathcal{N}_2) \neq \phi$. Then $(\mathcal{N}_1 \cup \mathcal{N}_2)$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$.*

Proof. Considering $\mathcal{N}_1$ and $\mathcal{N}_2$ are two $\mathcal{QC}$-ideals of $\mathcal{S}$. Let $y \in \mathcal{N}_1$ and $\mathcal{N}_1 \subseteq ((\mathcal{S}-\mathcal{N}_1)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}_1) \cup \mathcal{S}(\mathcal{S}-\mathcal{N}_1)\mathcal{S}]$. It implies that $\exists\ \alpha \in (\mathcal{S}-\mathcal{N}_1)$ s.t. $y \in (\alpha\mathcal{S} \cup \mathcal{S}\alpha \cup \mathcal{S}\alpha\mathcal{S}]$. Thus, There are two possibilities: (i) If $\alpha \in \big(\mathcal{S}-(\mathcal{N}_1 \cup \mathcal{N}_2)\big)$, then $y \in (\big(\mathcal{S}-(\mathcal{N}_1 \cup \mathcal{N}_2)\big)\mathcal{S} \cup \mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1 \cup \mathcal{N}_2)\big) \cup \mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1 \cup \mathcal{N}_2)\big)\mathcal{S}]$ (ii) If $\alpha \in (\mathcal{S}-\mathcal{N}_1) \cap \mathcal{N}_2$, then $\alpha \in \mathcal{N}_2 \subseteq ((\mathcal{S}-\mathcal{N}_2)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}_2) \cup \mathcal{S}(\mathcal{S}-\mathcal{N}_2)\mathcal{S}]$ so, $\exists\ \beta \in (\mathcal{S}-\mathcal{N}_2)$ s.t. $\alpha \in (\mathcal{S}\beta \cup \beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}]$. This implies $\alpha \in (\mathcal{S}\beta]$ or $\alpha \in (\beta\mathcal{S}]$ or $\alpha \in (\mathcal{S}\beta\mathcal{S}]$. Now, the element $\beta \notin \mathcal{N}_1$, otherwise $\alpha \in (\mathcal{S}\beta \cup \beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}] \subseteq (\mathcal{S}\mathcal{N}_1 \cup \mathcal{N}_1\mathcal{S} \cup \mathcal{S}\mathcal{N}_1\mathcal{S}] \subseteq \mathcal{N}_1$. This implies $\alpha \in \mathcal{N}_1$, which contradicts, so $\beta \in (\mathcal{S}-\mathcal{N}_1)$. Also $\beta \in (\mathcal{S}-\mathcal{N}_2)$, so $\beta \in \mathcal{S}-(\mathcal{N}_1 \cup \mathcal{N}_2)$. Now, we have three cases:

Case (i): If $\alpha \in (\mathcal{S}\beta]$, then $y \in (\mathcal{S}\mathcal{S}\beta \cup \mathcal{S}\beta\mathcal{S} \cup \mathcal{S}\mathcal{S}\beta\mathcal{S}] \subseteq (\mathcal{S}\beta \cup \mathcal{S}\beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}] \subseteq (\mathcal{S}\beta \cup \mathcal{S}\beta\mathcal{S}]$.

Case (ii): If $\alpha \in (\beta\mathcal{S}]$, then $y \in (\mathcal{S}\beta\mathcal{S} \cup \beta\mathcal{S}\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}\mathcal{S}] \subseteq (\mathcal{S}\beta\mathcal{S} \cup \beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}] \subseteq (\mathcal{S}\beta\mathcal{S} \cup \beta\mathcal{S}]$.

Case (iii): If $\alpha \in (\mathcal{S}\beta\mathcal{S}]$, then $y \in (\mathcal{S}\mathcal{S}\beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}\mathcal{S} \cup \mathcal{S}\mathcal{S}\beta\mathcal{S}\mathcal{S}] \subseteq (\mathcal{S}\beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}] \subseteq (\mathcal{S}\beta\mathcal{S}]$.

In all the three cases, we have $y \in (\mathcal{S}\beta \cup \beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}] \subseteq (\mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1 \cup \mathcal{N}_2)\big) \cup$

$\big(\mathcal{S}-(\mathcal{N}_1\cup\mathcal{N}_2)\big)\mathcal{S}\cup\mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1\cup\mathcal{N}_2)\big)\mathcal{S}]$. Thus, $\mathcal{N}_1\subseteq(\mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1\cup\mathcal{N}_2)\big)\cup\big(\mathcal{S}-(\mathcal{N}_1\cup\mathcal{N}_2)\big)\mathcal{S}\cup\mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1\cup\mathcal{N}_2)\big)\mathcal{S}]$. In the same way we can prove that $\mathcal{N}_2\subseteq(\mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1\cup\mathcal{N}_2)\big)\cup\big(\mathcal{S}-(\mathcal{N}_1\cup\mathcal{N}_2)\big)\mathcal{S}\cup\mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1\cup\mathcal{N}_2)\big)\mathcal{S}]$. Therefore $\mathcal{N}_1\cup\mathcal{N}_2\subseteq(\mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1\cup\mathcal{N}_2)\big)\cup\big(\mathcal{S}-(\mathcal{N}_1\cup\mathcal{N}_2)\big)\mathcal{S}\cup\mathcal{S}\big(\mathcal{S}-(\mathcal{N}_1\cup\mathcal{N}_2)\big)\mathcal{S}]$. Hence, $(\mathcal{N}_1\cup\mathcal{N}_2)$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. □

Theorem 2.17. *If $\mathcal{S}$ contains proper ideal. Then, there exists at least one $\mathcal{QC}$-ideal in $\mathcal{S}$.*

