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SEMI-SLANT LIGHTLIKE SUBMERSIONS WITH TOTALLY UMBILICAL FIBRES

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Abstract. We introduce the study of a semi-slant lightlike submersion from an indefinite Kaehler manifold onto an r-lightlike manifold. After giving the definition of a semi-slant lightlike submersion, we establish the existence Theorems for this class of lightlike submersions. Then, we derive the integrability conditions for the distributions D_1, D_2 and Δ associated with a semi-slant lightlike submersion. Consequently, we find some necessary and sufficient conditions for the foliations determined by the distributions D_1, D_2 and Δ. Subsequently, we examine the geometry of totally umbilical fibres of a semi-slant lightlike submersion.

1. Introduction

The concept of a lightlike submersion is one of the most fruitful area of research in semi-Riemannian geometry. Theoretically, a lightlike submersion is a smooth map that preserves the causal structure of the manifolds. In other words, a lightlike submersion is a map that preserves the light cones in two manifolds so that any two points in the domain, which are connected by a lightlike curve are mapped to two points in the co-domain, which are also connected by a lightlike curve. The theory of lightlike submersions is known to have extensive uses in mathematical physics, particularly, in the general theory of relativity. For instance, in physics, a lightlike submersion is used to describe the propagation of gravitational waves through spacetime, as these waves travel along null geodesics (paths that are tangent to the light cone) [2]-[1]. In addition, a lightlike submersion can also be used to construct non-locality conditions in quantum field theory, which are important for understanding the nature of entanglement and other quantum phenomena [4]. In string theory, lightlike submersions are used to describe the dynamics of strings moving in curved spacetimes. Moreover, a lightlike submersion has been used to map

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the worldsheet of the string onto the target spacetime [5]. Furthermore, lightlike submersions have been employed to describe the dynamics of particles and fields near the event horizon of a black hole, where the effect of gravity becomes extreme [10]. Comprehensively, a lightlike submersion is a valuable geometric tool for understanding various aspects of physics.

Initially, the concept of a Riemannian submersion was developed by Hermann [11] and O'Neill [15]. In [11], Hermann proved a sufficient condition for a mapping of a Riemannian manifold to be a fibre bundle. This motivated O'Neill to introduce the general notion of a Riemannian submersion [15]. Afterwards, various new generalizations of Riemannian submersions namely, invariant submersions, anti-invariant submersions, CR-submersions, generic submersions, semislant submersions, complex-contact and contact-complex submersions came into sight (for details, see [8]-[26], [18]). Further, with the development of semi-Riemannian geometry, O'Neill [17], introduced the notion of a semi-Riemannian submersion. It may be noted that for a Reimannian submersion $\phi : M \to B$, where M and B are Riemannian manifolds, the fibres of the Riemannian submersion ϕ are always Riemannian. However, when M and B are semi-Riemannian manifolds, the fibres of ϕ may not be semi-Riemannian. In this context, Sahin [22] studied the existence of a lightlike submersion defined from a semi-Riemannian manifold onto a lightlike manifold and illustrated how this idea differs from Riemannian and semi-Riemannian submersions. Further, in [21], Sahin used a semi-Riemannian manifold as the base and a Kaehler manifold as the total manifold to define a new type of lightlike submersion. Park and Prasad [18], introduced the notion of a semi-slant submersion from an almost indefinite Hermitian manifold onto a Riemannian manifold. Literature suggests that till date very few studies are available on the geometry of lightlike submersions. In view of wide variety of applications of a lightlike submersion, we define a new class of lightlike submersions, namely, semi-slant lightlike submersions following the similar approach as developed and used by Sahin in [22].

The paper is structured as follows: At first we give the general definition for a semi-slant lightlike submersion ϕ from an indefinite Kaehler manifold M_1 onto an r-lightlike manifold M_2 and present some Theorems for the existence of this class of lightlike submersions. Then we establish some conditions for the integrability of the distributions Δ, D_1 and D_2 arising in case of a semi-slant lightlike submersion. Further, we obtain some necessary and sufficient conditions for the leaves determined by the distributions on a semi-slant lightlike submersion to be totally geodesic foliations. We also discuss the requisite for a semi-slant lightlike submersion ϕ to be a totally geodesic map. Finally, we investigate the geometry of totally umbilical fibres of semi-slant submersion and give some geometric characterisation results.

2. Preliminaries

In this section, we refer to [22] for the basic notations and fundamental equations related to a lightlike submersion.
Assume that (M_1, g_1, J) be an indefinite almost Hermitian manifold. This indicates that a tensor field J of type $(1,1)$ on M_1 is admissible, such that

$$J^2 = -I, \; g_1(JX, JY) = g_1(X, Y), \tag{1}$$

for $X, Y \in \Gamma(TM_1)$. An indefinite almost Hermitian manifold M_1 is said to be an indefinite Kaehler manifold if

$$(\nabla_X J)Y = 0, \tag{2}$$

for any $X, Y \in \Gamma(TM_1)$.
On the other hand, the radical space Rad T_pM_1 of T_pM_1 is defined as

$$RadT_pM_1 = \{\xi \in T_pM_1 : g_1(\xi, X) = 0, \forall X \in T_pM_1\}.$$

Let (M_1, g_1, J) be a semi-Riemannian manifold and (M_2, g_2) be an r-lightlike manifold. Consider a smooth submersion $\phi : M_1 \to M_2$, then for $p \in M_2$, $\phi^{-1}(p)$ is a submanifold of M_1 of dimension dim M_1 - dim M_2 . For any point $p \in M_1$, the kernel of ϕ_*, denoted as $ker\ \phi_*$ and is given by

$$ker\ \phi_* = \{Z \in T_pM_1 : \phi_*(Z) = 0\}.$$

Now we define $(ker\ \phi_*)^\perp$ as

$$(ker\ \phi_*)^\perp = \{Y \in T_pM_1 : g_1(X, Y) = 0, \forall X \in ker\ \phi_*\}.$$

