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ASYMPTOTICALLY WIJSMAN LACUNARY SEQUENCES OF ORDER (α, β) BY USING IDEAL

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Abstract. In this paper, we propose the concepts of asymptotic equivalence, asymptotic statistical equivalence, lacunary statistical equivalence of order (α, β) in sense of Wijsman. We also make an effort to define these concepts by using modulus function with respect to ideal $\mathcal{I}$ and examine some algebraic and topological properties related to these concepts.

1. Introduction

Functional analysis and summability theory both primarily depend on the idea of statistical convergence. Schoenberg [30] introduced the connections between summability theory and statistical convergence. Ulusu and Nuray [35] introduced the idea of Wijsman lacunary statistical convergence. In accordance with admissible ideal $\mathcal{I}$ of subsets $\mathbb{N}$, Kostyrko et al. [15] generalised statistical convergence and called it $\mathcal{I}$-convergence, more effective convergence in metric space, whereas, for real sequences, Das et al. [7] combined this concept with statistical convergence. Savas et al. [29] has presented new concepts regarding $\mathcal{I}$-statistical convergence by combining the ideas of statistical convergence and $\mathcal{I}$-convergence.

Marouf [20] identified asymptotically equivalent real number sequences and investigated their connections to certain matrices-transformed sequences to examine the growth rates of two sequences. By suggesting an analogy between these definitions and the inherent regularity criteria for nonnegative summability matrices, Patterson [25] enlarged these ideas. Many authors have expressed interest in examining different challenges that have arisen in this field (see [1, 6, 9, 10, 11, 12, 23, 24, 26, 32, 33]).

The Lacunary sequence $\theta = (k_r)$ is an ascending positive integers sequence of the form $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$, where $k_0 = 0$. Let $J_r = (k_{r-1}, k_r]$ and $q_r = \frac{k_r}{k_{r-1}}$. ℓ_∞ is the collection of all bounded sequences. Fridy et al.

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[8] proposed generalized statistical convergence by using lacunary sequences. More information about these (see [13, 16, 34]) and references therein.
Nakano [22] introduced modulus function as f from $[0,\infty)$ to $[0,\infty)$ such that

1. $f(u) = 0$ if and only if $u = 0$;
2. $f(u+v) \leq f(u) + f(v)$ for all $u \geq 0, v \geq 0$;
3. f is increasing;
4. f is continuous from right at 0.

Thus f must be continuous on $[0,1)$. A modulus function may be bounded or unbounded. Modulus function was commonly used by several authors to construct some sequence spaces (see [19, 14, 27, 5]). Bilgin [3] investigated some of the relationships between asymptotically equivalent sequences and defined the concept using modulus function. More information about these (see [21, 4, 31]) and references therein. Throughout the article we will use α, β such that $0 < \alpha \leq \beta \leq 1$. Now we recall some definitions of statistical convergence, lacunary statistical convergence, Wijsman convergence of order (α, β) from [17, 18] and references therein.

Definition 1.1. *The nonnegative sequences $\xi = (\xi_k)$ and $\chi = (\chi_k)$ are known to be asymptotically equivalent to L such that*

$$\lim_{k\to\infty} \frac{\xi_k}{\chi_k} = L.$$

Definition 1.2. *The two nonnegative sequences $\xi = (\xi_k)$ and $\chi = (\chi_k)$ are asymptotically statistically equivalent to L of order (α, β) such that*

$$\lim_{m\to\infty} \frac{1}{m^\alpha} \left| \left\{ k \leq m : \left| \frac{\xi_k}{\chi_k} - L \right| \geq \epsilon \right\} \right|^\beta = 0, \text{ for every } \epsilon > 0.$$

Definition 1.3. *For lacunary sequence $\theta = k_r$, the nonnegative sequences $\xi = (\xi_k)$ and $\chi = (\chi_k)$ are asymptotically lacunarily statistically equivalent to L of order (α, β) such that*

$$\lim_r \frac{1}{h_r^\alpha} \left| \left\{ k \in J_r : \left| \frac{\xi_k}{\chi_k} - L \right| \geq \epsilon \right\} \right|^\beta = 0, \text{ for every } \epsilon > 0.$$

It is symbolised by $\xi_{(\alpha,\beta)} \sim^{S_\theta^L} \chi_{(\alpha,\beta)}$ and for $L = 1$, it is known as simply asymptotically lacunary statistically equivalent of order (α, β).

Consider $(X, \mathcal{J})$ be a metric space and for any nonempty subset A of X, the distance between u to A is

$$\delta(u, A) = \inf_{a\in A} \mathcal{J}(u, a), \; u \in X.$$

Definition 1.4. *Consider $(X, \mathcal{J})$ be a metric space and A, A_k be nonempty closed subsets of X, then $\{A_k\}$ is Wijsman convergent to A if $\lim_{k\to\infty} \delta(u, A_k) = \delta(u, A)$, for each $u \in X$. It is symbolised by $\mathcal{W} - \lim A_k = A$.*

Definition 1.5. *Consider $(X, \mathcal{J})$ be a metric space. For nonempty closed subsets A and A_k of X the sequence $\{A_k\}$ is Wijsman statistically convergent to A of order (α, β) if for $\epsilon > 0$,*

$$\lim_{m\to\infty} \frac{1}{m^\alpha} |\{k \leq m : |\delta(u, A_k) - \delta(u, A)| \geq \epsilon\}|^\beta = 0, \text{for each } u \in X.$$

It is symbolised by $St - \lim_{\mathcal{W}} A_k = A$ or $A_k \to A(\mathcal{W}S)$.