Proof. Let be $\mathcal{N}$ any proper ideal of $\mathcal{S}$ and constructing an ideal $\mathcal{N}_1=(\mathcal{S}(\mathcal{S}-\mathcal{N})\cup(\mathcal{S}-\mathcal{N})\mathcal{S}\cup\mathcal{S}(\mathcal{S}-\mathcal{N})\mathcal{S}]$ of $\mathcal{S}$. Then $(\mathcal{N}\cap\mathcal{N}_1)$ is an ideal of $\mathcal{S}$. Consider that $(\mathcal{N}\cap\mathcal{N}_1)=\mathcal{N}_2$, thus $\mathcal{N}_2\subset\mathcal{N}_1$ and $\mathcal{N}_2\subset\mathcal{N}$. Therefore, $\mathcal{N}_2\subset\mathcal{N}_1=(\mathcal{S}(\mathcal{S}-\mathcal{N})\cup(\mathcal{S}-\mathcal{N})\mathcal{S}\cup\mathcal{S}(\mathcal{S}-\mathcal{N})\mathcal{S}]$ and $\mathcal{N}_2\subset\mathcal{N}$. Thus, $\mathcal{N}_2\subset\mathcal{S}(\mathcal{S}-\mathcal{N}_2)\cup(\mathcal{S}-\mathcal{N}_2)\mathcal{S}\cup\mathcal{S}(\mathcal{S}-\mathcal{N}_2)\mathcal{S}$. Hence, $\mathcal{N}_2$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. □

3. THE GREATEST IDEALS AND THE QUASI COVERED IDEALS

In the following section, we provide the conditions for the greatest ideal to be an $\mathcal{QC}$-ideal of $\mathcal{S}$. If an ideal $\mathcal{N}\subset\mathcal{S}$ contains every proper ideal of $\mathcal{S}$ then, it is called greatest ideal of $\mathcal{S}$. It is denoted by $\mathcal{N}^*$ [7]. In Example (2.7), consider $\mathcal{N}_1=\{0,1,2\}$ and $\mathcal{N}_2=\{0,1\}$ are two ideals of $\mathcal{S}$ s.t. $\mathcal{N}_2\subset\mathcal{N}_1$. Clearly, $\mathcal{N}_1$ is the greatest ideal of $\mathcal{S}$.

Theorem 3.1. *Let there is only a maximal ideal $\mathcal{N}$ of $\mathcal{S}$, which is an $\mathcal{QC}$-ideal of $\mathcal{S}$, then $\mathcal{N}$ is the greatest ideal of $\mathcal{S}$.*

Proof. If $\mathcal{N}_1$ is proper ideal of $\mathcal{S}$, then $\mathcal{N}_1\subseteq\mathcal{N}$. By Theorem 2.14, $\mathcal{N}$ is the greatest ideal of $\mathcal{S}$. □

Theorem 3.2. *Let every proper ideal of $\mathcal{S}$ be an $\mathcal{QC}$-ideal. Then one of the conditions mentioned below is true:*
(i) $\mathcal{S}$ contains $\mathcal{N}^*$.
(ii) $\mathcal{S}=(\mathcal{S}^2]$, for any proper ideal $\mathcal{N}$ and for any principal ideal $\mathcal{I}(\alpha)\subseteq\mathcal{N}$, there exists a principal ideal $\mathcal{I}(\beta)$, such that $\mathcal{I}(\alpha)\subset\mathcal{I}(\beta)\subset\mathcal{S}$, where $\beta\in\mathcal{S}-\mathcal{N}$.

Proof. Consider that I^α, and I^β are two maximal $\mathcal{I}$- classes of $\mathcal{S}$, then I^α is not equal to I^β. Therefore from Lemma 1.2 [3], $\mathcal{N}^\alpha=\mathcal{S}-I^\alpha$ and $\mathcal{N}^\beta=\mathcal{S}-I^\beta$ are two different proper maximal ideals of $\mathcal{S}$ such that by Theorem 2.14 none of them is an $\mathcal{QC}$-ideal of $\mathcal{S}$. It contradicts, then $\mathcal{S}$ does not contain distinct maximal $\mathcal{I}$-classes, hence either no maximal $\mathcal{I}$-class is in $\mathcal{S}$ or one $\mathcal{I}$-class is contained in $\mathcal{S}$. If $\mathcal{S}$ having with one maximal $\mathcal{I}$-class. Then the maximal proper ideal is $\mathcal{N}^\alpha=\mathcal{S}-I^\alpha$ of $\mathcal{S}$. By supposition $\mathcal{N}^\alpha$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$, from Theorem 3.1, $\mathcal{N}^\alpha=\mathcal{N}^*$. Hence $\mathcal{S}$ contains $\mathcal{N}^*$. On the other hand, let us suppose that $\mathcal{S}$ has no any maximal $\mathcal{I}$-class. Then, we shall prove that $\mathcal{S}=(\mathcal{S}^2]$. Let $(\mathcal{S}^2]\subset\mathcal{S}$. Then, $\exists\ x\in\mathcal{S}-(\mathcal{S}^2]$. If $\mathcal{I}(x)=\mathcal{S}$, then $\mathcal{S}$

having a maximal $\mathcal{I}$-class, which is not possible. Then $\mathcal{I}(x) \subset \mathcal{S}$, therefore $\mathcal{I}(x) \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{I}(x)) \cup (\mathcal{S}-\mathcal{I}(x))\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{I}(x))\mathcal{S}] \subseteq (\mathcal{S}^2 \cup \mathcal{S}^2 \cup \mathcal{S}^3]$. This implies $x \in (\mathcal{S}^2]$. It is a contradiction. Therefore $\mathcal{S} = (\mathcal{S}^2]$. Let $\mathcal{N} \subset \mathcal{S}$, and suppose $\mathcal{I}(\alpha) \subseteq \mathcal{N}$. Since $\mathcal{N} \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{N}) \cup (\mathcal{S}-\mathcal{N})\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N})\mathcal{S}]$, $\exists$ $\beta \in \mathcal{S}-\mathcal{N}$ such that $\alpha \in (\mathcal{S}\beta \cup \beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}]$, therefore $\mathcal{I}(\alpha) \subseteq \mathcal{I}(\beta) \subseteq \mathcal{S}$. Since $\beta \in \mathcal{S}-\mathcal{N}$, so $\mathcal{I}(\alpha) \subset \mathcal{I}(\beta)$. Using the supposition that no maximum $\mathcal{I}$-class exists i.e. $\mathcal{I}(\beta) \subset \mathcal{S}$. It implies $\mathcal{I}(\alpha) \subset \mathcal{I}(\beta) \subset \mathcal{S}$. □