$(ker\ \phi_*)^\perp$ may not be complementary to $ker\ \phi_*$ as T_pM_1 is a semi-Riemannian vector space. Hence

$$\Delta = ker\ \phi_* \cap (ker\ \phi_*)^\perp \neq \{0\}.$$

For a lightlike submersion, there are four possible cases, which are discussed below:

Case (i). $0 < \dim \Delta < \min\{\dim(ker\ \phi_*), \dim(ker\ \phi_*)^\perp\}$: In this case Δ is the radical subspace of T_pM_1. Therefore we can find a quasi-orthonormal basis of M_1 along $ker\ \phi_*$. Assume that $S(ker\ \phi_*)$ is a complementary non-degenerate subspace to Δ in $ker\ \phi_*$. Then we have

$$ker\ \phi_* = \Delta \perp S(ker\ \phi_*).$$

In a similar way, we have

$$(ker\ \phi_*)^\perp = \Delta \perp S(ker\ \phi_*)^\perp,$$

where $S(ker\ \phi_*)^\perp$ is a complementary subspace of Δ in $(ker\ \phi_*)^\perp$. Moreover, $S(ker\ \phi_*)$ being non-degenerate in T_pM_1 gives

$$T_pM_1 = S(ker\ \phi_*) \perp (S(ker\ \phi_*))^\perp,$$

where $(S(ker\ \phi_*))^\perp$ is the complementary subspace of $S(ker\ \phi_*)$ in T_pM_1. As $S(ker\ \phi_*)$ and $(S(ker\ \phi_*))^\perp$ are non-degenerate in T_pM_1, therefore we get

$$(S(ker\ \phi_*))^\perp = S(ker\ \phi_*)^\perp \perp (S(ker\ \phi_*)^\perp)^\perp.$$

Now from ([6]), in view of proposition 2.4, we conclude that there exist a quasi-orthonormal basis of T_pM_1 along $ker\ \phi_*$, then we have

$$g_1(\xi_i, \xi_j) = g_1(N_i, N_j) = 0, \quad g_1(\xi_i, N_j) = \delta_{ij},$$
$$g_1(W_\alpha, \xi_j) = g_1(W_\alpha, N_j) = 0, \quad g_1(W_\alpha, W_\beta) = \epsilon_\alpha \delta_{\alpha\beta},$$

for any $i, j \in \{1, ...r\}$ and $\alpha, \beta \in \{1, ...t\}$, where $\{N_i\}$ are smooth null vector fields of $(S(ker\ \phi_*)^\perp)^\perp$, $\{W_\alpha\}$ is a basis of $S(ker\ \phi_*)^\perp$ and $\{\xi_i\}$ is a basis of Δ. Let the set of vector fields $\{N_i\}$ be denoted by $ltr(ker\ \phi_*)$, then consider the subspace as follows.

$$tr(ker\ \phi_*) = ltr(ker\ \phi_*) \perp S(ker\ \phi_*)^\perp.$$

Since $ltr(ker\ \phi_*)$ and $ker\ \phi_*$ are not orthogonal to each other, so we denote the vertical space of T_pM_1 as $\mathcal{V} = ker\ \phi_*$ and the horizontal space as $\mathcal{H} = tr(ker\ \phi_*)$. Thus we have

$$T_pM_1 = \mathcal{V}_p \oplus \mathcal{H}_p.$$

It is pertinent to highlight again that $\mathcal{V}$ and $\mathcal{H}$ are not orthogonal to each other. We are now equipped to define a lightlike submersion.

Definition 2.1. [22] *Consider a submersion $\phi : M_1 \to M_2$ defined from semi-Riemannian manifold (M_1, g_1) onto an r-lightlike manifold (M_2, g_2) such that*

(1) *for $0 < r < min\{dim(ker\ \phi_*), dim(ker\ \phi_*)^\perp\}$, $dim\ \Delta = dim\{(ker\ \phi_*) \cap (ker\ \phi_*)^\perp\} = r$,*
(2) *the length of horizontal vectors is preserved under ϕ_*, i.e., $g_1(Z, W) = g_2(\phi_*(Z), \phi_*(W))$ for $Z, W \in \Gamma(\mathcal{H})$.*

Then ϕ is called an r-lightlike submersion.

Case (ii). $\dim\ \Delta = \dim(ker\ \phi_*) < \dim(ker\ \phi_*)^\perp$. In this case $\mathcal{V} = \Delta$ and $\mathcal{H} = S(ker\ \phi_* \perp ltr(ker\ \phi_*)$ and ϕ is said to be an isotropic lightlike submersion.

Case (iii). $\dim\ \Delta = \dim(ker\ \phi_*)^\perp < \dim(ker\ \phi_*)$. Here $\mathcal{V} = S(ker\ \phi_*) \perp \Delta$ and $\mathcal{H} = ltr(ker\ \phi_*)$ and ϕ is said to be a co-isotropic lightlike submersion.

Case (iv). $\dim\ \Delta = \dim(ker\ \phi_*)^\perp = \dim(ker\ \phi_*)$. In this case $\mathcal{V} = \Delta$ and $\mathcal{H} = ltr(ker\ \phi_*)$ and ϕ is said to be a totally lightlike submersion.

Before proceeding further, at first we prove an essential lemma required to define the concept of a semi-slant lightlike submersion from an indefinite Kaehler manifold onto a lightlike manifold.