Definition 1.6. *Consider $(X, \mathcal{J})$ be a metric space and nonempty closed subset A_k of X, the sequence $\{A_k\}$ is bounded if*

$$\sup_k \delta(u, A_k) < \infty, \text{ for each } u \in X.$$

We can write it as $\{A_k\} \in \ell_\infty$.

Definition 1.7. *Consider $(X, \mathcal{J})$ be a metric space. For nonempty closed subsets A and A_k and lacunary sequence $\theta = \{k_r\}$, the sequence $\{A_k\}$ is Wijsman statistically convergent to A of order (α, β) if for $\epsilon > 0$,*

$$\lim_r \frac{1}{h_r^\alpha} |\{k \in J_r : |\delta(u, A_k) - \delta(u, A)| \geq \epsilon\}|^\beta = 0, \text{for each } u \in X.$$

We can write it as $S_\theta - \lim_{\mathcal{W}} = A$ or $A_k \to A(\mathcal{W}S_\theta)$.

A family $\mathcal{I} \in P(X)$ is known to be ideal if
(i) $\phi \in \mathcal{I}$;
(ii) $A \in \mathcal{I}$ and $B \subset A$ then $B \in \mathcal{I}$;
(iii) $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$.
If $\mathcal{I} \neq \phi$ and $X \notin \mathcal{I}$, then $\mathcal{I}$ is known to be non-trivial ideal.

Definition 1.8. *Consider $(X, \mathcal{J})$ be a metric space. For a non-trivial ideal $\mathcal{I} \in P(X)$, the sequence $\xi = (\xi_k)$ in X is $\mathcal{I}$-statistically convergence to μ of order (α, β) such that for each $\epsilon > 0$,*

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha} |\{k \leq m : |\xi_m - \mu| \geq \epsilon\}|^\beta \geq \varrho\right\} \in \mathcal{I}, \text{ for every } \varrho > 0.$$

We can write it as $\mathcal{I} - S - \lim_{m\to\infty} \xi_m = \mu$ or $\xi_m \to \mu(\mathcal{I} - S)$.

Definition 1.9. *Consider a non-trivial ideal $\mathcal{I} \in P(X)$ the nonnegative sequences $\xi = (\xi_k)$ and $\chi = (\chi_k)$ are known to be asymptotically $\mathcal{I} - [\mathcal{C}, 1]$-equivalent to L of order (α, β) such that*

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha} \left[\sum_{n=1}^m \left|\frac{\xi_m}{\chi_m} - L\right| \geq \varrho\right]^\beta\right\} \in \mathcal{I}, \text{ for each } \varrho > 0.$$

It is symbolised by $\xi_{(\alpha,\beta)} \sim^{[\mathcal{C},1]^{L_\mathcal{I}}} \chi_{(\alpha,\beta)}$ and for $L = 1$, it is known as simply asymptotically $\mathcal{I} - [\mathcal{C}, 1]$-equivalent of order (α, β).

Definition 1.10. *For a lacunary sequence $\theta = \{k_r\}$ and a non-trivial ideal $\mathcal{I} \in P(X)$, the sequences $\xi = (\xi_k)$ and $\chi = (\chi_k)$ are known to be asymptotically lacunarily statistically equivalent to L of order (α, β) such that for each $\epsilon > 0$ and $\varrho \geq 0$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left| \left\{ k \in J_r : \left| \frac{\xi_m}{\chi_m} - L \right| \geq \epsilon \right\} \right|^\beta \geq \varrho \right\} \in \mathcal{I}.$$

It is symbolised by $\xi_{(\alpha,\beta)} \sim^{\mathcal{I}(S_\theta)} \chi_{(\alpha,\beta)}$ and if $L = 1$, then it is known as simply asymptotically lacunarily statistically equivalent of order (α, β).

Definition 1.11. *For non-trivial ideal $\mathcal{I} \in P(X)$ the sequences $\xi = (\xi_k)$ and $\chi = (\chi_k)$ are known to be asymptotically $\mathcal{I}$-statistically equivalent to L of order (α, β) such that for every $\epsilon > 0$,*

$$\left\{ m \in \mathbb{N} : \frac{1}{m^\alpha} \left| \left\{ k \leq m : \left| \frac{\xi_m}{\chi_m} - L \right| \geq \epsilon \right\} \right|^\beta \geq \varrho \right\} \in \mathcal{I}, \text{ for each } \varrho \geq 0.$$

It is symbolised by $\xi_{(\alpha,\beta)} \sim^{S^{L(\mathcal{I})}} \chi_{(\alpha,\beta)}$ and for $L = 1$, it is known as simply asymptotically $\mathcal{I}$-statistically equivalent of order (α, β).