Theorem 3.3. *Let $\mathcal{N}$ be a proper ideal of $\mathcal{S}$, then*
(i) If $\mathcal{S}$ contains $\mathcal{N}^*$, then $\mathcal{N}$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$.
(ii) If $\mathcal{S} = (\mathcal{S}^2]$, for any principal ideal $\mathcal{I}(\alpha) \subseteq \mathcal{N}$, there exists a proper principal ideal $\mathcal{I}(\beta)$, where $\beta \in \mathcal{S}-\mathcal{N}$ and $\mathcal{I}(\alpha) \subseteq \mathcal{I}(\beta)$. Then $\mathcal{N}$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$.

Proof. Consider $\mathcal{N}$ is any proper ideal of $\mathcal{S}$. If (i) is satisfied. Then $\mathcal{N} \subseteq \mathcal{N}^*$. It implies $\mathcal{S}-\mathcal{N}^* \subseteq \mathcal{S}-\mathcal{N}$. Thus $\mathcal{N} \subseteq \mathcal{N}^* \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{N}^*) \cup (\mathcal{S}-\mathcal{N}^*)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}^*)\mathcal{S}]$. It implies $\mathcal{N} \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{N}) \cup (\mathcal{S}-\mathcal{N})\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N})\mathcal{S}]$. Hence $\mathcal{N}$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. Let (ii) be satisfied. If $\alpha_1 \in \mathcal{N}$, thus $\mathcal{I}(\alpha_1) \subset \mathcal{N}$, then there exists $\beta \in \mathcal{S}-\mathcal{N}$, as it is given $\mathcal{I}(\alpha_1) \subset \mathcal{I}(\beta)$. As we have, $\mathcal{S} = (\mathcal{S}^2]$ implies $\mathcal{S} = (\mathcal{S}^3]$, and $\beta \in \mathcal{S}$, then $\beta \in (\mathcal{S}^3]$. Thus, we have $\beta \in (\mathcal{S}^3] \cup (\mathcal{S}^3] \cup (\mathcal{S}^3]$, then $\beta \in (\mathcal{S}^2] \cup (\mathcal{S}^2] \cup (\mathcal{S}^3]$. Thus $\beta \in (\mathcal{S}\mathcal{S}] \cup (\mathcal{S}\mathcal{S}] \cup (\mathcal{S}\mathcal{S}\mathcal{S}]$ which implies $\beta \in (\mathcal{S}\beta_1] \cup (\beta_1\mathcal{S}] \cup (\mathcal{S}\beta_1\mathcal{S}]$, for some $\beta_1 \in \mathcal{S}$. Thus $\beta \in (\mathcal{S}\beta_1]$ or $\beta \in (\beta_1\mathcal{S}]$ or $\beta \in (\mathcal{S}\beta_1\mathcal{S}]$, for some $\beta_1 \in \mathcal{S}$. Let $\beta \in (\mathcal{S}\beta_1]$, for some $\beta_1 \in \mathcal{S}$, then $(\mathcal{S}\beta_1] \subset \mathcal{N}$ and $\beta \in \mathcal{S}\beta_1 \subset \mathcal{N}$. Hence $\beta \in \mathcal{N}$, which is a contradiction as $\beta \in \mathcal{S}-\mathcal{N}$. Therefore, for arbitrary $\alpha_1 \in \mathcal{N}$, $\exists$ $\beta_1 \in \mathcal{S}-\mathcal{N}$ s.t. $\alpha_1 \in (\mathcal{S}\beta_1]$. Thus, $\alpha_1 \in (\mathcal{S}(\mathcal{S}-\mathcal{N})]$. Similarly, $\alpha_1 \in (\beta_1\mathcal{S}]$ implies, $\alpha_1 \in ((\mathcal{S}-\mathcal{N})\mathcal{S}]$ and $\alpha_1 \in (\mathcal{S}\beta_1\mathcal{S}]$ implies $\alpha_1 \in (\mathcal{S}(\mathcal{S}-\mathcal{N})\mathcal{S}]$. Thus, we get $\alpha_1 \in (\mathcal{S}(\mathcal{S}-\mathcal{N}) \cup (\mathcal{S}-\mathcal{N})\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N})\mathcal{S}]$. It implies that $\mathcal{N} \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{N}) \cup (\mathcal{S}-\mathcal{N})\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N})\mathcal{S}]$. Hence, $\mathcal{N}$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. □

Theorem 3.4. *Assuming that $\mathcal{N}^*$ is the greatest ideal of $\mathcal{S}$ s.t. $\mathcal{S} = (\mathcal{S}^2]$, then $\mathcal{N}^*$ is an $\mathcal{QC}$-ideal.*

Proof. Suppose that an ideal of $\mathcal{S}$ is $(\mathcal{S}(\mathcal{S}-\mathcal{N}^*) \cup (\mathcal{S}-\mathcal{N}^*)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}^*)\mathcal{S}]$ and $\mathcal{N}^*$ is the greatest ideal of $\mathcal{S}$, then either $(\mathcal{S}(\mathcal{S}-\mathcal{N}^*) \cup (\mathcal{S}-\mathcal{N}^*)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}^*)\mathcal{S}] = \mathcal{S}$ or $(\mathcal{S}(\mathcal{S}-\mathcal{N}^*) \cup (\mathcal{S}-\mathcal{N}^*)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}^*)\mathcal{S}] \subseteq \mathcal{N}^*$. Therefore, three cases are arise:

Case(i): If $(\mathcal{S}(\mathcal{S}-\mathcal{N}^*) \cup (\mathcal{S}-\mathcal{N}^*)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}^*)\mathcal{S}] = \mathcal{S}$, then $\mathcal{N}^* \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{N}^*) \cup (\mathcal{S}-\mathcal{N}^*)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}^*)\mathcal{S}]$. It implies $\mathcal{N}^*$ is an $\mathcal{QC}$-ideal.