Lemma 2.2. *Consider a r-lightlike submersion $\phi : M_1 \to M_2$ from an indefinite Kaehler manifold (M_1, g_1, J) (where g_1 is a semi-Riemannian metric* *of index 2r) onto an r-lightlike manifold* (M_2, g_2). *If* $J\Delta$ *is a distribution on* M_1 *such that* $\Delta \cap J\Delta = 0$, *then any distribution complementary to* $J\Delta \oplus Jltr(ker\ \phi_*)$ *in* $S(ker\ \phi_*)$ *is Riemannian.*

Proof. If possible,assume that $J(ltr(ker\ \phi_*)$ is invariant with respect to J, then for $Z \in \Gamma(\Delta), N \in \Gamma(ltr(ker\ \phi_*))$, we have $g_1(Z,N) = 1$, which gives $g_1(JZ,JN) = 1$, this further gives $0 = 1$, which is a contradiction. Hence $J(ltr(ker\ \phi_*)$ is not invariant with respect to J. Furthermore, on contrary suppose that $J(ltr(ker\ \phi_*))$ is contained in $S(ker\ \phi_*)$, then we have $0 = g_1(JZ,JN) = g_1(Z,N) = 1$, which is also a contradiction. Thus, $J(ltr(ker\ \phi_*))$ is a distribution on M_1. Moreover, $J(ltr(ker\ \phi_*)$ is not contained in Δ. Because if so, then for $JN \in \Gamma(\Delta)$, we have $J^2N = -N \in \Gamma(J\Delta)$, which is again a contradiction. In a similar way, we can prove that $J(ltr(ker\ \phi_*)$ is not contained in $J\Delta$. Hence, $J(ltr(ker\ \phi_*) \subset S(ker\ \phi_*)$ such that $J\Delta \cap J(ltr(ker\ \phi_*) = 0$. Let D denotes a distribution which is complementary to $J\Delta \oplus J(ltr(ker\ \phi_*)$ in $S(ker\ \phi_*)$. Then, for the local quasi-orthonormal frames on M_1, $\xi_1, ..., \xi_r, J\xi_1, ..., J\xi_r, N_1, ..., N_r, JN_1 ..., JN_r$ forms an orthonormal basis of $\Delta \oplus J\Delta \oplus ltr(ker\ \phi_*) \oplus Jltr(ker\ \phi_*)$. Next we define $U_1, ..., U_{2r}, V_1, ..., V_{2r}$ as

$$
\begin{aligned}
U_1 &= \frac{1}{\sqrt{2}}(\xi_1 + N_1), U_2 = \frac{1}{\sqrt{2}}(\xi_1 - N_1), U_3 = \frac{1}{\sqrt{2}}(\xi_2 + N_2),\\
U_4 &= \frac{1}{\sqrt{2}}(\xi_2 + N_2), . . ., U_{2r-1} = \frac{1}{\sqrt{2}}(\xi_r + N_r), U_{2r} = \frac{1}{\sqrt{2}}(\xi_r + N_r),\\
V_1 &= \frac{1}{\sqrt{2}}(J\xi_1 + JN_1), V_2 = \frac{1}{\sqrt{2}}(J\xi_1 - JN_1), V_3 = \frac{1}{\sqrt{2}}(J\xi_2 + JN_2),\\
V_4 &= \frac{1}{\sqrt{2}}(J\xi_2 - JN_1), . . ., V_{2r-1} = \frac{1}{\sqrt{2}}(J\xi_r + JN_r), V_{2r}\\
&= \frac{1}{\sqrt{2}}(J\xi_r - JN_r).
\end{aligned}
$$

Hence, Span $\{\xi_i, N_i, J\xi_i, JN_i\}$ is a non-degenerate space of constant index $2r$, that is $\Delta \oplus J\Delta \oplus ltr(ker\ \phi_*) \oplus Jltr(ker\ \phi_*)$ is non-degenerate and has a constant index $2r$ on M_1. Since index(TM_1) = index$(\Delta \oplus ltr(ker\ \phi_*))$ + index$(J\Delta \oplus Jltr(ker\ \phi_*)$ + index$(D \perp S(ker\ \phi_*)^{\perp})$, we obtain, $2r = 2r +$ index$(J\Delta \oplus Jltr(ker\ \phi_*)$ + index$(D \perp S(ker\ \phi_*)^{\perp})$. This implies that $(D \perp S(ker\ \phi_*)^{\perp}))$ is Reimannian and, therefore, D is Riemannian. □

3. Semi-Slant Lightlike Submersions

In this section, at first, we define the concept of a semi-slant lightlike submersion from an indefinite Kaehler manifold onto an r-lightlike manifold.

Definition 3.1. *Consider an r-lightlike submersion $\phi : M_1 \to M_2$ defined from an indefinite Kaehler manifold (M_1, g_1, J), where g_1 is a semi-Riemannian metric of index 2r, where $2r < dim(M_1)$, onto an r-lightlike manifold (M_2, g_2). Then ϕ is said to be a semi-slant lightlike submersion if the following conditions hold:*

(1) *$J\Delta$ is a distribution on $ker\ \phi_*$ such that $\Delta \cap J\Delta = 0$.*
(2) *There exist two non-degenerate distributions D_1 and D_2 on M_1, such that*

$$ker\ \phi_* = \Delta \perp \{J\Delta \oplus Jltr(ker\ \phi_*)\} \oplus_{ortho} D_1 \oplus_{ortho} D_2,$$

where $JD_1 = D_1$.
(3) *For any non-zero vector field X tangent to D_2, the angle $\theta_p(X)$ between JX and the vector space $(D_2)_p$ is constant for any given point $p \in U \subset M_1$, where D_2 is the complementary distribution to $J\Delta \oplus Jltr(ker\ \phi_*) \oplus_{orth} D_1$ in $S(ker\ \phi_*)$. This implies that angle $\theta_p(X)$ does not depend on the choice of X.*

Here the angle θ on M_1 is known as a semi-slant angle. If $\phi : M_1 \to M_2$ is a semi-slant lightlike submersion, then the decomposition of *ker* ϕ_* is as follows:

$$ker\ \phi_* = \Delta \perp \{J\Delta \oplus Jltr(ker\ \phi_*)\} \oplus_{ortho} D_1 \oplus_{ortho} D_2. \quad (3)$$

Hence we get

$$\begin{aligned} T_pM_1 &= \mathcal{V}_p \oplus \mathcal{H}_p \\ &= \{\Delta \perp \{J\Delta \oplus Jltr(ker\ \phi_*)\} \oplus_{ortho} D_1 \oplus_{ortho} D_2\} \oplus \{\phi(D_2) \perp \\ &\quad \eta \perp ltr(ker\ \phi_*)\}, \quad (4) \end{aligned}$$

where η is the orthogonal subbundle complimentary to $\phi(D_2)$ in $(ker\ \phi_*)^{\perp}$.

