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ON SOME REFINING INEQUALITIES VIA BEREZIN SYMBOLS

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Abstract. In recent years, several inequalities have been established when comparing the features of the Berezin transform directly. We examine other inequities associated with them in this work. For the Berezin numbers of a reproducing kernel Hilbert space operator, we therefore derived a variety of upper estimates. A few uses for the outcomes are also provided.

1. Introduction

In the present article, we prove some upper estimates for the Berezin symbols on a reproducing kernel Hilbert space operators. Recall that the reproducing kernel Hilbert space (shortly RKHS) is the Hilbert space $\mathcal{H} = \mathcal{H}(\digamma)$ of complex-valued functions on some set $\digamma$ such that the evaluation functionals $\varphi_\rho(f) = f(\rho)$, $\rho \in \digamma$, are continuous on $\mathcal{H}$ and for every $\rho \in \digamma$ there exist a function $f_\rho \in \mathcal{H}$ such that $f_\rho(\rho) \neq 0$ or, equivalently, there is no $\rho_0 \in \digamma$ such that $f(\rho_0) = 0$ for all $f \in \mathcal{H}$. Then by the Riesz representation theorem for each $\rho \in \digamma$ there exists a unique function $k_\rho \in \mathcal{H}$, which is called the reproducing kernel of the space $\mathcal{H}$, such that $f(\rho) = \langle f, k_\rho \rangle$ for all $f \in \mathcal{H}$. The function $K_\rho := \frac{k_\rho}{||k_\rho||}, \rho \in \digamma$, is called the normalized reproducing kernel of $\mathcal{H}$. The prototypical RKHSs are the Hardy space $H^2(\mathbb{D})$, the Bergman space $L^2_a(\mathbb{D})$, the Dirichlet space $\mathcal{D}^2(\mathbb{D})$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ is the unit disc, and the Fock space $\mathcal{F}(\mathbb{C})$. A detailed presentation of the theory of reproducing kernels and RKHSs is given, for instance in Aronzajn [1]. Reproducing kernels play important role in many branches of pure and applied mathematics including frame theory, wavelets, signals, fractals theories (see for instance, Jorgensen's book [19] and its references).

Note that for a bounded linear operator X on $\mathcal{H}$ (i.e., for $X \in \mathcal{B}(\mathcal{H})$ its Berezin symbol $\tilde{X}$ is defined on $\digamma$ by (see Berezin [6])

$$\widetilde{X}(\rho) := \langle XK_\rho(z), K_\rho(z) \rangle, \rho \in \digamma.$$

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In other words, Berezin symbol $\widetilde{X}$ is the function on $\digamma$ defined by restriction of the quadratic form $\langle Xx, x\rangle$ with $x \in \mathcal{H}$ to the subset of all normalized reproducing kernels of the unit sphere in $\mathcal{H}$. It is clear from the Cauchy-Schwarz inequality that $\widetilde{X}$ is the bounded function on $\digamma$ whose values lie in the numerical range of the operator X. So, the Berezin number $\mathrm{b}(X)$ and the Berezin set $\mathrm{B}(X)$ of operator X are defined respectively by

$$\mathrm{b}(X) := \sup_{\rho\in\digamma} \left|\widetilde{X}(\rho)\right|$$

and

$$\mathrm{B}(X) := \mathrm{Range}\left(\widetilde{X}\right) = \{\widetilde{X}(\rho) : \rho \in \digamma\}.$$

Note also that the numerical range and numerical radius of operator X is defined respectively by

$$W(X) := \{\langle Xx, x\rangle : x \in \mathcal{H}(\digamma) \text{ and } \|x\| = 1\}$$

and

$$w(X) := \sup_{\|x\|=1} |\langle Xx, x\rangle|.$$

See [7, 11, 16, 22, 23, 24, 29] for further details on numerical radius. Clearly, $\mathrm{B}(X) \subset W(X)$ and $\mathrm{b}(X) \leq w(X)$. For the relations between $\mathrm{B}(X)$ and $W(X)$, see for instance, [1, 18, 20] and references therein.

Moreover, the Berezin number of an operator X satisfies the following properties:

(i) $\mathrm{b}(X) = \mathrm{b}(X^*)$.

(ii) $\mathrm{b}(\alpha X) = |\alpha|\mathrm{b}(X)$ for all $\alpha \in \mathbb{C}$.

(iii) $\mathrm{b}(X+Y) \leq \mathrm{b}(X) + \mathrm{b}(Y)$ for all $X, Y \in \mathcal{B}(\mathcal{H})$.