Case(ii): If $(\mathcal{S}(\mathcal{S}-\mathcal{N}^*) \cup (\mathcal{S}-\mathcal{N}^*)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}^*)\mathcal{S}] = \mathcal{N}^*$, then $\mathcal{N}^*$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$.

Case(iii): If $(\mathcal{S}(\mathcal{S}-\mathcal{N}^*) \cup (\mathcal{S}-\mathcal{N}^*)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}^*)\mathcal{S}] \subset \mathcal{N}^*$, as given $(\mathcal{S}^2] = \mathcal{S}$ implies $(\mathcal{S}^3] = \mathcal{S}$, then $(\mathcal{S}^3] = \mathcal{S} \cup \mathcal{S}\mathcal{N}^*\mathcal{S} = (\mathcal{S}(\mathcal{S}-\mathcal{N}^*) \cup (\mathcal{S}-\mathcal{N}^*)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}^*)\mathcal{S} \cup \mathcal{S}\mathcal{N}^*\mathcal{S}] \subset \mathcal{N}^* \cup \mathcal{N}^* = \mathcal{N}^* \subset \mathcal{S}$. It implies $(\mathcal{S}^3] \subset \mathcal{S}$, which is a contradiction. Hence by case(i), case(ii) and case(iii), $\mathcal{N}^*$ is an $\mathcal{QC}$-ideal. □

Theorem 3.5. *Every proper ideal of $\mathcal{S}$ with identity 1 is an $\mathcal{QC}$-ideal. Specifically, if there exist any greatest ideal $\mathcal{N}^*$ in $\mathcal{S}$, then $\mathcal{N}^*$ is an $\mathcal{QC}$-ideal.*

Proof. Let $\mathcal{N}$ be proper ideal of $\mathcal{S}$. Then, $1 \notin \mathcal{N}$. If possible $1 \in \mathcal{N}$. Then, $\mathcal{S}$=$\mathcal{S}$.1 $\subseteq \mathcal{SN} \subseteq \mathcal{N}$. i.e. $\mathcal{S} = \mathcal{N}$, it is contradict. Then $1 \in \mathcal{S} - \mathcal{N}$, it follows that $(\mathcal{S}(\mathcal{S}-\mathcal{N}) \cup \mathcal{S}(\mathcal{S}-\mathcal{N})\mathcal{S} \cup (\mathcal{S}-\mathcal{N})\mathcal{S}] = (\mathcal{S}]$. Hence, $\mathcal{N} \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{N}) \cup \mathcal{S}(\mathcal{S}-\mathcal{N})\mathcal{S} \cup (\mathcal{S}-\mathcal{N})\mathcal{S}]$. It implies that $\mathcal{N}$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. If exists, any greatest ideal $\mathcal{N}^*$ of $\mathcal{S}$, then $\mathcal{N}^*$ is an $\mathcal{QC}$-ideal. □

4. QUASI BASE AND THE GREATEST QUASI COVERED IDEALS

In this section, we define quasi base and the greatest quasi covered ideal of $\mathcal{S}$ with the support of some examples. Also, we have proved some results based on quasi base and the greatest $\mathcal{QC}$-ideal of $\mathcal{S}$.

Definition 4.1. *A non-empty proper subset $\mathcal{N}$ of $\mathcal{S}$ is called quasi base of $\mathcal{S}$. If (i) $\mathcal{S} = (\mathcal{N} \cup (\mathcal{SN} \cap \mathcal{NS}) \cup \mathcal{SNS}]$.*
(ii) there does not exist any proper subset $\mathcal{M} \subset \mathcal{N}$ such that

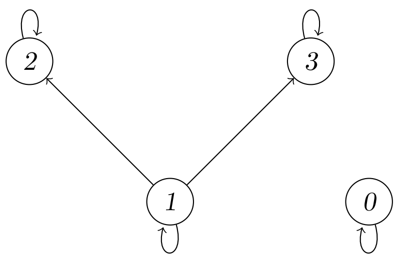
$$\mathcal{S} = (\mathcal{M} \cup (\mathcal{SM} \cap \mathcal{MS}) \cup \mathcal{SMS}].$$

Example 4.2. *Let $\mathcal{S} = \{0,1,2,3\}$ and $(\mathcal{S},\cdot,\leq)$ be an ordered semigroup with the binary operation $'\cdot'$ and the poset $'\leq'$ defined as follows:*

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	0
3	0	1	0	3

$$\leq = \{(0,0),(1,1),(2,2),(3,3),(1,2),(1,3)\}.$$

Also, we define the poset $'\leq'$ with the help of figure as follows:



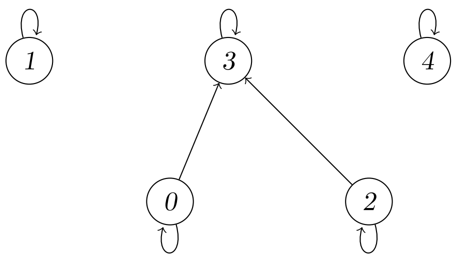
Then $\{2,3\}$ is a quasi base of $\mathcal{S}$.

Example 4.3. *Let $\mathcal{S} = \{0,1,2,3,4\}$ and $(\mathcal{S},\cdot,\leq)$ be an ordered semigroup with the binary operation $'\cdot'$ and the poset $'\leq'$ defined as follows:*

·	0	1	2	3	4
0	0	4	2	3	4
1	0	4	2	3	4
2	0	4	2	3	4
3	0	4	2	3	4
4	0	1	2	3	4

$$\leq = \{(0,0),(1,1),(2,2),(3,3),(4,4),(0,3),(2,3)\}.$$

Also, we define the poset $' \leq'$ with the help of figure as follows:



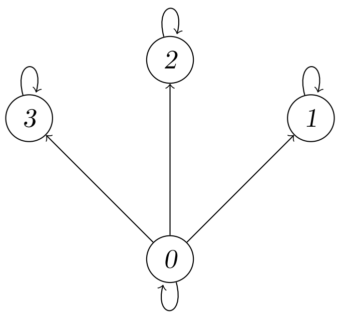
Then $\{4\}$ is a quasi base of $\mathcal{S}$.