Definition 1.12. *Consider $(X, \mathcal{J})$ be a metric space and A_k and B_k be nonempty closed subsets of X such that $\delta(u, A_k) > 0$ and $\delta(u, B_k) > 0$ for each $u \in X$, then $\{A_k\}, \{B_k\}$ are asymptotically Wijsman statistically equivalent to L of order (α, β) if*

$$\lim_m \frac{1}{m^\alpha} \left| \left\{ k \leq m : \left| \frac{\delta(u, A_k)}{\delta(u, B_k)} - L \right| \geq \epsilon \right\} \right|^\beta = 0, \text{ for each } u \in X.$$

It is symbolised by $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{W}S_L} \{B_k\}_{(\alpha,\beta)}$ and for $L = 1$, it is known as simply asymptotically Wijsman statistically equivalent of order (α, β).

Definition 1.13. *Consider $(X, \mathcal{J})$ be a metric space and A_k and B_k be nonempty closed subsets of X and a lacunary sequence $\theta = \{k_r\}$ such that $\delta(u, A_k) > 0$ and $\delta(u, B_k) > 0$ for each $u \in X$, then $\{A_k\}, \{B_k\}$ are asymptotically Wijsman lacunarily statistically equivalent to L of order (α, β) if*

$$\lim_r \frac{1}{h_r^\alpha} \left| \left\{ k \in J_r : \left| \frac{\delta(u, A_k)}{\delta(u, B_k)} - L \right| \geq \epsilon \right\} \right|^\beta = 0, \text{ for each } u \in X.$$

It is symbolised by $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{W}S_\theta^L} \{B_k\}_{(\alpha,\beta)}$ and for $L = 1$, it is known as simply asymptotically Wijsman lacunarily statistically equivalent of order (α, β).

Motivated by above notions, we define new some notions in the proceeding section and obtain several interesting results.

2. Main results

We now consider our main results and begin with the following definitions

Definition 2.1. *Consider $(X, \mathcal{J})$ be a metric space and A_k and B_k be nonempty closed subsets of X and a non-trivial ideal $\mathcal{I} \in P(X)$ such that $\delta(u, A_k) > 0$ and $\delta(u, B_k) > 0$ for each $u \in X$, then the sequences $\{A_k\}, \{B_k\}$ are strongly asymptotically Wijsman equivalent with respect to ideal $\mathcal{I}$ to L of order (α, β) such that*

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left[\sum_{k=1}^{m}\left|\frac{\delta(u, A_k)}{\delta(u, B_k)} - L\right| \geq \varrho\right]^\beta\right\} \in \mathcal{I}, \text{ for each } \varrho \geq 0.$$

It is symbolised by $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}_L)} \{B_k\}_{(\alpha,\beta)}$ and if $L = 1$, then it is known as simply strongly asymptotically Wijsman equivalent with respect to ideal $\mathcal{I}$ of order (α, β).

Definition 2.2. *Consider $(X, \mathcal{J})$ be a metric space and A_k and B_k be nonempty closed subsets of X and a non-trivial ideal $\mathcal{I} \in P(X)$ such that $\delta(u, A_k) > 0$ and $\delta(u, B_k) > 0$ for each $u \in X$, then the sequences $\{A_k\}, \{B_k\}$ are asymptotically Wijsman statistically equivalent with respect to ideal $\mathcal{I}$ to L of order (α, β) such that for every $\epsilon > 0$,*

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left|\left\{k \leq m : \left|\frac{\delta(u, A_k)}{\delta(u, B_k)} - L\right| \geq \epsilon\right\}\right|^\beta \geq \varrho\right\} \in \mathcal{I}, \text{ for each } \varrho \geq 0.$$

It is symbolised by $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}S^L)} \{B_k\}_{(\alpha,\beta)}$ and if $L = 1$, then it is known as simply asymptotically Wijsman statistically equivalent with respect to ideal $\mathcal{I}$ of order (α, β).

Definition 2.3. *For lacunary sequence $\theta = \{k_r\}$ and $(X, \mathcal{J})$ be a metric space. Suppose that A_k and B_k be nonempty closed subsets of X such that $\delta(u, A_k) > 0$ and $\delta(u, B_k) > 0$ for each $u \in X$ and a non-trivial ideal $\mathcal{I} \in P(X)$, then the sequences $\{A_k\}, \{B_k\}$ are asymptotically Wijsman lacunarily statistically equivalent with respect to ideal $\mathcal{I}$ to L of order (α, β) such that for every $\epsilon > 0$,*

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left|\left\{k \in J_r : \left|\frac{\delta(u, A_k)}{\delta(u, B_k)} - L\right| \geq \epsilon\right\}\right|^\beta \geq \varrho\right\} \in \mathcal{I}, \text{ for each } \varrho \geq 0.$$

It is symbolised by $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}S_\theta^L)} \{B_k\}_{(\alpha,\beta)}$ and for $L = 1$, it is known as simply asymptotically Wijsman lacunarily statistically equivalent with respect to ideal $\mathcal{I}$ of order (α, β).