Notice that, in general the Berezin number does not define a norm. However, if $\mathcal{H}$ is a reproducing kernel Hilbert space of analytic functions (for instance on the unit disc $\mathbb{D}$), then $\mathrm{b}(\cdot)$ defines a norm on $\mathcal{B}(\mathcal{H}(\mathbb{D}))$ (see [4, 8, 21]). The Berezin symbol has been studied in detail for Toeplitz and Hankel operators on Hardy and Bergman spaces. For example, the Berezin symbol $\widetilde{T_\varphi}$ on the Toeplitz operator $T_\varphi(\varphi \in L^\infty(\delta\mathbb{D}))$ on $(\mathcal{H}^2(\mathcal{D})$ coincides with harmonic extension $\widetilde{\varphi}$ of function φ into the unit disc $\mathbb{D}$; in particular, if $\varphi \in H^\infty(\mathbb{D})$, i.e., if the symbol function φ is a bounded analytic function on $\mathbb{D}$, then $\widetilde{T_\varphi} = \varphi$. Also it is well known that the Toeplitz operator on the Bergman space $L_a^2(\mathbb{D})$ is compact if and only if its Berezin symbol $\widetilde{T_\varphi}$ vanishes on the boundary $\partial\mathbb{D}$, i.e., if $\lim_{\rho\to\mathfrak{z}} \widetilde{T_\varphi}(\rho) = 0$ for all $\mathfrak{z} \in \partial\mathbb{D}$ (see [2]). A one more nice property of the Berezin symbol is the following:

If $\widetilde{X} = \widetilde{Y}$, then $X = Y$. Therefore, the Berezin symbol uniquely determines the operator. The Berezin symbol, Berezin set and Berezin number have been investigated by many authors over the years, a few of them are [3], [5], [10], [15], [20], [26], [27]).

For any operators $X \in \mathcal{B}(\mathcal{H})$ we define the Berezin norm of X as follows

$$||X||_{\mathrm{B}} := \sup_{\rho \in F} ||XK_\rho||.$$

It is obvious that

$$\mathrm{b}(X) \le ||X||_{\mathrm{B}} \le ||X|| \tag{1}$$

Since $\mathrm{span}\{k_\rho : \rho \in F\} = \mathcal{H}(F)$, it is elementary that $||X||_{\mathrm{B}} = 0$ iff $X = 0$. Then it is easy to see that $||X||_{\mathrm{B}}$ share the above properties (i)-(iii), and therefore $||\cdot||_{\mathrm{B}}$ is the norm in $\mathcal{B}(\mathcal{H})$. As improvements of inequality (1), Huban et al. came up with the following upper bounds for Berezin radius:

$$\mathrm{ber}\,(X) \le \frac{1}{2}\left(\|X\|_{\mathrm{b}} + \left\|X^2\right\|_{\mathrm{b}}^{1/2}\right) \tag{2}$$

see [17, Theorem 3.1]

For $X \in \mathcal{B}(\mathcal{H})$ its so-called Aluthge transform $\widehat{X}$ is defined by $\widehat{X} := |X|^{\frac{1}{2}}\, U\, |X|^{\frac{1}{2}}$, where $|X| := (X^*X)^{\frac{1}{2}}$ and U is the partial isometry associated with the polar decomposition $X = U|X|$ and $\ker U = \ker X$. The generalized Aluthge transform, denoted by $\widehat{X}_\varsigma$, is defined as $\widehat{X}_\varsigma = |X|^\varsigma\, U\, |X|^{1-\varsigma}$, $0 \le \varsigma \le 1$. In particular, $\widehat{X}_0 = U^*UU\,|X| = U\,|X| = X$, $\widehat{X}_1 = |X|\,UU^*U = |X|\,U$ and $\widehat{X}_{1/2} = |X|^{\frac{1}{2}}\, U\, |X|^{\frac{1}{2}} = \widehat{X}$. Here $|X|^0$ is defined as U^*U.

In [13, Corollary 1], Garayev et al. proved that $\mathrm{b}(X) \le \frac{1}{2}\left(\|X\|_{\mathrm{b}} + \mathrm{b}(\widehat{X})\right)$.

Concerning the product of two operators, Huban et al. [18, Corollary 2.10] has shown the following estimate of $\mathrm{b}(Y^*X)$,

$$\mathrm{b}(Y^*X) \le \frac{1}{2}\left\||X|^2 + |Y|^2\right\|_{\mathrm{b}} \tag{3}$$

and

$$\mathrm{b}^{2\varsigma}(Y^*X) \le \frac{1}{2}\left\||X|^{4\varsigma} + |Y|^{4\varsigma}\right\|_{\mathrm{b}},\ \varsigma \ge 1.$$

In this paper, we prove some upper bounded inequalities. Some new applications of obtained results are also presented.

2. Main results

In this part, we now present the first outcome.

Theorem 2.1. *Let $X \in \mathcal{B}(\mathcal{H})$ be a operator on the RKHS $\mathcal{H} = \mathcal{H}(F)$. Then for $0 \le \varsigma \le 1$, we have*

$$4\mathrm{b}(X) \le \left\||X|^{2\varsigma} + |X^*|^{2(1-\varsigma)} - (X + X^*)\right\|_{\mathrm{b}} + \left\||X|^{2\varsigma} + |X^*|^{2(1-\varsigma)} + X + X^*\right\|_{\mathrm{b}}.$$

Proof. From the inequality (3), the operator parallelogram law and the triangle inequality, we obtain

$$\begin{aligned}4\mathrm{b}(Y^*X) &\le 2\left\| |X|^2+|Y|^2 \right\|_{\mathrm{b}} \\ &= \left\| |X+Y|^2+|X-Y|^2 \right\|_{\mathrm{b}} \\ &\le \left\| |X+Y|^2 \right\|_{\mathrm{b}} + \left\| |X-Y|^2 \right\|_{\mathrm{b}} \\ &= \left\| |X|^2+|Y|^2+X^*Y+Y^*X \right\|_{\mathrm{b}} + \left\| |X|^2+|Y|^2-X^*Y-Y^*X \right\|_{\mathrm{b}}.\end{aligned}$$