Example 4.4. *Let $\mathcal{S} = \{0,1,2,3,4\}$ and $(\mathcal{S},\cdot,\leq)$ be an ordered semigroup with the binary operation $'\cdot'$ and the poset $' \leq'$ defined as follows:*

·	0	1	2	3	4
0	0	1	2	3	4
1	1	1	3	3	4
2	2	3	2	3	4
3	3	3	3	3	3
4	4	4	4	4	4

$$\leq = \{(0,0),(1,1),(2,2),(3,3),(4,4),(0,1),(0,2),(0,3)\}.$$

Also, we define the poset $' \leq'$ with the help of figure as follows:



Then $\{0\}$ is a quasi base of $\mathcal{S}$.

The following theorem shows the existence of quasi base of an ordered semigroup.

Theorem 4.5. *Consider that $\mathcal{N}(\neq \phi) \subset \mathcal{S}$ is a quasi base of $\mathcal{S}$ if and only if it satisfies: (i) For any $\alpha \in \mathcal{S}$, $\exists\, \beta \in \mathcal{N}$ s.t. $\mathcal{I}(\alpha) \subseteq \mathcal{I}(\beta)$ (ii) For any $\alpha_1, \alpha_2 \in \mathcal{N}$ if $\mathcal{I}^{(\alpha_1)} \preceq \mathcal{I}^{(\alpha_2)}$, then $\alpha_1 = \alpha_2$.*

Proof. From the definition of quasi base, condition (i) can be obtained. Now for (ii), if for any $\alpha_1, \alpha_2 \in \mathcal{N}$, $\mathcal{I}^{(\alpha_1)} \preceq \mathcal{I}^{(\alpha_2)}$, then $\mathcal{I}(\alpha_1) \subseteq \mathcal{I}(\alpha_2)$. Suppose $\alpha_1 \neq \alpha_2$. Consider $\mathcal{M} = \mathcal{N} - \{\alpha_1\}$, then $(\mathcal{N} \cup (\mathcal{SN} \cap \mathcal{NS}) \cup \mathcal{SNS}] \subseteq (\mathcal{M} \cup (\mathcal{SM} \cap \mathcal{MS}) \cup \mathcal{SMS}]$. Therefore $(\mathcal{M} \cup (\mathcal{SM} \cap \mathcal{MS}) \cup \mathcal{SMS}] = \mathcal{S}$. Which is a contradiction, thus $\alpha_1 = \alpha_2$. Conversely, suppose that $\mathcal{N}$ is not a quasi base of $\mathcal{S}$, therefore $\exists\, \mathcal{M} \subset \mathcal{N}$ such that $(\mathcal{M} \cup (\mathcal{SM} \cap \mathcal{MS}) \cup \mathcal{SMS}] = \mathcal{S}$. Let $\alpha_1 \in \mathcal{N} - \mathcal{M}$, then $\exists\, m_1 \in \mathcal{M}$, $s \in \mathcal{S}$ such that $\alpha_1 \leq m_1$ or $\alpha_1 \leq sm_1$ or $\alpha_1 \leq m_1 s$ or $\alpha \leq sm_1 s$, it implies that $\mathcal{I}(\alpha_1) \subseteq \mathcal{I}(m_1)$ i.e. $\mathcal{I}^{\alpha_1} \preceq \mathcal{I}^{(m_1)}$. It is a contradicts of (ii). Hence $\mathcal{N}$ is a quasi base of $\mathcal{S}$. □

Remark 4.6. *From Theorem 4.5, we can observe that if $\mathcal{N}$ is quasi base of $\mathcal{S}$, then each element of $\mathcal{N}$ belongs to some maximal $\mathcal{I}$-class and for each maximal $\mathcal{I}$-class there is exactly one element in $\mathcal{N}$. If any $\mathcal{QC}$- ideal $\mathcal{N}$, contains every $\mathcal{QC}$-ideals of $\mathcal{S}$. Then $\mathcal{N}$ is called the greatest $\mathcal{QC}$-ideal of $\mathcal{S}$. If it exist, then it is denoted $\mathcal{N}^g$.*

Remark 4.7. *If the maximal ideals of $\mathcal{S}$ are $\{\mathcal{M}_\lambda, \lambda \in N\}$ and define $\hat{\mathcal{M}} = \bigcap_{\lambda \in N} \mathcal{M}_\lambda \neq \phi$. If $\mathcal{M}^g$ exists in $\mathcal{S}$, it implies that $\mathcal{M}^g \subset \hat{\mathcal{M}}$. But, if there exist even one $\mathcal{M}_\lambda$ s.t. $\mathcal{M}^g \nsubseteq \mathcal{M}_\lambda$, then $\mathcal{M}^g \cup \mathcal{M}_\lambda = \mathcal{S}$. By Theorem 2.12, we can say that the greatest $\mathcal{QC}$-ideal is not an $\mathcal{QC}$-ideal of $\mathcal{S}$.*

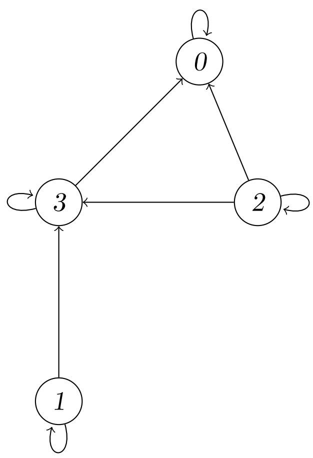
We can see in the below example that the existence of maximal ideals does not imply the existence of the greatest $\mathcal{QC}$-ideal of an ordered semigroup.