Definition 2.4. *For lacunary sequence $\theta = \{k_r\}$ and $(X, \mathcal{J})$ be a metric space. Suppose that A_k and B_k be nonempty closed subsets of X such that $\delta(u, A_k) > 0$ and $\delta(u, B_k) > 0$ for each $u \in X$ and a non-trivial ideal $\mathcal{I} \in P(X)$,* *then the sequences $\{A_k\}, \{B_k\}$ are strongly asymptotically Wijsman lacunarily statistically equivalent with respect to ideal $\mathcal{I}$ to L of order (α, β) such that*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left[\sum_{k \in J_r} \left| \frac{\delta(u, A_k)}{\delta(u, B_k)} - L \right| \geq \epsilon \right]^\beta \right\} \in \mathcal{I}, \text{ for each } \varrho \geq 0.$$

It is symbolised by $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}[\mathcal{WN}]_\theta^L} \{B_k\}_{(\alpha,\beta)}$ and for $L = 1$, it is known as simply strongly asymptotically Wijsman lacunarily statistically equivalent with respect to ideal $\mathcal{I}$ of order (α, β).

Definition 2.5. *For lacunary sequence $\theta = \{k_r\}$ and $(X, \mathcal{J})$ be a metric space. Suppose that A_k and B_k be nonempty closed subsets of X such that $\delta(u, A_k) > 0$ and $\delta(u, B_k) > 0$ for each $u \in X$. Let f be a modulus function and a non-trivial ideal $\mathcal{I} \in P(X)$, then the sequences $\{A_k\}, \{B_k\}$ are strongly f-asymptotically Wijsman lacunarily equivalent with respect to ideal $\mathcal{I}$ to L of order (α, β) such that for each $\epsilon > 0$,*

$$\left\{ r \in \mathbb{N} : \frac{1}{h_r^\alpha} \left[\sum_{k \in J_r} f\left(\left| \frac{\delta(u, A_k)}{\delta(u, B_k)} - L \right| \right) \geq \epsilon \right]^\beta \right\} \in \mathcal{I}.$$

It is symbolised by $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f_\theta^{\mathcal{W}})} \{B_k\}_{(\alpha,\beta)}$ and if $L = 1$, then it is known as simply strongly f-asymptotically Wijsman lacunarily equivalent with respect to ideal $\mathcal{I}$, of order (α, β).

Definition 2.6. *Consider $(X, \mathcal{J})$ be a metric space and A_k and B_k be nonempty closed subsets of X and a a non-trivial ideal $\mathcal{I} \in P(X)$ such that $\delta(u, A_k) > 0$ and $\delta(u, B_k) > 0$ for each $u \in X$, then the sequences $\{A_k\}, \{B_k\}$ are strongly f-asymptotically Wijsman equivalent with respect to ideal $\mathcal{I}$ to L of order (α, β) such that for each $\varrho \geq 0$,*

$$\left\{ m \in \mathbb{N} : \frac{1}{m^\alpha} \left[\sum_{k=1}^{m} f\left(\left| \frac{\delta(u, A_k)}{\delta(u, B_k)} - L \right| \right) \geq \varrho \right]^\beta \right\} \in \mathcal{I}.$$

It is symbolised by $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f^L\mathcal{W})} \{B_k\}_{(\alpha,\beta)}$ and if $L = 1$, then it is known as simply strongly f-asymptotically Wijsman equivalent with respect to ideal $\mathcal{I}$, of order (α, β).

Theorem 2.7. *Consider $(X, \mathcal{J})$ be a metric space and non-trivial ideal $\mathcal{I} \in P(X)$ and modulus function f. Then*
(a) If $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}_L)} \{B_k\}_{(\alpha,\beta)}$, then $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f^L\mathcal{W})} \{B_k\}_{(\alpha,\beta)}$;
(b) $\lim_{\kappa\to\infty} \frac{f(\kappa)}{\kappa} = \gamma > 0$ and $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}_L)} \{B_k\}_{(\alpha,\beta)}$ if and only if $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f^L\mathcal{W})} \{B_k\}_{(\alpha,\beta)}$.

Proof. (a) For given $\epsilon \geq 0$ and $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}_L)} \{B_k\}_{(\alpha,\beta)}$. Choose $0 < \varrho < 1$ such that $f(t) < \epsilon$ for $0 \leq \kappa \leq \varrho$. Thus, we have

$$\frac{1}{m^{\alpha}}\left[\sum_{k=1}^{m} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\right]^{\beta}$$
$$=\quad \frac{1}{m^{\alpha}}\left[\sum_{k=1} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\right]^{\beta}+\frac{1}{m^{\alpha}}\left[\sum_{k=2} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\right]^{\beta},$$

where the first summation runs over $\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\le\varrho$, and the complement of first summation is $\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|>\varrho$. Also by using modulus function f, we get

$$\frac{1}{m^{\alpha}}\left[\sum_{k=1}^{m} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\right]^{\beta}<\epsilon+\left(\frac{2f(1)}{\varrho}\right)\frac{1}{m^{\alpha}}\left[\sum_{k=1}^{m}\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right]^{\beta}.$$

Thus for $\lambda>0$,

$$\left\{m\in\mathbb{N}:\frac{1}{m^{\alpha}}\left[\sum_{k=1}^{m} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\ge\lambda\right]^{\beta}\right\}$$
$$\subseteq\left\{m\in\mathbb{N}:\frac{1}{m^{\alpha}}\left[\sum_{k=1}^{m}\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right]^{\beta}\ge\frac{(\lambda-\epsilon)\varrho}{2f(1)}\right\}.$$

Since $\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}(\mathcal{W}_L)}\{B_k\}_{(\alpha,\beta)}$ and hence, the expression belongs to $\mathcal{I}$, it follows the result. Therefore $\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}(f^L\mathcal{W})}\{B_k\}_{(\alpha,\beta)}$.