And so,

$$\begin{aligned}4\mathrm{b}(Y^*X) &\le \left\| |X|^2+|Y|^2+X^*Y+Y^*X \right\|_{\mathrm{b}} \\ &\quad + \left\| |X|^2+|Y|^2-(X^*Y+Y^*X) \right\|_{\mathrm{b}}.\end{aligned} \tag{4}$$

Let $X = U|X|$ be the polar decomposition of X. On choosing $X = |X|^{\varsigma}$ and $Y = |X|^{1-\varsigma}U^*$ in the inequality (4), we deduce that

$$4\mathrm{b}(X) \le \left\| |X|^{2\varsigma}+|X^*|^{2(1-\varsigma)}-(X+X^*) \right\|_{\mathrm{b}} + \left\| |X|^{2\varsigma}+|X^*|^{2(1-\varsigma)}+X+X^* \right\|_{\mathrm{b}}.$$

and where the above inequality follows from the inequality in [7], i.e.,

$$|Y|^2 = Y^*Y = U|X|^{1-\varsigma}|X|^{1-\varsigma}U^* = U|X|^{2(1-\varsigma)}U^* = |X^*|^{2(1-\varsigma)}$$

and completes the proof of the theorem. □

In this step, we set up an additional upper estimate.

Theorem 2.2. *Let $X \in \mathcal{B}(\mathcal{H})$ be a operator on the RKHS $\mathcal{H}(F)$. Then, for $0 \le \varsigma \le 1$ and any mean σ, we have*

$$\mathrm{b}(X) \le \frac{1}{2}\left(\left\| |X|^{2(1-\varsigma)}+|X^*|^{2\varsigma} \right\|_{\mathrm{b}} \sigma \left\| |X|^{2\varsigma}+|X^*|^{2(1-\varsigma)} \right\|_{\mathrm{b}} \right).$$

Proof. We obtain

$$\begin{aligned}|\langle XK_\rho, K_\rho\rangle| &= |\langle U|X|K_\rho, K_\rho\rangle| = \left|\left\langle U|X|^{1-\varsigma}|X|^{\varsigma}K_\rho, K_\rho\right\rangle\right| \\ &= \left|\left\langle |X|^{\varsigma}K_\rho, |X|^{1-\varsigma}U^*K_\rho\right\rangle\right| \le \||X|^{\varsigma}K_\rho\| \left\| |X|^{1-\varsigma}U^*K_\rho \right\|\end{aligned}$$

and

$$\begin{aligned}|\langle XK_\rho, K_\rho\rangle| &= |\langle U|X|K_\rho, K_\rho\rangle| = \left|\left\langle U|X|^{\varsigma}|X|^{1-\varsigma}K_\rho, K_\rho\right\rangle\right| \\ &= \left|\left\langle |X|^{1-\varsigma}K_\rho, |X|^{\varsigma}U^*K_\rho\right\rangle\right| \le \left\| |X|^{1-\varsigma}K_\rho \right\| \||X|^{\varsigma}U^*K_\rho\|\end{aligned}$$

from the Cauchy-Schwarz inequality. It follows from the first relation in the above that by the AM-GM inequality

$$\begin{aligned}
|\langle XK_\rho, K_\rho\rangle| &\le \| |X|^\varsigma K_\rho\| \left\| |X|^{1-\varsigma} U^* K_\rho \right\| \\
&= \sqrt{\langle |X|^\varsigma K_\rho, |X|^\varsigma K_\rho\rangle \left\langle |X|^{1-\varsigma} U^* K_\rho, |X|^{1-\varsigma} U^* K_\rho \right\rangle} \\
&= \sqrt{\left\langle |X|^{2\varsigma} K_\rho, K_\rho \right\rangle \left\langle U |X|^{2(1-\varsigma)} U^* K_\rho, K_\rho \right\rangle} \\
&= \sqrt{\left\langle |X|^{2\varsigma} K_\rho, K_\rho \right\rangle \left\langle |X^*|^{2(1-\varsigma)} K_\rho, K_\rho \right\rangle} \\
&\le \frac{1}{2} \left(\left\langle |X|^{2\varsigma} K_\rho, K_\rho \right\rangle + \left\langle |X^*|^{2(1-\varsigma)} K_\rho, K_\rho \right\rangle \right) \\
&= \frac{1}{2} \left\langle \left(|X|^{2\varsigma} + |X^*|^{2(1-\varsigma)} \right) K_\rho, K_\rho \right\rangle
\end{aligned}$$

and where the fourth inequality follows from the inequality in [12], i.e., for any positive number q,

$$|X^*|^q = U\,|X|^q\, U^*.$$

In a similar manner, we get

$$|\langle XK_\rho, K_\rho\rangle| \le \frac{1}{2} \left\langle \left(|X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} \right) K_\rho, K_\rho \right\rangle.$$