Example 4.8. *Let $\mathcal{S} = \{0, 1, 2, 3\}$ and $(\mathcal{S}, \cdot, \leq)$ be an ordered semigroup with the binary operation $'\cdot'$ and the poset $'\leq'$ defined as follows:*

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	1

$$\leq = \{(0,0), (1,1), (2,2), (3,3), (1,3), (2,3), (2,0), (3,0)\}.$$

Also, we define the poset $' \leq'$ with the help of figure as follows:



Here, $\mathcal{N} = \{0, 1, 2\}$ *and* $\mathcal{M} = \{0, 1, 3\}$ *are two maximal ideals of* $\mathcal{S}$*. However, the greatest* $\mathcal{QC}$*-ideal is not contained in* $\mathcal{S}$*.*

With the help of following results, we define the relationship between quasi base and the greatest $\mathcal{QC}$-ideal of $\mathcal{S}$.

Theorem 4.9. *Suppose* $\mathcal{S}$ *contains a quasi base* $\mathcal{N}$*, then it contains the greatest* $\mathcal{QC}$*-ideal* $\mathcal{N}^g$*. Furthermore* $\mathcal{N}^g = (\mathcal{S}^3] \cap \hat{\mathcal{N}}$*, where* $\hat{\mathcal{N}} = \bigcap_{\alpha \in \wedge} \mathcal{N}_\alpha$ *and* $\{\mathcal{N}_\alpha | \alpha \in \wedge\}$ *is the collection of maximal ideals of* $\mathcal{S}$*.*

Proof. For any $n \in \mathcal{N}$, $\mathcal{I}^{(n)}$ is a maximal $\mathcal{I}$-class. Also $\mathcal{S} - \mathcal{I}^{(n)}$ is a maximal ideal and $\hat{\mathcal{N}} \neq \phi$, since $\hat{\mathcal{N}}$ and $(\mathcal{S}^3]$ are ideals of $\mathcal{S}$, $\hat{\mathcal{N}} \cap (\mathcal{S}^3] \neq \phi$. Construct $\mathcal{M} = \hat{\mathcal{N}} \cap (\mathcal{S}^3]$. Then for any $n_1 \in \mathcal{M}$, $\exists\, \alpha \in \mathcal{S}$ such that $n_1 \in (\mathcal{S}^3]$, it implies $n_1 \in (\mathcal{S}^3] \cup (\mathcal{S}^3] \cup (\mathcal{S}^3] = (\mathcal{S}^2] \cup (\mathcal{S}^2] \cup (\mathcal{S}^3]$. It implies $n_1 \in (\mathcal{S}\alpha \cup \alpha\mathcal{S} \cup \alpha\mathcal{S}\alpha]$. If $\alpha \notin \mathcal{N}$, $\exists\, \beta \in \mathcal{N}$ such that $\mathcal{I}(\alpha) \subseteq \mathcal{I}(\beta)$. Therefore $\alpha \in (\beta \cup \mathcal{S}\beta \cup \beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}]$ and we have $\alpha \neq \beta$. Now, two cases arise: (i) When $\alpha < \beta$, then $(\mathcal{S}\alpha\mathcal{S}] \subseteq (\mathcal{S}\beta\mathcal{S}]$, it implies $(\mathcal{S}\alpha \cup \alpha\mathcal{S} \cup \mathcal{S}\alpha\mathcal{S}] \subseteq (\mathcal{S}\beta \cup \beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}]$, thus $n_1 \in (\mathcal{S}\beta \cup \beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}]$. (ii) When $\alpha \in (\mathcal{S}\beta \cup \beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}]$. Therefore $\alpha \in (\mathcal{S}\beta]$ or $\alpha \in (\beta\mathcal{S}]$ or $\alpha \in (\mathcal{S}\beta\mathcal{S}]$. Then three subcases arise here: (a) If $\alpha \in (\mathcal{S}\beta]$, then $(\mathcal{S}\alpha] \subseteq (\mathcal{S}^2\beta]$. (b) If $\alpha \in (\beta\mathcal{S}]$, then$(\alpha\mathcal{S}] \subseteq (\beta\mathcal{S}^2]$. (c) If $\alpha \in (\mathcal{S}\beta\mathcal{S}]$, then $(\mathcal{S}\alpha\mathcal{S}] \subseteq (\mathcal{S}^2\beta\mathcal{S}^2]$. From (a), (b) and (c) we get, $(\mathcal{S}\alpha \cup \alpha\mathcal{S} \cup \mathcal{S}\alpha\mathcal{S}] \subseteq (\mathcal{S}^2\beta \cup \beta\mathcal{S}^2 \cup \mathcal{S}^2\beta\mathcal{S}^2] \subseteq (\mathcal{S}\beta \cup \beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}]$. Thus $n_1 \in (\mathcal{S}\beta \cup \beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}]$. From (i) and (ii) we have, $n_1 \in (\mathcal{S}\beta \cup \beta\mathcal{S} \cup \mathcal{S}\beta\mathcal{S}] \subseteq (\mathcal{S}\mathcal{N} \cup \mathcal{N}\mathcal{S} \cup \mathcal{S}\mathcal{N}\mathcal{S}] \subseteq (\mathcal{S}(\mathcal{S} - \hat{\mathcal{N}}) \cup (\mathcal{S} - \hat{\mathcal{N}})\mathcal{S} \cup \mathcal{S}(\mathcal{S} - \hat{\mathcal{N}})\mathcal{S}] \subseteq (\mathcal{S}(\mathcal{S} - \mathcal{M}) \cup (\mathcal{S} - \mathcal{M})\mathcal{S} \cup \mathcal{S}(\mathcal{S} - \mathcal{M})\mathcal{S}]$. Hence $\mathcal{M}$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. Now, we will show that $\mathcal{M}$ is the greatest $\mathcal{QC}$-ideal. Consider that $\mathcal{P}$ is any $\mathcal{QC}$-ideal of $\mathcal{S}$. Then $\mathcal{P} \subseteq (\mathcal{S}(\mathcal{S} - \mathcal{P}) \cup (\mathcal{S} - \mathcal{P})\mathcal{S} \cup \mathcal{S}(\mathcal{S} - \mathcal{P})\mathcal{S}]$. By Theorem 3.1, the intersection all maximal ideals contain every $\mathcal{QC}$-ideal of $\mathcal{S}$. Thus $\mathcal{P} \subseteq \hat{\mathcal{N}}$. Then $\mathcal{P} \subseteq (\mathcal{S}^3] \cap \hat{\mathcal{N}}$, but $\mathcal{M} = (\mathcal{S}^3] \cap \hat{\mathcal{N}}$. Hence, $\mathcal{M}$ is the greatest $\mathcal{QC}$-ideal $\mathcal{N}^g$. □