Example 3.2. *Let (R^{18}, g_1, J) and (R^8, g_2) be an indefinite Kaehler manifold and a lightlike manifold, endowed with semi-reimannian metric g_1 with signature $(-,+,-,+,+,+,+,+,+,+,+,+)$ and degenerate metric g_2 with signature $(+,+,+,+,+,+,+,+)$.*

Define a map ϕ from $R^{18} \to R^8$ as

$$\phi(x_1, ..., x_8) = (x_1, -x_2, x_3, x_2 + x_4, x_1 + x_4, x_2 + x_3, x_5, x_6, x_6, x_5x_6, \frac{(x_5)^2}{2} + \frac{(x_6)^2}{2}, x_7, x_7, x_8, x_8, x_7, x_8, x_7x_8, \frac{x_7^2}{2} + \frac{x_8^2}{2}).$$

Then we can easily see that ϕ is a 2-lightlike submersion with

$$\Delta = ker\phi_* \cap (ker\phi_*)^{\perp} = Span\{Z_1 = -\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_5}, Z_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}\},$$

$$J\Delta = Span\{Z_2 = -\frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_6}, Z_4 = -\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}\},$$

$$Jltr(ker\phi_*) = Span\{Z_4 = -\frac{\partial}{\partial x_4} + \frac{\partial}{\partial x_5}\},$$

$$D_1 = Span\{Z_5 = \frac{\partial}{\partial x_5}, Z_8 = \frac{\partial}{\partial x_8}\},$$
$$D_2 = Span\{Z_7 = \frac{1}{\sqrt{2}}(\frac{\partial}{\partial x_9} + \frac{\partial}{\partial x_{12}}), Z_8 = \frac{\partial}{\partial x_8}\}.$$

Since $JZ_1 = Z_2, JZ_3 = Z_4$, *therefore* $\Delta \cap J\Delta = 0$. *By easy calculation, we can see that* D_1 *is invariant w.r.t.* J *and* D_2 *is a slant distribution with the slant angle* $\theta = \frac{\pi}{4}$. *Thus,* ϕ *is a proper semi-slant lightlike submersion.*

For any vector field $X \in \mathcal{V}_p$, we may write

$$X = Q_1X + Q_2X + Q_3X + Q_4X + Q_5X, \tag{5}$$

where Q_1, Q_2, Q_3, Q_4 and Q_5 denote the projections of X onto the distributions Δ, $J\Delta$, $J(ltr(ker\ \phi_*)$, D_1, and D_2 respectively. Applying J to Eq. (5), we get

$$JX = fX + \omega X, \tag{6}$$

where fX and ωX are the tangential and transversal components of JX, respectively. This further gives

$$\begin{aligned} JX &= JQ_1X + JQ_2X + JQ_3X + JQ_4X + JQ_5X \\ &= fQ_1X + fQ_2X + \omega Q_3X + fQ_4X + fQ_5X + \omega Q_5X, \end{aligned} \tag{7}$$

then clearly, we have $fQ_1X \in \Gamma(J\Delta)$, $fQ_2X \in \Gamma(\Delta)$, $\omega Q_3X \in \Gamma(ltr(ker\ \phi_*))$, $fQ_4X \in \Gamma(D_1)$, $fQ_5X \in \Gamma(D_2)$, $\omega Q_5X \in \Gamma(\phi(D_2))$. Further for $X \in \Gamma(ker\ \phi_*)$, Therefore in view of Eqs. (6) and (7), we have

$$fX = fQ_1X + fQ_2X + fQ_4X + fQ_5X, \quad \omega X = \omega Q_3X + \omega Q_5X.$$

In a similar way, we call P_1 and P_2 as the projections of $ltr(ker\ \phi_*)$ and $S(ker\ \phi_*)^\perp$ respectively. Therefore for $Z \in \Gamma((ker\ \phi_*)^\perp)$, we have

$$Z = P_1Z + P_2Z, \tag{8}$$

then on applying J, the above equation reduces to

$$JZ = JP_1Z + JP_2Z = JP_1Z + BP_2Z + CP_2Z, \tag{9}$$

where BP_2Z and CP_2Z represent the tangential and transversal components of JP_2Z. Thus we get, $JP_1Z \in \Gamma(Jltr(ker\ \phi_*)), BP_2Z \in \Gamma(D_2)$ and $CP_2Z \in \Gamma(\eta)$. Now we define O'Neill [15] tensors $\mathcal{T}$ and $\mathcal{A}$ as

$$\mathcal{T}_EF = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{10}$$
$$\mathcal{A}_EF = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{11}$$

where E and F are the vector fields on M_1 and ∇ is the Levi-Civita connection of g_1. It may be observed that $\mathcal{T}$ and $\mathcal{A}$ are skew symmetric tensors in Riemannian submersions, but this is not true for a lightlike submersion since the horizontal and vertical subspaces are not orthogonal to each other. The horizontal and vertical subspaces are reversed by both the tensors $\mathcal{T}$ and $\mathcal{A}$ and moreover, $\mathcal{T}$ is symmetric, that is, for each $U, V \in \Gamma(ker\ \phi_*)$, we have $\mathcal{T}_UV = \mathcal{T}_VU$.