(b) If $\lim\limits_{\kappa\to\infty}\dfrac{f(\kappa)}{\kappa}=\gamma>0$, then $f(\kappa)\ge\kappa\gamma$, for all $\kappa>0$. Assume that $\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}(f^L\mathcal{W})}\{B_k\}_{(\alpha,\beta)}$, then

$$\frac{1}{m^{\alpha}}\left[\sum_{k=1}^{m} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\right]^{\beta}\ \ge\ \frac{1}{m^{\alpha}}\left[\sum_{k=1}^{m}\gamma\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\right]^{\beta}$$
$$=\ \gamma\left(\frac{1}{m^{\alpha}}\left[\sum_{k=1}^{m}\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right]^{\beta}\right)$$

it follows that for each $\epsilon>0$, we have

$$\left\{m\in\mathbb{N}:\frac{1}{m^{\alpha}}\left[\sum_{k=1}^{m}\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\ge\epsilon\right]^{\beta}\right\}$$
$$\subseteq\left\{m\in\mathbb{N}:\frac{1}{m^{\alpha}}\left[\sum_{k=1}^{m} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\ge\gamma\epsilon\right]^{\beta}\right\}.$$

Since $\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}(f^L\mathcal{W})}\{B_k\}_{(\alpha,\beta)}$ and $\epsilon>0$, hence, the set belongs to $\mathcal{I}$. This completes the proof. □

Theorem 2.8. [2, 28] *Let $0<\varrho<1$ and f be a modulus function. Then for $z\neq0$ and each $(\frac{x}{z})>\varrho$, we have $f(\frac{x}{z})\le\frac{2f(1)}{\varrho}(\frac{x}{z})$.*

Theorem 2.9. *Consider $(X, \mathcal{J})$ be a metric space, a modulus function f and a non-trivial ideal $\mathcal{I} \subset P(X)$. Then*
(a) If $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f^L\mathcal{W})} \{B_k\}_{(\alpha,\beta)}$, then $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}S^L)} \{B_k\}_{(\alpha,\beta)}$;
(b) If f is bounded, then $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f^L\mathcal{W})} \{B_k\}_{(\alpha,\beta)}$ if and only if $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}S^L)} \{B_k\}_{(\alpha,\beta)}$.

Proof. For given $\epsilon > 0$ and $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f^L\mathcal{W})} \{B_k\}_{(\alpha,\beta)}$, we have

$$\frac{1}{m^\alpha}\left[\sum_{k=1}^{m} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right|\right)\right]^\beta \geq \frac{1}{m^\alpha}\left[\sum_{\substack{k=1\\|\Psi|^\beta \geq \epsilon}}^{m} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right|\right)\right]^\beta$$
$$\geq \frac{f(\epsilon)}{m^\alpha}\left|\left\{k \leq m : \left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right| \geq \epsilon\right\}\right|^\beta,$$

where $\Psi = \left(\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right)$.

Thus, for $\lambda > 0$, we have

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left|\left\{k \leq m : \left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right| \geq \epsilon\right\}\right|^\beta \geq \frac{\lambda}{f(\epsilon)}\right\}$$
$$\subseteq \left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left[\sum_{k=1}^{m} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right|\right)\right]^\beta \geq \lambda\right\}.$$

Since $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f^L\mathcal{W})} \{B_k\}_{(\alpha,\beta)}$, and from Definition 2.6,

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left[\sum_{k=1}^{m} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right|\right)\right]^\beta \geq \lambda\right\} \in \mathcal{I}.$$

Hence, $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}S^L)} \{B_k\}_{(\alpha,\beta)}$.
(b) Assume that $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}S^L)} \{B_k\}_{(\alpha,\beta)}$ and f is bounded. Since f is bounded, there exists a real number Q such that $\sup f(\kappa) \leq Q$. For $\epsilon > 0$, we have

$$\left\{m \in \mathbb{N} : \frac{1}{m^\alpha}\left[\sum_{k=1}^{m} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right|\right)\right]^\beta\right.$$
$$= \frac{1}{m^\alpha}\left\{\left[\sum_{\substack{k=1\\|\Psi|^\beta \geq \epsilon}}^{m} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right|\right)\right]^\beta\right.$$
$$\left. + \left[\sum_{\substack{k=1\\|\Psi|^\beta < \epsilon}}^{m} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right|\right)\right]^\beta\right\}, \text{ where } \Psi = \left(\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right)$$
$$\leq \frac{Q}{m^\alpha}\left|\left\{k \leq m : \left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right| \geq \frac{\epsilon}{2}\right\}\right|^\beta + f(\frac{\epsilon}{2}).$$

Put $B(\epsilon)=\left\{m\in\mathbb{N}:\dfrac{1}{m^{\alpha}}\left[\sum_{k=1}^{m}f\left(\left|\dfrac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\right]^{\beta}\right\}$ and