Lemma 4.10. *Suppose $\mathcal{S}$ contains the greatest $\mathcal{QC}$-ideal $\mathcal{N}^g$ and $\mathcal{N}^g \subset (\mathcal{S}^3]$. Then $\mathcal{I}(e) = (\mathcal{S}e \cup e\mathcal{S} \cup \mathcal{S}e\mathcal{S}]$, $\forall\ e \in (\mathcal{S}^3] - \mathcal{N}^g$.*

Proof. We have $\mathcal{N}^g \subset (\mathcal{S}^3]$. This implies that $(\mathcal{S}^3] - \mathcal{N}^g \neq \phi$. Suppose that $e \in (\mathcal{S}^3] - \mathcal{N}^g$ and $\mathcal{N}^g$ is an ideal of $\mathcal{S}$. $\mathcal{I}^{(e)} \subseteq (\mathcal{S}^3] - \mathcal{N}^g$. Then $e \in (\mathcal{S}f \cup f\mathcal{S} \cup \mathcal{S}f\mathcal{S}]$, for some $f \in \mathcal{S}$. This implies that $e \in (\mathcal{S}f]$ or $e \in (f\mathcal{S}]$ or $e \in (\mathcal{S}f\mathcal{S}]$. (i) If $e \in (\mathcal{S}f]$, then $(\mathcal{S}e] \subseteq (\mathcal{S}\mathcal{S}f] \subseteq (\mathcal{S}f]$. (ii) If $e \in (f\mathcal{S}]$, then $(e\mathcal{S}] \subseteq (f\mathcal{S}]$. (iii) If $e \in (\mathcal{S}f\mathcal{S}]$, then $(\mathcal{S}e\mathcal{S}] \subseteq (\mathcal{S}f\mathcal{S}]$. From (i), (ii) and (iii), $(\mathcal{S}e \cup e\mathcal{S} \cup \mathcal{S}e\mathcal{S}] \subseteq (\mathcal{S}f \cup f\mathcal{S} \cup \mathcal{S}f\mathcal{S}]$. Since $(\mathcal{S}f \cup f\mathcal{S} \cup \mathcal{S}f\mathcal{S}] \subseteq \mathcal{I}(f)$ and therefore $\mathcal{I}(e) \subseteq \mathcal{I}(f)$. Also $f \notin \mathcal{I}(e) \Rightarrow f \notin \mathcal{I}^{(e)}$. Thus $\mathcal{I}(e) \neq \mathcal{I}(f) \Rightarrow \mathcal{I}^{(e)} \neq \mathcal{I}^{(f)}$. For if $f \in \mathcal{I}(e) \Rightarrow \mathcal{I}(e) = \mathcal{I}(f)$ and so $\mathcal{I}^{(e)} = \mathcal{I}^{(f)}$, which is not possible. Then $f \in \mathcal{S} - \mathcal{I}(e) \Rightarrow \mathcal{I}(e) \subseteq \big(\mathcal{S}\big(\mathcal{S} - \mathcal{I}(e)\big) \cup \big(\mathcal{S} - \mathcal{I}(e)\big)\mathcal{S} \cup \mathcal{S}\big(\mathcal{S} - \mathcal{I}(e)\big)\mathcal{S}\big]$ and so $\mathcal{I}(e)$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. By Theorem 2.16, $\mathcal{N}^g \cup \mathcal{I}(e)$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. Since $e \notin \mathcal{N}^g$. Therefore $\mathcal{N}^g \subset \mathcal{N}^g \cup \mathcal{I}(e)$, which is a contradiction. Thus $f \in \mathcal{I}^{(e)}$ and so $\mathcal{I}(e) \subseteq (\mathcal{S}f \cup f\mathcal{S} \cup \mathcal{S}f\mathcal{S}] \subseteq \mathcal{I}(f) = \mathcal{I}(e)$. Then $\mathcal{I}(e) = (\mathcal{S}f \cup f\mathcal{S} \cup \mathcal{S}f\mathcal{S}] = \mathcal{I}(f)$ clearly, $(\mathcal{S}e \cup e\mathcal{S} \cup \mathcal{S}e\mathcal{S}] \subseteq \mathcal{I}(e)$. If $f \leq e$, then $\mathcal{I}(e) = (\mathcal{S}f \cup f\mathcal{S} \cup \mathcal{S}f\mathcal{S}] \subseteq (\mathcal{S}e \cup e\mathcal{S} \cup \mathcal{S}e\mathcal{S}]$. Hence, $\mathcal{I}(e) \subseteq (\mathcal{S}e \cup e\mathcal{S} \cup \mathcal{S}e\mathcal{S}]$. If $f \nleq e$, then $f \in (\mathcal{S}e \cup e\mathcal{S} \cup \mathcal{S}e\mathcal{S}]$. Similarly from (i), (ii) and (iii) we can say $(\mathcal{S}f \cup f\mathcal{S} \cup \mathcal{S}f\mathcal{S}] \subseteq (\mathcal{S}e \cup e\mathcal{S} \cup \mathcal{S}e\mathcal{S}]$, therefore $\mathcal{I}(e) = \mathcal{I}(f) = (\mathcal{S}f \cup f\mathcal{S} \cup \mathcal{S}f\mathcal{S}] \subseteq (\mathcal{S}e \cup e\mathcal{S} \cup \mathcal{S}e\mathcal{S}]$. Hence, $\mathcal{I}(e) = (\mathcal{S}e \cup e\mathcal{S} \cup \mathcal{S}e\mathcal{S}]$. □