Now, we obtain

$$\begin{aligned}
|\langle XK_\rho, K_\rho\rangle| &= |\langle XK_\rho, K_\rho\rangle| \,\sigma\, |\langle XK_\rho, K_\rho\rangle| \\
&\le \frac{1}{2} \left\langle \left(|X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} \right) K_\rho, K_\rho \right\rangle \sigma \left\langle \left(|X|^{2\varsigma} + |X^*|^{2(1-\varsigma)} \right) K_\rho, K_\rho \right\rangle
\end{aligned}$$

and

$$\sup_{\rho\in F} |\langle XK_\rho, K_\rho\rangle| \le \sup_{\rho\in F} \frac{1}{2} \left\langle \left(|X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} \right) K_\rho, K_\rho \right\rangle \sigma \left\langle \left(|X|^{2\varsigma} + |X^*|^{2(1-\varsigma)} \right) K_\rho, K_\rho \right\rangle.$$

By combining the inequalities (2) and (3), using the homogeneity of σ and the monotonicity condition of the mean for positive values, we obtain

$$\mathrm{b}(X) \le \frac{1}{2} \left(\left\| |X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} \right\|_{\mathrm{b}} \sigma \left\| |X|^{2\varsigma} + |X^*|^{2(1-\varsigma)} \right\|_{\mathrm{b}} \right)$$

which desired inequality. □

The inequality

$$\mathrm{b}^2(X) \le \frac{1}{2} \left\| |X|^2 + |X^*|^2 \right\|_{\mathrm{b}}$$

(see [17, Theorem 3.2]) is extended by the following theorem.

Theorem 2.3. *Let $X \in \mathcal{B}(\mathcal{H})$ be a operator on the RKHS $\mathcal{H}(F)$. Then, for $0 \leq \varsigma \leq 1$, we have*

$$\mathrm{b}^2(X) \leq \int_0^1 \left\| \varsigma\, |X|^2 + (1-\varsigma)\, |X^*|^2 \right\|_{\mathrm{b}}^{1-v} \left\| (1-\varsigma)\, |X|^2 + \varsigma\, |X^*|^2 \right\|_{\mathrm{b}}^{v} dv.$$

Proof. Utilizing the Hölder-McCarthly inequality in [25, Theorem 1.4], the logarithmic-geometric mean inequality and the weighted AM-GM mean inequality, one may write

$$\begin{aligned}
&|\langle XK_\rho, K_\rho\rangle|^2\\
&\leq \left\| |X|^\varsigma K_\rho \right\| \left\| |X|^{1-\varsigma} U^* K_\rho \right\| \left\| |X|^{1-\varsigma} K_\rho \right\| \left\| |X|^\varsigma U^* K_\rho \right\|\\
&= \sqrt{\left\langle |X|^{2\varsigma} K_\rho, K_\rho \right\rangle \left\langle |X|^{2(1-\varsigma)} K_\rho, K_\rho \right\rangle} \sqrt{\left\langle U\, |X|^{2(1-\varsigma)}\, U^* K_\rho, K_\rho \right\rangle \left\langle U\, |X|^{2\varsigma}\, U^* K_\rho, K_\rho \right\rangle}\\
&= \sqrt{\left\langle |X|^{2\varsigma} K_\rho, K_\rho \right\rangle \left\langle |X|^{2(1-\varsigma)} K_\rho, K_\rho \right\rangle} \sqrt{\left\langle |X^*|^{2(1-\varsigma)} K_\rho, K_\rho \right\rangle \left\langle |X^*|^{2\varsigma} K_\rho, K_\rho \right\rangle}\\
&= \sqrt{\left\langle |X|^{2\varsigma} K_\rho, K_\rho \right\rangle \left\langle |X^*|^{2(1-\varsigma)} K_\rho, K_\rho \right\rangle} \sqrt{\left\langle |X|^{2(1-\varsigma)} K_\rho, K_\rho \right\rangle \left\langle |X^*|^{2\varsigma} K_\rho, K_\rho \right\rangle}\\
&\leq \int_0^1 \left(\left\langle |X|^{2\varsigma} K_\rho, K_\rho \right\rangle \left\langle |X^*|^{2(1-\varsigma)} K_\rho, K_\rho \right\rangle \right)^{1-v}\\
&\left(\left\langle |X|^{2(1-\varsigma)} K_\rho, K_\rho \right\rangle \left\langle |X^*|^{2\varsigma} K_\rho, K_\rho \right\rangle \right)^{v} dv\\
&\leq \int_0^1 \left(\left\langle |X|^2 K_\rho, K_\rho \right\rangle^{\varsigma} \left\langle |X^*|^2 K_\rho, K_\rho \right\rangle^{1-\varsigma} \right)^{1-v}\\
&\left(\left\langle |X|^2 K_\rho, K_\rho \right\rangle^{1-\varsigma} \left\langle |X^*|^2 K_\rho, K_\rho \right\rangle^{\varsigma} \right)^{v} dv\\
&\leq \int_0^1 \left(\varsigma \left\langle |X|^2 K_\rho, K_\rho \right\rangle + (1-\varsigma) \left\langle |X^*|^2 K_\rho, K_\rho \right\rangle \right)^{1-v}\\
&\left((1-\varsigma) \left\langle |X|^2 K_\rho, K_\rho \right\rangle + \varsigma \left\langle |X^*|^2 K_\rho, K_\rho \right\rangle \right)^{v} dv\\
&= \int_0^1 \left\langle \left(\varsigma\, |X|^2 + (1-\varsigma)\, |X^*|^2 \right) K_\rho, K_\rho \right\rangle^{1-v}\\
&+ \left\langle \left((1-\varsigma)\, |X|^2 + \varsigma\, |X^*|^2 \right) K_\rho, K_\rho \right\rangle^{v} dv\\
&\leq \int_0^1 \left\| \varsigma\, |X|^2 + (1-\varsigma)\, |X^*|^2 \right\|^{1-v} \left\| (1-\varsigma)\, |X|^2 + \varsigma\, |X^*|^2 \right\|^{v} dv.
\end{aligned}$$