$A(\epsilon)=\left\{m\in\mathbb{N}:\frac{1}{m^{\alpha}}\left|\left\{k\leq m:\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\frac{\epsilon}{2}\right\}\right|^{\beta}\geq\frac{f(\epsilon)}{2Q}\right\}$. Thus we have $B(\epsilon)\subset A(\epsilon)$. Since $\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}(\mathcal{W}S^L)}\{B_k\}_{(\alpha,\beta)}$, as a result $A(\epsilon)\in\mathcal{I}$ and therefore $B(\epsilon)\in\mathcal{I}$, by using $\epsilon\to 0$. Hence, $\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}(f^L\mathcal{W})}\{B_k\}_{(\alpha,\beta)}$. □

Theorem 2.10. *Consider $(X,\mathcal{J})$ be a metric space and lacunary sequence $\theta=\{k_r\}$, a modulus function f and a non-trivial ideal $\mathcal{I}\subset P(X)$. If $\liminf_r q_r>1$ and $\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}(f^L\mathcal{W})}\{B_k\}_{(\alpha,\beta)}$ then $\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}(f_\theta^{\mathcal{W}})}\{B_k\}_{(\alpha,\beta)}$.*

Proof. If $\liminf_r q_r>1$, then there exists $\varrho>0$ such that $q_r=\frac{k_r}{k_{r-1}}\geq 1+\varrho$, it implies that $\frac{h_r^{\alpha}}{k_r^{\alpha}}\geq\frac{\varrho}{1+\varrho}$. Suppose that $\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}(f^L\mathcal{W})}\{B_k\}_{(\alpha,\beta)}$. For sufficiently large r, we have

$$\begin{aligned}\frac{1}{k_r^{\alpha}}\left[\sum_{k=1}^{k_r}f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\right]^{\beta} &\geq \frac{1}{k_r^{\alpha}}\left[\sum_{k\in J_r}f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\right]^{\beta}\\ &= \left(\frac{h_r^{\alpha}}{k_r^{\alpha}}\right)\frac{1}{h_r^{\alpha}}[\sum_{k\in J_r}f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)]^{\beta}\\ &\geq \left(\frac{\varrho}{1+\varrho}\right)\frac{1}{h_r^{\alpha}}\left[\sum_{k\in J_r}f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\right]^{\beta}.\end{aligned}$$

For $\epsilon>0$,

$$\begin{aligned}&\left\{r\in\mathbb{N}:\frac{1}{h_r^{\alpha}}\left[\sum_{k\in J_r}f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\geq\epsilon\right]^{\beta}\right\}\\ \subseteq&\left\{r\in\mathbb{N}:\frac{1}{k_r^{\alpha}}\left[\sum_{k=1}^{k_r}f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\frac{\epsilon\varrho}{1+\varrho}\right]^{\beta}\right\}.\end{aligned}$$

Since $\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}(f^L\mathcal{W})}\{B_k\}_{(\alpha,\beta)}$, it follows that

$$\left\{r\in\mathbb{N}:\frac{1}{k_r^{\alpha}}\left[\sum_{k=1}^{k_r}f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\frac{\epsilon\varrho}{1+\varrho}\right]^{\beta}\right\}\in\mathcal{I}.$$

Hence,

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^{\alpha}}\left[\sum_{k\in J_r}f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\right)\geq\epsilon\right]^{\beta}\right\}\in\mathcal{I}.$$

Therefore, $\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}(f_\theta^{\mathcal{W}})}\{B_k\}_{(\alpha,\beta)}$. □

Theorem 2.11. *Consider $(X,\mathcal{J})$ be a metric space and $\theta=\{k_r\}$ be a lacunary sequence. Let f be a modulus function and a non-trivial ideal $\mathcal{I}\subset P(X)$. Then*

(a) If $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}[\mathcal{WN}]^L_\theta} \{B_k\}_{(\alpha,\beta)}$, *then* $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f^{\mathcal{W}}_\theta)} \{B_k\}_{(\alpha,\beta)}$*;*
(b) If $\lim_{\kappa\to\infty} \frac{f(\kappa)}{\kappa} = \gamma > 0$, $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}[\mathcal{WN}]^L_\theta} \{B_k\}_{(\alpha,\beta)}$ *if and only if* $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f^{\mathcal{W}}_\theta)} \{B_k\}_{(\alpha,\beta)}$.

Proof. The proof is straight forward and it follows from Theorem 2.7. □

Theorem 2.12. *Consider* $(X, \mathcal{J})$ *be a metric space and a modulus function* f*. For lacunary sequence* $\theta = \{k_r\}$ *and* $\mathcal{I} \subset P(X)$ *be a non-trivial ideal. Then*
(a) If $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f^{\mathcal{W}}_\theta)} \{B_k\}_{(\alpha,\beta)}$, *then* $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}S^L_\theta)} \{B_k\}_{(\alpha,\beta)}$*;*
(b) If f *is bounded,* $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f^{\mathcal{W}}_\theta)} \{B_k\}_{(\alpha,\beta)}$ *if and only if* $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}S^L_\theta)} \{B_k\}_{(\alpha,\beta)}$.