Theorem 4.11. *Suppose $\mathcal{S}$ contains the greatest $\mathcal{QC}$-ideal $\mathcal{N}^g$. If $\mathcal{S} \neq (\mathcal{S}^2]$ and any two elements of $\mathcal{S} - (\mathcal{S}^2]$ are incredible. Then $\mathcal{S}$ contains a quasi base.*

Proof. Firstly, we are given that $\mathcal{N}^g$ is an $\mathcal{QC}$-ideal of $\mathcal{S}$. Then $\mathcal{N}^g \subseteq (\mathcal{S}(\mathcal{S}-\mathcal{N}^g) \cup (\mathcal{S}-\mathcal{N}^g)\mathcal{S} \cup \mathcal{S}(\mathcal{S}-\mathcal{N}^g)\mathcal{S}] \subseteq (\mathcal{S}^2 \cup \mathcal{S}^2 \cup \mathcal{S}^3] \subseteq (\mathcal{S}^2] \subseteq \mathcal{S}$. Let $\mathcal{N}^\alpha$, $\mathcal{N}^\beta$ and $\mathcal{N}^\gamma$ are the $\mathcal{I}$-class of $(\mathcal{S}-\mathcal{S}^2]$, $(\mathcal{S}^2-\mathcal{S}^3]$ and $(\mathcal{S}-\mathcal{N}^g]$ respectively. (i) From Lemma 2[8], we have $\mathcal{I}(\alpha) = \mathcal{I}(\beta) \Leftrightarrow (\alpha] = (\beta] \Leftrightarrow \alpha = \beta$, $\forall\ \alpha,\ \beta \in \mathcal{S} - (\mathcal{S}^2]$. Thus $\mathcal{N}^\alpha$ contains only one element. Since any two elements of $\mathcal{S} - \mathcal{S}^2$ are incredible, $\mathcal{N}^\alpha$ is maximal. (ii) Let for any arbitrary $\alpha \in \mathcal{N}^\beta \subseteq (\mathcal{S}^2 - \mathcal{S}^3]$, $\exists$ $x, y \in \mathcal{S}$ such that $\alpha \leq xy$. Thus $x, y \in (\mathcal{S} - \mathcal{S}^2]$. Otherwise $x, y \in (\mathcal{S}^2]$, which contradiction, therefore $\alpha \in \mathcal{I}(x) = (x]$. (iii) From Lemma 4.10, we get $\mathcal{N}^\gamma$ is maximal. Since $\mathcal{N}^g \subseteq (\mathcal{S}(\mathcal{S} - \mathcal{N}^g) \cup (\mathcal{S} - \mathcal{N}^g)\mathcal{S} \cup \mathcal{S}(\mathcal{S} - \mathcal{N}^g)\mathcal{S}]$. Therefore $\exists\ \alpha_1 \in \mathcal{S} - \mathcal{N}^g$ such that $x \in \mathcal{I}(\alpha_1)$ for any $x \in \mathcal{N}^g$ accordingly $x \in \mathcal{N}^\alpha$ or $\mathcal{N}^\beta$ or $\mathcal{N}^\gamma$. Let us construct a set B. (a) If $\alpha_1 \in \mathcal{N}^\alpha$ or $\alpha_1 \in \mathcal{N}^\gamma$, then we can choose α_1 into B. (b) If $\alpha_1 \in \mathcal{N}^\beta$, then $\exists\ \beta_1 \in \mathcal{N}^\alpha$ such that $\alpha_1 \in \mathcal{I}(\beta_1)$. Thus we can choose any β_1 into B. Now, $\mathcal{N}^g \subseteq (B \cup (\mathcal{S}B \cap B\mathcal{S}) \cup \mathcal{S}B\mathcal{S}]$. $(\mathcal{S}^2] - (\mathcal{S}^3] \subseteq (B \cup (\mathcal{S}B \cap B\mathcal{S}) \cup \mathcal{S}B\mathcal{S}]$, $\mathcal{S} - (\mathcal{S}^2] \subseteq (B \cup (\mathcal{S}B \cap B\mathcal{S}) \cup \mathcal{S}B\mathcal{S}]$, $(\mathcal{S}^3] - (\mathcal{N}^g] \subseteq (B \cup (\mathcal{S}B \cap B\mathcal{S}) \cup \mathcal{S}B\mathcal{S}]$. Hence $\mathcal{S} \subseteq (B \cup (\mathcal{S}B \cap B\mathcal{S}) \cup \mathcal{S}B\mathcal{S}]$, i.e. $(B \cup (\mathcal{S}B \cap B\mathcal{S}) \cup \mathcal{S}B\mathcal{S}] = \mathcal{S}$. To show that B is quasi base of $\mathcal{S}$. It is easy to prove that there is no proper subset $D \subset B$ with $(D \cup (\mathcal{S}D \cap D\mathcal{S}) \cup \mathcal{S}D\mathcal{S}] = \mathcal{S}$. This is obvious because B is constructed by the element of maximal $\mathcal{I}$-classes of $\mathcal{S}$ and from every maximal $\mathcal{I}$-class just choose one element into B. Hence B is quasi base of $\mathcal{S}$. □

5. CONCLUSION

The notion of quasi covered ideals of an ordered semigroup generalizes the concept of an ideal, incorporating the order structure into its definition. In this paper, we have introduced quasi covered ideal, the greatest quasi covered ideal and quasi base of an ordered semigroup and discussed some properties of the same. Also, we have characterized them and discussed the relation with each other. Finally, we have provided the conditions of covered ideal and the greatest ideal to be quasi covered ideal of an ordered semigroup.

In summary, the investigation of quasi covered ideal and quasi base of an ordered semigroup not only contributes to the theoretical foundations of an ideal theory in abstract algebra but also opens up avenues for interdisciplinary research and real world applications. This work sets the stage for future explorations, inviting researchers to delve deeper into the intricate structures of an ordered semigroup and their implications across various mathematical and computational domains.

CONFLICT OF INTERESTS

According to the authors, there are no conflict of interest.

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