Lemma 3.3. *Let ϕ be a semi-slant lightlike submersion from an indefinite Kaehler manifold (M_1, g_1, J) onto an r-lightlike manifold (M_2, g_2), then for $X, Y \in \Gamma(ker\ \phi_*)$ and $U, V \in \Gamma(ker\ \phi_*)^{\perp}$, we have*

$$\nabla_X Y = \mathcal{T}_X Y + \hat{\nabla}_X Y, \tag{12}$$

$$\nabla_X V = \mathcal{H}\nabla_X V + \mathcal{T}_X V, \tag{13}$$

$$\nabla_U X = \mathcal{A}_U X + \hat{\nabla}_U X, \tag{14}$$

$$\nabla_U V = \mathcal{H}\nabla_U V + \mathcal{A}_U V, \tag{15}$$

where $\hat{\nabla}_X Y = \mathcal{V}\nabla_X Y$.

Then we have the following lemma:

Lemma 3.4. *Suppose that $\phi : (M_1, g_1, J) \to (M_2, g_2)$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold M_1 onto an r-lightlike manifold M_2, then*

$$\begin{aligned} &Q_1(\hat{\nabla}_X fQ_1Y + \hat{\nabla}_X fQ_2Y + \mathcal{T}_X\omega Q_3Y + \hat{\nabla} fQ_4Y + \nabla fQ_5Y + \\ &\mathcal{T}_X\omega Q_5Y) = fQ_2(\hat{\nabla}_X Q_1Y + \hat{\nabla}_X Q_2Y + \hat{\nabla}_X Q_4Y + \hat{\nabla}_X Q_5Y), \end{aligned} \tag{16}$$

$$\begin{aligned} &Q_2(\hat{\nabla}_X fQ_1Y + \hat{\nabla}_X fQ_2Y + \mathcal{T}_X\omega Q_3Y + \hat{\nabla} fQ_4Y + \nabla fQ_5Y + \\ &\mathcal{T}_X\omega Q_5Y) = fQ_1(\hat{\nabla}_X Q_1Y + \hat{\nabla}_X Q_2Y + \hat{\nabla}_X Q_4Y + \hat{\nabla}_X Q_5Y), \end{aligned} \tag{17}$$

$$\begin{aligned} &Q_3(\hat{\nabla}_X fQ_1Y + \hat{\nabla}_X fQ_2Y + \mathcal{T}_X\omega Q_3Y + \hat{\nabla} fQ_4Y + \nabla fQ_5Y + \\ &\mathcal{T}_X\omega Q_5Y) = JP_1(\mathcal{T}_X Q_1Y + \mathcal{T}_X Q_2Y + \mathcal{T}_X Q_3Y + \\ &\mathcal{T}_X Q_4y + \mathcal{T}_X Q_5Y), \end{aligned} \tag{18}$$

$$\begin{aligned} &Q_4(\hat{\nabla}_X fQ_1Y + \hat{\nabla}_X fQ_2Y + \mathcal{T}_X\omega Q_3Y + \hat{\nabla} fQ_4Y + \nabla fQ_5Y + \\ &\mathcal{T}_X\omega Q_5Y) = fQ_4(\hat{\nabla}_X Q_1Y + \hat{\nabla}_X Q_2Y + \hat{\nabla}_X Q_4Y + \hat{\nabla}_X Q_5Y), \end{aligned} \tag{19}$$

$$\begin{aligned} &Q_5(\hat{\nabla}_X fQ_1Y + \hat{\nabla}_X fQ_2Y + \mathcal{T}_X\omega Q_3Y + \hat{\nabla} fQ_4Y + \nabla fQ_5Y + \\ &\mathcal{T}_X\omega Q_5Y) = fQ_5(\hat{\nabla}_X Q_1Y + \hat{\nabla}_X Q_2Y + \hat{\nabla}_X Q_4Y + \hat{\nabla}_X Q_5Y) + \\ &+BP_2(\mathcal{T}_X Q_1Y + \mathcal{T}_X Q_2Y + \mathcal{T}_X Q_3Y + \mathcal{T}_X Q_4y + \mathcal{T}_X Q_5Y), \end{aligned} \tag{20}$$

$$\begin{aligned} &P_1(\mathcal{T}_X fQ_1Y + \mathcal{T}X fQ_2Y + \mathcal{H}\nabla_X\omega Q_3Y + \mathcal{T}_X fQ_4y + \mathcal{T}_X fQ_5Y + \\ &\mathcal{H}\nabla_X\omega Q_5Y) = \omega Q_3(\hat{\nabla}_X Q_1Y + \hat{\nabla}_X Q_2Y + \hat{\nabla}_X Q_4Y + \hat{\nabla}_X Q_5Y), \end{aligned} \tag{21}$$

$$\begin{aligned} &P_2(\mathcal{T}_X Q_1Y + \mathcal{T}_X Q_2Y + \mathcal{T}_X Q_3Y + \mathcal{T}_X Q_4y + \mathcal{T}_X Q_5Y) \\ &= CP_2(\mathcal{T}_X Q_1Y + \mathcal{T}_X Q_2Y + \mathcal{T}_X Q_3Y + \mathcal{T}_X Q_4y + \mathcal{T}_X Q_5Y) \\ &+\omega Q_5(\hat{\nabla}_X Q_1Y + \hat{\nabla}_X Q_2Y + \hat{\nabla}_X Q_4Y + \hat{\nabla}_X Q_5Y), \end{aligned} \tag{22}$$

where $X, Y \in \Gamma(ker\ \phi_*)$ and $U, V \in \Gamma(ker\ \phi_*)^{\perp}$.