by following the same steps as in the proof of Theorem 2.2. Thus, we can get

$$\begin{aligned}
\sup_{\rho \in F} |\langle XK_\rho, K_\rho\rangle|^2 \leq \sup_{\rho \in F} \int_0^1 &\left\langle \left(\varsigma\, |X|^2 + (1-\varsigma)\, |X^*|^2 \right) K_\rho, K_\rho \right\rangle^{1-v}\\
&+ \left\langle \left((1-\varsigma)\, |X|^2 + \varsigma\, |X^*|^2 \right) K_\rho, K_\rho \right\rangle^{v} dv
\end{aligned}$$

and

$$\mathrm{b}^2(X) \le \int_0^1 \left\| \varsigma \left|X\right|^2 + (1-\varsigma) \left|X^*\right|^2 \right\|_{\mathrm{b}}^{1-v} \left\| (1-\varsigma) \left|X\right|^2 + \varsigma \left|X^*\right|^2 \right\|_{\mathrm{b}}^{v} dv$$

which desired inequality. □

The next result provides a refinement of the well-known inequality

$$\mathrm{b}^2(X) \le \frac{1}{4} \left\| |X|^2 + |X^*|^2 \right\|_{\mathrm{b}} + \frac{1}{2}\mathrm{b}(X^2)$$

(see [18, Corollary 2.11]). Buzano's inequality [9] asserts that

$$|\langle z, x\rangle| \, |\langle z, y\rangle| \le \frac{\|z\|^2}{2} \left(|\langle x, y\rangle| + \|x\| \, \|y\| \right) \tag{5}$$

for any $x, y, z \in \mathcal{H}$.

Theorem 2.4. *Let $X \in \mathcal{B}(\mathcal{H})$ be a operator on the RKHS $\mathcal{H}(F)$. Then we have*

$$\mathrm{b}^2(X) \le \frac{1}{2}\mathrm{b}(X^2) + \frac{1}{4} \left\| |X|^2 + |X^*|^2 \right\|_{\mathrm{b}} - \frac{1}{2} \inf_{\rho \in F} \left(\left(\widetilde{|X|^2}(\rho) \right)^{1/2} - \left(\widetilde{|X^*|^2}(\rho) \right)^{1/2} \right)^2 .$$

Proof. Let $x = XK_\rho$, $y = X^*K_\rho$ and $z = K_\rho$, $\rho \in F$, in the inequality (5). Then, we obtain

$$\begin{aligned}
&|\langle XK_\rho, K_\rho\rangle|^2 \\
&\le \frac{1}{2} \left(\left(\left\langle |X|^2 K_\rho, K_\rho \right\rangle \left\langle |X^*|^2 K_\rho, K_\rho \right\rangle \right)^{1/2} + \left| \left\langle X^2 K_\rho, K_\rho \right\rangle \right| \right) \\
&= \frac{1}{2} \left(\left(\langle X^*XK_\rho, K_\rho\rangle \, \langle XX^*, K_\rho\rangle \right)^{1/2} + \left| \left\langle X^2 K_\rho, K_\rho \right\rangle \right| \right) \\
&= \frac{1}{2} \left(\left(\langle XK_\rho, XK_\rho\rangle \, \langle X^*, X^*K_\rho\rangle \right)^{1/2} + \left| \left\langle X^2 K_\rho, K_\rho \right\rangle \right| \right) \\
&= \frac{1}{2} \left(\|XK_\rho\| \, \|X^*K_\rho\| \right) + \left| \left\langle X^2 K_\rho, K_\rho \right\rangle \right| \\
&= \frac{1}{2} (\frac{1}{2} \left(\|XK_\rho\|^2 + \|X^*K_\rho\|^2 - \left(\|XK_\rho\| - \|X^*K_\rho\| \right)^2 \right) + \left| \left\langle X^2 K_\rho, K_\rho \right\rangle \right|) \\
&= \frac{1}{4} \left(\left\langle |X|^2 K_\rho, K_\rho \right\rangle + \left\langle |X^*|^2 K_\rho, K_\rho \right\rangle \right) - \frac{1}{2} \left(\left(\left\langle |X|^2 K_\rho, K_\rho \right\rangle \right)^{1/2} - \left(\left\langle |X^*|^2 K_\rho, K_\rho \right\rangle \right)^{1/2} \right)^2 \\
&+ \frac{1}{2} \left| \left\langle X^2 K_\rho, K_\rho \right\rangle \right|) \\
&= \frac{1}{4} \left\langle \left(|X|^2 + |X^*|^2 \right) K_\rho, K_\rho \right\rangle - \frac{1}{2} \inf_{\rho \in F} \left(\left(\left\langle |X|^2 K_\rho, K_\rho \right\rangle \right)^{1/2} - \left(\left\langle |X^*|^2 K_\rho, K_\rho \right\rangle \right)^{1/2} \right)^2 \\
&+ \frac{1}{2} \left| \left\langle X^2 K_\rho, K_\rho \right\rangle \right| .
\end{aligned}$$