Proof. (a) For given $\epsilon > 0$ and $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f^{\mathcal{W}}_\theta)} \{B_k\}_{(\alpha,\beta)}$, we have

$$\begin{aligned}\frac{1}{h_r^\alpha}\left[\sum_{k\in J_r} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right|\right) \geq \epsilon\right]^\beta &\geq \frac{1}{h_r^\alpha}\Bigg\{\Bigg[\sum_{\substack{k=1\\ |\Psi|^\beta \geq \epsilon}} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right|\right)\Bigg]^\beta \\ &+ \Bigg[\sum_{\substack{k=1\\ |\Psi|^\beta < \epsilon}} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right|\right)\Bigg]^\beta\Bigg\}, \\ &\quad \text{where } \Psi = \left(\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right) \\ &\geq \frac{f(\epsilon)}{h_r^\alpha}\left|\left\{k \in J_r : \left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right| \geq \epsilon\right\}\right|^\beta.\end{aligned}$$

For $\lambda > 0$, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left|\left\{k \in J_r : \left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right| \geq \epsilon\right\}\right|^\beta > \lambda\right\}$$
$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left[\sum_{k\in J_r} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right|\right) \geq \lambda f(\epsilon)\right]^\beta\right\}.$$

Since $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f^{\mathcal{W}}_\theta)} \{B_k\}_{(\alpha,\beta)}$, we have

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left[\sum_{k\in J_r} f\left(\left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right|\right) \geq \lambda f(\epsilon)\right]^\beta\right\} \in \mathcal{I}.$$

By definition of ideal,

$$\left\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\left|\left\{k \in J_r : \left|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\right| \geq \epsilon\right\}\right|^\beta > \lambda\right\} \in \mathcal{I}.$$

Hence, $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}S^L_\theta)} \{B_k\}_{(\alpha,\beta)}$.
(b) Assume that $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}S^L_\theta)} \{B_k\}_{(\alpha,\beta)}$ and f is bounded therefore,

there exists a real number $Q > 0$ such that $|f(\kappa)| \leq Q$ for all $\kappa > 0$, we have

$$\frac{1}{h_r^\alpha}\Big[\sum_{k\in J_r} f\Big(\Big|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\Big|\Big)\Big]^\beta$$

$$= \frac{1}{h_r^\alpha}\Big\{\Big[\sum_{\substack{k\in J_r\\ |\Psi|^\beta \geq \epsilon}} f\Big(\Big|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\Big|\Big)\Big]^\beta$$

$$+\Big[\sum_{\substack{k\in J_r\\ |\Psi|^\beta < \epsilon}} f\Big(\Big|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\Big|\Big)\Big]^\beta\Big\}, \text{ where } \Psi = \Big(\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\Big)$$

$$\leq \frac{Q}{h_r^\alpha}\Big|\Big\{k \in J_r : \Big|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\Big| \geq \frac{\epsilon}{2}\Big\}\Big|^\beta + f(\frac{\epsilon}{2}).$$

Put $B(\epsilon) = \Big\{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\Big[\sum_{k\in J_r} f\Big(\Big|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\Big|\Big)\Big]^\beta\Big\}$ and

$A(\epsilon) = \{r \in \mathbb{N} : \frac{1}{h_r^\alpha}\Big|\Big\{k \in J_r : \big|\frac{\delta(u,A_k)}{\delta(u,B_k)} - L\big| \geq \frac{\epsilon}{2}\Big\}\Big|^\beta \geq \frac{f(\epsilon)}{2Q}\Big\}$.

Thus we have inclusion $B(\epsilon) \subset A(\epsilon)$. Since $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{WS}_\theta^L)} \{B_k\}_{(\alpha,\beta)}$, as a result $A(\epsilon) \in \mathcal{I}$, therefore $B(\epsilon) \in \mathcal{I}$, by using $\epsilon \to 0$.
Hence, $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(f_\theta^{\mathcal{W}})} \{B_k\}_{(\alpha,\beta)}$. □

Theorem 2.13. *Consider $(X, \mathcal{J})$ be a metric space and a modulus function f. Suppose that A_k and B_k be nonempty closed subsets of X and a lacunary sequence $\theta = \{k_r\}$ and a non-trivial ideal $\mathcal{I} \subset P(X)$. Then*
(a) If $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}[\mathcal{WN}]_\theta^L} \{B_k\}_{(\alpha,\beta)}$ then $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{WS}_\theta^L)} \{B_k\}_{(\alpha,\beta)}$ and inclusion is proper.
(b) If $\{A_k\}, \{B_k\} \in \ell_\infty$ and $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{WS}_\theta^L)} \{B_k\}_{(\alpha,\beta)}$ then $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}[\mathcal{WN}]_\theta^L} \{B_k\}_{(\alpha,\beta)}$.
(c) $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{WS}_\theta^L)} \{B_k\}_{(\alpha,\beta)} \cap \ell_\infty = \{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}[\mathcal{WN}]_\theta^L} \{B_k\}_{(\alpha,\beta)} \cap \ell_\infty$.