Proof. In view of Eqs. (2), (5), (7) and Lemma (3.3), the result follows. □

Theorem 3.5. *(Existence Theorem) The necessary and sufficient condition for a lightlike submersion $\phi : M_1 \to M_2$ from an indefinite Kaehler manifold M_1 onto an r-lightlike manifold M_2 to be a semi-slant lightlike submersion is as follows:*

(i) *$J\Delta$ is a distribution on M_1 such that $\Delta \cap J\Delta = \{0\}$,*

(ii) *the screen distribution $S(ker\ \phi_*)$ can be decomposed as a direct sum as*

$$S(ker\ \phi_*) = (J\Delta \oplus Jltr(ker\ \phi_*)) \oplus_{ortho} D_1 \oplus_{ortho} D_2,$$

(iii) *there exists a constant $\lambda \in [0,1)$ such that $f^2(Z) = -\lambda Z$, for all $Z \in \Gamma(D_2)$. Here $\lambda = cos^2\theta$, where θ is known as a semi-slant angle of D_2.*

Proof. Let ϕ be a semi-slant lightlike submersion from an indefinite Kaehler manifold (M_1, g_1, J) onto an r-lightlike manifold (M_2, g_2). Then by virtue of Definition (2.1) and Lemma (2.2), the distribution D_1 is invariant with respect to J and $J\Delta$ is a distribution on M_1 such that $\Delta \cap J\Delta = \{0\}$. This proves (i) and (ii).
Now for any $Z \in \Gamma(D_2)$, we have

$$cos\theta = \frac{|fZ|}{|JZ|} \tag{23}$$

which gives

$$cos^2\theta = \frac{|fZ|^2}{|JZ|^2} = \frac{g_1(fZ, fZ)}{g_1(JZ, JZ)} = \frac{g_1(Z, f^2Z)}{g_1(Z, J^2Z)}.$$

this further implies

$$g_1(Z, f^2Z) = cos^2\theta g_1(Z, J^2Z). \tag{24}$$

Since ϕ is a semi-slant lightlike submersion, therefore $cos^2\theta = \lambda$(constant) $\in [0,1)$ and then from Eq. (24), we get

$$g_1(Z, f^2Z) = \lambda g_1(Z, J^2Z) = g_1(Z, \lambda J^2Z),$$

for all $Z \in \Gamma(D_2)$ which further yields that

$$g_1(Z, (f^2 - \lambda J^2)Z) = 0. \tag{25}$$

Since $(f^2 - \lambda J^2)Z \in \Gamma(D_2)$ and D_2 is a non-degenerate distribution of $S(ker\ \phi_*)$, therefore from Eq. (25), we have $(f^2 - \lambda J^2)Z = 0$, that is $f^2Z = \lambda J^2Z = -\lambda Z$ for all $Z \in \Gamma(D_2)$, which proves (iii).
Conversely, let ϕ be a lightlike submersion satisfying the conditions (i), (ii) and (iii). Then from (iii) we obtain

$$f^2Z = \lambda J^2Z,$$

for all $Z \in \Gamma(D_2)$, where $\lambda \in [0,1)$. Now

$$\begin{aligned} cos\theta &= \frac{g_1(JZ, fZ)}{|JZ||fZ|} = \frac{-g_1(Z, f^2Z)}{|JZ||fZ|} = \frac{-\lambda g_1(Z, J^2Z)}{|JZ||fZ|} = \lambda\frac{g_1(JZ, JZ)}{|JZ||fZ|} \\ &= \lambda\frac{|fZ|}{|JZ|}. \end{aligned} \tag{26}$$

Thus using Eq. (23) in Eq. (26), we get $cos^2\theta = \lambda$ (constant).

□

Theorem 3.6. *(Existence Theorem) The necessary and sufficient condition for a lightlike submersion $\phi : M_1 \to M_2$ from an indefinite Kaehler manifold M_1 onto an r-lightlike manifold M_2 to be a semi-slant lightlike submersion is as follows:*

(i) *$J\Delta$ is a distribution on M_1 such that $\Delta \cap J\Delta = \{0\}$,*
(ii) *the screen distribution $S(ker\ \phi_*)$ can be decomposed as a direct sum $S(ker\ \phi_*) = (J\Delta \oplus Jltr(ker\ \phi_*)) \oplus_{ortho} D_1 \oplus_{ortho} D_2$,*
(iii) *there exists a constant $\gamma \in (0, 1]$ such that $B\omega Z = -\gamma Z$, for all $Z \in \Gamma(D_2)$. In this case, $\gamma = sin^2\theta$, where θ is the semi-slant angle of D_2.*

Proof. Assume that ϕ be a semi-slant lightlike submersion, therefore by virtue of its Definition (2.1) and Lemma (2.2), the distribution D_1 is invariant with respect to J and $J\Delta$ is a distribution on M_1 such that $\Delta \cap J\Delta = \{0\}$. As for any vector $Z \in \Gamma(D_2)$, we have

$$JZ = fZ + \omega Z, \tag{27}$$

where fZ and ωZ are tangential and transversal components of JZ respectively. Applying J to Eq. (27) and comparing the tangential components, we get

$$-Z = f^2 Z + B\omega Z, \tag{28}$$

for all $Z \in \Gamma(D_2)$. As ϕ is a semi-slant submersion, so using Theorem (3.5), we have

$$f^2 Z = -\lambda Z,$$

for all $Z \in \Gamma(D_2)$, where $\lambda \in [0, 1)$ and therefore from Eq. (28), we obtain

$$B\omega Z = -\gamma Z,$$

for all $Z \in \Gamma(D_2)$, where $\gamma = 1 - \lambda \in (0, 1]$. This proves (iii).
Conversely, let ϕ be a lightlike submersion such that the three conditions (i), (ii) and (iii) hold. Then from Eq. (28), we acquire

$$-Z = f^2 Z - \gamma Z,$$

for all $Z \in \Gamma(D_2)$, which further implies

$$f^2 Z = -\lambda Z,$$

for all $Z \in \Gamma(D_2)$. Further the proof follows directly from Theorem (3.5). □

Corollary 3.7. *Suppose that $\phi : M_1 \to M_2$ be a semi-slant lightlike submersion with a semi-slant angle θ, then for any $X, Y \in \Gamma(ker\ \phi_*)$, we have*

$$g_1(fX, fY) = cos^2\theta g_1(X, Y), \tag{29}$$

$$g_1(\omega X, \omega Y) = sin^2\theta g_1(X, Y). \tag{30}$$

Lemma 3.8. *Consider a semi-slant lightlike submersion $\phi : M_1 \to M_2$ with a semi-slant angle θ from an indefinite Kaehler manifold M_1 onto an r-lightlike manifold M_2. Then for any unit tangent vector $Z \in \Gamma(D_2)$, we have*

$$fZ = cos\theta . Z^*, \tag{31}$$

where Z^* is a unit tangent vector orthogonal to Z such that $Z^* \in \Gamma(D_2)$.