Taking supremum over all $\rho \in F$, we achieve our desired inequality

$$\sup_{\rho\in F} |\langle XK_\rho, K_\rho\rangle|^2 \le \sup_{\rho\in F} \frac{1}{2} \left|\left\langle X^2K_\rho, K_\rho\right\rangle\right| + \sup_{\rho\in F} \frac{1}{4} \left\langle \left(|X|^2 + |X^*|^2\right) K_\rho, K_\rho \right\rangle - \frac{1}{2}\inf_{\rho\in F} \left(\left(\left\langle |X|^2 K_\rho, K_\rho\right\rangle\right)^{1/2} - \left(\left\langle |X^*|^2 K_\rho, K_\rho\right\rangle\right)^{1/2} \right)^2$$

and

$$\mathrm{b}^2(X) \le \frac{1}{2}\mathrm{b}(X^2) + \frac{1}{4} \left\| |X|^2 + |X^*|^2 \right\|_{\mathrm{ber}} - \frac{1}{2}\inf_{\rho\in F} \left(\left(\widetilde{|X|^2}(\rho)\right)^{1/2} - \left(\widetilde{|X^*|^2}(\rho)\right)^{1/2} \right)^2,$$

which completes the proof. □

The following well-known the polarization identity for the inner product $\langle x, y\rangle$ will be used in the sequel. Let $X \in \mathcal{B}(\mathcal{H})$ and $x, y \in \mathcal{H}$. Then

$$\langle x, y\rangle = \frac{1}{4}\sum_{k=0}^{3} i^k \left\|x + i^k y\right\|^2 .$$

Theorem 2.5. *Let $X \in \mathcal{B}(\mathcal{H})$ be a operator on the RKHS $\mathcal{H} = \mathcal{H}(F)$. Then, for $0 \le \varsigma \le 1$, we have*

$$4\mathrm{b}(X) \le \left(4\mathrm{b}^2(X) + 2\left\| |X|^{4(1-\varsigma)} + |X^*|^{4\varsigma} \right\|_{\mathrm{B}} + 4\left\| |X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} \right\|_{\mathrm{B}} \mathrm{b}(X)\right)^{1/2}.$$

Proof. We deduce that

$$\begin{aligned}
&\left\| \left(\left(|X|^{2(1-\varsigma)} + |X^*|^{2\varsigma}\right) + \left(2\,\mathrm{Re}\left(e^{i\theta}X\right)\right)\right)^2 \right\|_{\mathrm{B}} \\
&= \Big\| \left(|X|^{2(1-\varsigma)} + |X^*|^{2\varsigma}\right)^2 + 4\left(\mathrm{Re}\left(e^{i\theta}X\right)\right)^2 \\
&+ \left(|X|^{2(1-\varsigma)} + |X^*|^{2\varsigma}\right)\left(2\,\mathrm{Re}\left(e^{i\theta}X\right)\right) + \left(2\,\mathrm{Re}\left(e^{i\theta}X\right)\right)\left(|X|^{2(1-\varsigma)} + |X^*|^{2\varsigma}\right) \Big\|_{\mathrm{B}} \\
&\le \left\| \left(|X|^{2(1-\varsigma)} + |X^*|^{2\varsigma}\right)^2 \right\|_{\mathrm{B}} + 4\left\| \left(\mathrm{Re}\left(e^{i\theta}X\right)\right)^2 \right\|_{\mathrm{B}} \\
&+ 2\left\| \left(|X|^{2(1-\varsigma)} + |X^*|^{2\varsigma}\right)\left(\mathrm{Re}\left(e^{i\theta}X\right)\right) \right\|_{\mathrm{B}} + 2\left\| \left(\mathrm{Re}\left(e^{i\theta}X\right)\right)\left(|X|^{2(1-\varsigma)} + |X^*|^{2\varsigma}\right) \right\|_{\mathrm{B}} \\
&\le \left\| \left(|X|^{2(1-\varsigma)} + |X^*|^{2\varsigma}\right)^2 \right\|_{\mathrm{B}} + 4\left\| \left(\mathrm{Re}\left(e^{i\theta}X\right)\right)^2 \right\|_{\mathrm{B}} + 4\left\| |X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} \right\|_{\mathrm{B}} \left\|\mathrm{Re}\left(e^{i\theta}X\right)\right\|_{\mathrm{B}} \\
&= \left\| \left(\frac{2\,|X|^{2(1-\varsigma)} + 2\,|X^*|^{2\varsigma}}{2} \right)^2 \right\|_{\mathrm{B}} + 4\left\|\mathrm{Re}\left(e^{i\theta}X\right)\right\|_{\mathrm{B}}^2 + 4\left\| |X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} \right\|_{\mathrm{B}} \left\|\mathrm{Re}\left(e^{i\theta}X\right)\right\|_{\mathrm{B}} \\
&\le 2\left\| |X|^{4(1-\varsigma)} + |X^*|^{4\varsigma} \right\|_{\mathrm{B}} + 4\left\|\mathrm{Re}\left(e^{i\theta}X\right)\right\|_{\mathrm{B}}^2 + 4\left\| |X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} \right\|_{\mathrm{B}} \left\|\mathrm{Re}\left(e^{i\theta}X\right)\right\|_{\mathrm{B}} \\
&\le 2\left\| |X|^{4(1-\varsigma)} + |X^*|^{4\varsigma} \right\|_{\mathrm{B}} + 4\mathrm{b}^2(X) + 4\left\| |X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} \right\|_{\mathrm{B}} \mathrm{b}(X),
\end{aligned}$$