Proof. (a) Since $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}[\mathcal{WN}]_\theta^L} \{B_k\}_{(\alpha,\beta)}$ and nonempty closed subsets A_k, B_k of X. We have to show $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}[\mathcal{WN}]_\theta^L} \{B_k\}_{(\alpha,\beta)}$. For $\epsilon > 0$, we have

$$\begin{aligned}\frac{1}{h_r^\alpha}\Bigg[\sum_{k\in J_r}\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\epsilon\Bigg]^\beta &\geq \frac{1}{h_r^\alpha}\Bigg\{\Bigg[\sum_{\substack{k=1\\|\Psi|^\beta\geq\epsilon}}\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\Bigg]^\beta\\ &+ \Bigg[\sum_{\substack{k=1\\|\Psi|^\beta<\epsilon}}\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\Bigg]^\beta\Bigg\},\\ &\quad \text{where } \Psi=\left(\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right)\\ &\geq \epsilon\cdot\frac{1}{h_r^\alpha}\left|\left\{k\in J_r:\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\epsilon\right\}\right|,\end{aligned}$$

For given $\varrho>0$,

$$\frac{1}{h_r^\alpha}\left|\left\{k\in J_r:\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\epsilon\right\}\right|^\beta \text{implies that}\frac{1}{h_r^\alpha}\Bigg[\sum_{k\in J_r}\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\epsilon\varrho\Bigg]^\beta.$$

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^\alpha}\left|\left\{k\in J_r:\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\epsilon\right\}\right|^\beta\geq\varrho\right\}$$

$$\subseteq\left\{r\in\mathbb{N}:\frac{1}{h_r^\alpha}\Bigg[\sum_{k\in J_r}\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\epsilon\varrho\Bigg]^\beta\right\}. \tag{1}$$

Since $\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}[\mathcal{WN}]_\theta^L}\{B_k\}_{(\alpha,\beta)}$, by Definition 2.4, the equation (1) belongs to $\mathcal{I}$ which implies

$$\left\{r\in\mathbb{N}:\frac{1}{h_r^\alpha}\left|\left\{k\in J_r:\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\epsilon\right\}\right|^\beta\geq\varrho\right\}\in\mathcal{I}.$$

Hence, $\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}(\mathcal{W}S_\theta^L)}\{B_k\}_{(\alpha,\beta)}$.
Next to show the inclusion $\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}[\mathcal{WN}]_\theta^L}\{B_k\}_{(\alpha,\beta)}\subset\{A_k\}_{(\alpha,\beta)}\sim^{\mathcal{I}(\mathcal{W}S_\theta^L)}\{B_k\}_{(\alpha,\beta)}$ is proper, we give an example. Let

$$A_k=\begin{cases}\{k\}, \text{ if } k_{r-1}<k<k_{r-1}+[\sqrt{h_r}],\ r=1,2,\cdots\\ \{0\},\ \text{or else}\end{cases}$$

$\{B_k\}=0$ for all k. Then clearly $\{A_k\}\not\subset\ell_\infty$. Thus for every $\epsilon>0$ and for each $x\in X$, we have

$$\frac{1}{h_r^\alpha}\left|\left\{k\in J_r:\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\epsilon\right\}\right|^\beta=\frac{[\sqrt{h_r}]}{h_r^\alpha}\to 0 \text{ as } r\to\infty.$$

i.e., $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}[\mathcal{WN}]^L_\theta} \{B_k\}_{(\alpha,\beta)}$. On other hand,

$$\frac{1}{h_r^\alpha}\left[\sum_{k\in J_r}\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\epsilon\right]^\beta \nrightarrow 0 \text{ as } r\to\infty.$$

Therefore, $\{A_k\}_{(\alpha,\beta)} \nsim^{\mathcal{I}[\mathcal{WN}]^L_\theta} \{B_k\}_{(\alpha,\beta)}$

(b) Assume that $\{A_k\},\{B_k\}\in\ell_\infty$ and $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}(\mathcal{W}S^L_\theta)} \{B_k\}_{(\alpha,\beta)}$. We have to show that $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}[\mathcal{WN}]^L_\theta} \{B_k\}_{(\alpha,\beta)}$. For this consider a real number Q such that $\frac{1}{h_r^\alpha}\left[\sum_{k\in J_r}\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\epsilon\right]^\beta \leq Q$ for all $k\in\mathbb{N}$. For given $\epsilon>0$, we have

$$\begin{aligned}
\frac{1}{h_r^\alpha}\left[\sum_{k\in J_r}\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\epsilon\right]^\beta &= \frac{1}{h_r^\alpha}\Bigg\{\Bigg[\sum_{\substack{k=1\\|\Psi|^\beta\geq\epsilon}}\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\epsilon\Bigg]^\beta \\
&+ \Bigg[\sum_{\substack{k=1\\|\Psi|^\beta<\epsilon}}\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\epsilon\Bigg]^\beta\Bigg\}, \\
&\quad \text{where } \Psi=\left(\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right) \\
&\leq \frac{Q}{h_r^\alpha}\left|\left\{k\in J_r:\left|\frac{\delta(u,A_k)}{\delta(u,B_k)}-L\right|\geq\frac{\epsilon}{2}\right\}\right|^\beta+\frac{\epsilon}{2}.
\end{aligned}$$

Thus, $\{A_k\}_{(\alpha,\beta)} \sim^{\mathcal{I}[\mathcal{WN}]^L_\theta} \{B_k\}_{(\alpha,\beta)}$.

(c) It directly follows from (a) and (b). □

3. Conclusion

We defined the notions of asymptotic equivalence, asymptotic statistical equivalence, lacunary statistical equivalence of order (α,β) in sense of Wijsman. Moreover, we made an effort to define these notions by means of modulus function with respect to ideal $\mathcal{I}$ and determined some algebraic and topological properties related to these concepts.

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