Proof. For a unit tangent vector $Z \in \Gamma(D_2)$, we have

$$|fZ| = cos\theta(Z)|Z|.$$

Consider another unit tangent vector $Z^* = \frac{fZ}{|fZ|}$ in the direction of fZ, then we have

$$fZ = Z^*.|fZ| = Z^*.cos\theta(Z).$$

Also $g_1(JZ, Z) = 0$, then $g_1(fZ, Z) = 0$ and this further gives $g_1(Z^*, Z) = \frac{1}{|fZ|}.g_1(fZ, Z) = 0$. □

Now, we will investigate some conditions for the integrability of distributions of $ker(\phi_*)$.

Theorem 3.9. *Let $\phi : M_1 \to M_2$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold M_1 onto an r-lightlike manifold M_2. Then Δ is integrable if and only if*

(i) $Q_1(\hat{\nabla}_Z fW) = Q_1(\hat{\nabla}_W fZ)$,
(ii) $Q_4(\hat{\nabla}_Z fW) = Q_4(\hat{\nabla}_W fZ)$,
(iii) $Q_5(\hat{\nabla}_Z fW) = Q_5(\hat{\nabla}_W fZ)$,

for any $Z, W \in \Gamma(\Delta)$

Proof. Let $Z, W \in \Gamma(\Delta)$, then from Eq. (16), we get

$$Q_1\hat{\nabla}_Z fW = fQ_2\hat{\nabla}_Z W. \tag{32}$$

On interchanging Z and W in the preceding equation, we obtain

$$Q_1\hat{\nabla}_W fZ = fQ_2\hat{\nabla}_W Z. \tag{33}$$

Subtracting Eq. (33) from Eq. (32), we get

$$Q_1\hat{\nabla}_Z fW - Q_1\hat{\nabla}_W fZ = fQ_2\mathcal{V}[Z, W]. \tag{34}$$

Also for $Z, W \in \Gamma(\Delta)$, Eq. (19) gives

$$Q_4\hat{\nabla}_W fZ = fQ_4\hat{\nabla}_W Z. \tag{35}$$

On reversing the role of Z and W in the above equation, then we have

$$Q_4\hat{\nabla}_W fZ = fQ_4\hat{\nabla}_W Z. \tag{36}$$

Subtracting Eq. (36) from Eq. (35), we acquire

$$Q_4\hat{\nabla}_Z fW - Q_4\hat{\nabla}_W fZ = fQ_4\mathcal{V}[Z, W]. \tag{37}$$

From Eq. (20), we have

$$Q_5(\hat{\nabla}_Z fW) = Q_5(\hat{\nabla}_W fZ) + BP_2(\mathcal{T}_Z W). \tag{38}$$

If Z and W are interchanged in the above equation, then we get

$$Q_5(\hat{\nabla}_W fZ) = Q_5(\hat{\nabla}_Z fW) + BP_2(\mathcal{T}_W Z). \tag{39}$$

Then from Eqs. (38) and (39), we further obtain

$$Q_5(\hat{\nabla}_Z fW) - Q_5(\hat{\nabla}_W fZ) = fQ_5(\mathcal{V}[Z, W]) + BP_2(\mathcal{T}_Z W - \mathcal{T}_W Z).$$

Since the tensor $\mathcal{T}$ is symmetric, therefore $\mathcal{T}_Z W = \mathcal{T}_W Z$, hence we get

$$Q_5(\hat{\nabla}_Z fW) - Q_5(\hat{\nabla}_W fZ) = fQ_5(\mathcal{V}[Z, W]). \tag{40}$$

Thus the proof follows from Eqs. (34), (37) and (40). □

Theorem 3.10. *Consider a semi-slant lightlike submersion $\phi : M_1 \to M_2$ from an indefinite Kaehler manifold M_1 onto an r-lightlike manifold M_2. Then D_1 is integrable if and only if*

(i) $Q_1(\hat{\nabla}_Z fW) = Q_1(\hat{\nabla}_W fZ)$,
(ii) $Q_2(\hat{\nabla}_Z fW) = Q_2(\hat{\nabla}_W fZ)$,
(iii) $Q_5(\hat{\nabla}_Z fW) = Q_5(\hat{\nabla}_W fZ)$,

for any $Z, W \in \Gamma(D_1)$

Proof. Let ϕ be a semi-slant submersion, then for $Z, W \in \Gamma(D_1)$, Eq. (16) gives

$$Q_1 \hat{\nabla}_Z fW = fQ_2 \hat{\nabla}_Z W, \tag{41}$$

then reversing the role of Z and W, we get

$$Q_1 \hat{\nabla}_W fZ = fQ_2 \hat{\nabla}_W Z. \tag{42}$$

Further subtracting Eqs. (41) and (42), we obtain

$$Q_1 \hat{\nabla}_Z fW - Q_1 \hat{\nabla}_W fZ = fQ_2 \mathcal{V}[Z, W]. \tag{43}$$

Again for $Z, W \in \Gamma(D_1)$ and from Eq. (17), we have

$$Q_2(\hat{\nabla}_Z fW) = fQ_1(\hat{\nabla}_Z W) + BP_2(\mathcal{T}_Z W).$$

Similarly, we acquire

$$Q_2(\hat{\nabla}_W fZ) = fQ_1(\hat{\nabla}_W Z) + BP_2(\mathcal{T}_W Z).$$

Then subtracting last two Eqs., we obtain

$$Q_2(\hat{\nabla}_Z fW) - Q_2(\hat{\nabla}_W fZ) = fQ_1(\mathcal{V}[Z, W] - BP_2(\mathcal{T}_Z W - \mathcal{T}_W Z),$$

then using the symmetry of $\mathcal{T}$, the above equation reduces to

$$Q_2(\hat{\nabla}_Z fW) - Q_2(\hat{\nabla}_W fZ) = fQ_1(\mathcal{V}[Z, W]. \tag{44}$$