and

$$\left\| \left(\left(|X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} \right) + \left(2\,\mathrm{Re}\left(e^{i\theta} X \right) \right) \right)^2 \right\|_{\mathrm{B}}$$
$$\leq 2 \left\| |X|^{4(1-\varsigma)} + |X^*|^{4\varsigma} \right\|_{\mathrm{B}} + 4\mathrm{b}^2(X) + 4 \left\| |X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} \right\|_{\mathrm{B}} \mathrm{b}(X). \tag{6}$$

Let $X = U\,|X|$ be the polar decomposition of X. By employing the inequality (6), we have

$$\begin{aligned}
&4\,\mathrm{Re}\left\langle e^{i\theta} X K_\rho, K_\rho \right\rangle \\
&= 4\,\mathrm{Re}\left\langle e^{i\theta} U\,|X|\,K_\rho, K_\rho \right\rangle = 4\,\mathrm{Re}\left\langle e^{i\theta} U\,|X|^{\varsigma}\,|X|^{1-\varsigma} K_\rho, K_\rho \right\rangle \\
&= 4\,\mathrm{Re}\left\langle e^{i\theta}\,|X|^{1-\varsigma} K_\rho, |X|^{\varsigma}\,U^* K_\rho \right\rangle \\
&= \left\| \left(e^{i\theta}\,|X|^{1-\varsigma} + |X|^{\varsigma}\,U^* \right) K_\rho \right\|^2 - \left\| \left(e^{i\theta}\,|X|^{1-\varsigma} - |X|^{\varsigma}\,U^* \right) K_\rho \right\|^2 \\
&\leq \left\| \left(e^{i\theta}\,|X|^{1-\varsigma} + |X|^{\varsigma}\,U^* \right) K_\rho \right\|^2 \leq \left\| e^{i\theta}\,|X|^{1-\varsigma} + |X|^{\varsigma}\,U^* \right\|_{\mathrm{B}}^2 \\
&= \left\| \left(e^{i\theta}\,|X|^{1-\varsigma} + |X|^{\varsigma}\,U^* \right)^* \left(e^{i\theta}\,|X|^{1-\varsigma} + |X|^{\varsigma}\,U^* \right) \right\|_{\mathrm{B}} \\
&= \left\| |X|^{2(1-\varsigma)} + U\,|X|^{2\varsigma}\,U^* + 2\,\mathrm{Re}\left(e^{i\theta} U\,|X| \right) \right\|_{\mathrm{B}} \\
&= \left\| |X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} + 2\,\mathrm{Re}\left(e^{i\theta} U\,|X| \right) \right\|_{B} \\
&= \left\| |X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} + 2\,\mathrm{Re}\left(e^{i\theta} X \right) \right\|_{\mathrm{B}} \\
&= \left\| \left(\left(|X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} \right) + \left(2\,\mathrm{Re}\left(e^{i\theta} X \right) \right) \right)^2 \right\|_{\mathrm{B}}^{1/2}. \\
&\leq \left(2 \left\| |X|^{4(1-\varsigma)} + |X^*|^{4\varsigma} \right\|_{\mathrm{B}} + 4\mathrm{b}^2(X) + 4 \left\| |X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} \right\|_{\mathrm{B}} \mathrm{b}(X) \right)^{1/2}.
\end{aligned}$$

Taking supremum over all $\rho \in F$, we can obtain our desired inequality

$$4\mathrm{b}(X) \leq \left(4\mathrm{b}^2(X) + 2 \left\| |X|^{4(1-\varsigma)} + |X^*|^{4\varsigma} \right\|_{\mathrm{B}} + 4 \left\| |X|^{2(1-\varsigma)} + |X^*|^{2\varsigma} \right\|_{\mathrm{B}} \mathrm{b}(X) \right)^{1/2}.$$

□

If we put $\varsigma = \frac{1}{2}$ in Theorem 2.5, then we reach the inequality

$$4\mathrm{ber}(X) \leq \left(4\mathrm{ber}^2(X) + 2 \left\| |X|^2 + |X^*|^2 \right\|_{\mathrm{Ber}} + 4 \left\| |X| + |X^*| \right\|_{\mathrm{Ber}} \mathrm{ber}(X) \right)^{1/2}.$$

Theorem 2.6. *Let $X \in \mathcal{B}(\mathcal{H})$ be a non-zero operator on the RKHS $\mathcal{H} = \mathcal{H}(F)$. Then we have*

$$2\mathrm{b}(X) \leq \left\| |X|^{2(1-\varsigma)} + |X|^{2(\varsigma-1)} \right\|_{\mathrm{b}} \left\| \widehat{X}_{\varsigma} \right\|_{\mathrm{b}}.$$

Proof. Let $\rho \in \digamma$. Utilizing the Cauchy-Schwarz inequality and the arithmetic-geometric mean inequality, we can write

$$\begin{aligned}
|\langle XK_\rho, K_\rho\rangle| &= |\langle U|X|K_\rho, K_\rho\rangle| = \left|\left\langle |X|^{\varsigma-1}|X|^{1-\varsigma}U|X|^{\varsigma}|X|^{1-\varsigma}K_\rho, K_\rho\right\rangle\right| \\
&= \left|\left\langle |X|^{\varsigma-1}\tilde{A}_\varsigma|X|^{1-\varsigma}K_\rho, K_\rho\right\rangle\right| = \left|\left\langle \widehat{X}_\varsigma|X|^{1-\varsigma}K_\rho, |X|^{\varsigma-1}K_\rho\right\rangle\right| \\
&\le \left\|\widehat{X}_\varsigma\right\|\left(\left\langle |X|^{2(1-\varsigma)}K_\rho, K_\rho\right\rangle\left\langle |X|^{2(\varsigma-1)}K_\rho, K_\rho\right\rangle\right)^{1/2} \\
&\le \frac{1}{2}\left\|\widehat{X}_\varsigma\right\|\left(\left\langle |X|^{2(1-\varsigma)}K_\rho, K_\rho\right\rangle + \left\langle |X|^{2(\varsigma-1)}K_\rho, K_\rho\right\rangle\right) \\
&= \frac{1}{2}\left\langle\left(|X|^{2(1-\varsigma)} + |X|^{2(\varsigma-1)}\right)K_\rho, K_\rho\right\rangle\left\|\widehat{X}_\varsigma\right\|.
\end{aligned}$$

We deduce that

$$\sup_{\rho\in\digamma} 2|\langle XK_\rho, K_\rho\rangle| \le \sup_{\rho\in\digamma}\left\langle\left(|X|^{2(1-\varsigma)} + |X|^{2(\varsigma-1)}\right)K_\rho, K_\rho\right\rangle\left\|\widehat{X}_\varsigma\right\|$$

which is equivalent to

$$2\mathrm{b}(X) \le \left\||X|^{2(1-\varsigma)} + |X|^{2(\varsigma-1)}\right\|_{\mathrm{b}}\left\|\widehat{X}_\varsigma\right\|$$

and completes the proof of the theorem. □

For $X \in \mathcal{B}(\mathcal{H})$, its the number $\widetilde{c}(X)$ is defined by $\widetilde{c}(X) := \inf\left\{\widetilde{X}(\rho) : \rho \in \digamma\right\}$ (see [28]), which is used to give the next result.

Theorem 2.7. *Let $X \in \mathcal{B}(\mathcal{H})$ be a operator on the RKHS $\mathcal{H} = \mathcal{H}(\digamma)$. Then we have*

$$\left(\mathrm{b}\left(|X|^2 + |X^*|^2\right) + \widetilde{c}\left(X^2 + (X^*)^2\right)\right)^{1/2} \le \|X + X^*\|.$$

Proof. We have

$$\begin{aligned}
\|X + X^*\| &= \left\|(X + X^*)^2\right\|^{1/2} = \left\|X^2 + |X^*|^2 + |X|^2 + (X^*)^2\right\|^{1/2} \\
&\ge \left(\left|\left\langle\left(X^2 + |X^*|^2 + |X|^2 + (X^*)^2\right)K_\rho, K_\rho\right\rangle\right|\right)^{1/2} \\
&= \left(\left|\left\langle\left(|X|^2 + |X^*|^2\right)K_\rho, K_\rho\right\rangle + \left\langle\left(X^2 + (X^*)^2\right)K_\rho, K_\rho\right\rangle\right|\right)^{1/2} \\
&= \left(\left|\left\langle\left(|X|^2 + |X^*|^2\right)K_\rho, K_\rho\right\rangle\right| + \left|\widetilde{\left(X^2 + (X^*)^2\right)}(\rho)\right|\right)^{1/2} \\
&\ge \left(\left|\left\langle\left(|X|^2 + |X^*|^2\right)K_\rho, K_\rho\right\rangle\right| + \inf_{\rho\in\digamma}\left|\widetilde{\left(X^2 + (X^*)^2\right)}(\rho)\right|\right)^{1/2}
\end{aligned}$$

and

$$\left(\left|\left\langle\left(|X|^2 + |X^*|^2\right)K_\rho, K_\rho\right\rangle\right| + \widetilde{c}\left(X^2 + (X^*)^2\right)\right)^{1/2} \le \|X + X^*\|.$$

Taking supremum over all $\rho \in \digamma$, we can get our desired inequality

$$\sup_{\rho \in \digamma} \left(\left| \left\langle \left(|X|^2 + |X^*|^2 \right) K_\rho, K_\rho \right\rangle \right| + \widetilde{c}\left(X^2 + (X^*)^2 \right) \right)^{1/2} \leq \|X + X^*\|$$

and

$$\left(\mathrm{b}\left(|X|^2 + |X^*|^2 \right) + \widetilde{c}\left(X^2 + (X^*)^2 \right) \right)^{1/2} \leq \|X + X^*\|,$$

which completes the proof. □

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