Next from Eq. (20), we obtain

$$Q_5(\hat{\nabla}_Z fW) = fQ_5(\hat{\nabla}_Z W).$$

If we interchange the role of Z and W in last equation, then we get

$$Q_5(\hat{\nabla}_W fZ) = fQ_5(\hat{\nabla}_W Z),$$

this further gives

$$Q_5(\hat{\nabla}_Z fW) - Q_5(\hat{\nabla}_W fZ) = fQ_5(\mathcal{V}[Z,W]. \tag{45}$$

Thus the proof follows from the Eqs. (43), (44) and (45). □

Theorem 3.11. *If $\phi : M_1 \to M_2$ is a semi-slant lightlike submersion from an indefinite Kaehler manifold M_1 onto an r-lightlike manifold M_2, then D_2 is integrable if and only if*

(i) $Q_1(\hat{\nabla}_Z fW - \hat{\nabla}_W fZ) = Q_1(\mathcal{T}_W \omega Z - \mathcal{T}_Z \omega W)$,
(ii) $Q_2(\hat{\nabla}_Z fW - \hat{\nabla}_W fZ) = Q_2(\mathcal{T}_W \omega Z - \mathcal{T}_Z \omega W)$,
(iii) $Q_4(\hat{\nabla}_Z fW - \hat{\nabla}_W fZ) = Q_4(\mathcal{T}_W \omega Z - \mathcal{T}_Z \omega W)$,
(iv) $P_1(\mathcal{T}_Z fW - \mathcal{T}_W fZ) = P_1(\mathcal{H}\nabla_W \omega Z - \mathcal{H}\nabla_Z \omega W)$,

for any $Z, W \in \Gamma(D_2)$.

Proof. Let ϕ be a semi-slant submersion and $Z, W \in \Gamma(D_2)$, then from Eq. (16), we obtain

$$Q_1(\hat{\nabla}_Z fW) + Q_1(\mathcal{T}_Z \omega W) = fQ_2(\hat{\nabla}_Z W).$$

If we interchange Z and W in the above equation, then we get

$$Q_1(\hat{\nabla}_W fZ) + Q_1(\mathcal{T}_W \omega Z) = fQ_2(\hat{\nabla}_W Z),$$

this further gives

$$Q_1(\hat{\nabla}_Z fW - \hat{\nabla}_W fZ) + Q_1(\mathcal{T}_Z \omega W - \mathcal{T}_W \omega Z) = fQ_2(\mathcal{V}[Z,W]). \tag{46}$$

For $Z, W \in \Gamma(D_2)$, Eq. (17) reduces to

$$Q_2(\hat{\nabla}_Z fW) + Q_2(\mathcal{T}_Z \omega W) = fQ_1(\hat{\nabla}_Z W).$$

If we interchange Z and W in the above equation, then we get

$$Q_2(\hat{\nabla}_W fZ) + Q_2(\mathcal{T}_W \omega Z) = fQ_1(\hat{\nabla}_W Z),$$

which yields that

$$Q_2(\hat{\nabla}_Z fW - \hat{\nabla}_W fZ) + Q_2(\mathcal{T}_Z \omega W - \mathcal{T}_W \omega Z) = fQ_1(\mathcal{V}[Z,W]). \tag{47}$$

For $Z, W \in \Gamma(D_2)$, Eq. (19) gives

$$Q_4(\hat{\nabla}_Z fW) + Q_4(\mathcal{T}_Z \omega W) = fQ_4(\hat{\nabla}_Z W).$$

On reversing the role of Z and W in the above equation, we get

$$Q_4(\hat{\nabla}_W fZ) + Q_4(\mathcal{T}_W \omega Z) = fQ_4(\hat{\nabla}_W Z),$$

above equation further becomes

$$Q_4(\hat{\nabla}_Z fW - \hat{\nabla}_W fZ) + Q_4(\mathcal{T}_Z \omega W - \mathcal{T}_W \omega Z) = fQ_4(\mathcal{V}[Z,W]). \tag{48}$$

For $Z, W \in \Gamma(D_2)$ and from Eq. (21), we obtain

$$P_1(\mathcal{T}_Z fW) + P_1(\mathcal{H}\nabla_Z \omega W) = \omega Q_3(\hat{\nabla}_Z W).$$

On interchanging the role of Z and W in the above equation, we obtain

$$P_1(\mathcal{T}_W fZ) + P_1(\mathcal{H}\nabla_W \omega Z) = \omega Q_3(\hat{\nabla}_W Z),$$

above equation further implies that

$$P_1(\mathcal{T}_Z fW - \mathcal{T}_W fZ) + P_1(\mathcal{H}\nabla_Z \omega W - \mathcal{H}\nabla_W \omega Z) = \omega Q_3(\mathcal{V}[Z, W]). \tag{49}$$

Thus the result follows from Eqs. (46), (47), (48) and (49). □

4. Foliations Determined By Distributions

In this section, we will establish some necessary and sufficient conditions for the totally geodesic foliations determined by distributions on a semi-slant lightlike submersions.

Theorem 4.1. *Let $\phi : M_1 \to M_2$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold (M_1, g_1, J) onto an r-lightlike manifold M_2. Then Δ defines a totally geodesic foliation if and only if*

$$g_1(\hat{\nabla}_W JQ_2 Z + \hat{\nabla}_W JQ_4 Z + \hat{\nabla}_W JQ_5 Z, JY) = -g_1(\mathcal{T}_W \omega Q_3 Z + \mathcal{T}_W \omega Q_5 Z, JY),$$

for all $W, Y \in \Gamma(\Delta)$ and $Z \in \Gamma(S(ker\ \phi_*))$.

Proof. Let $\phi : M_1 \to M_2$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold (M_1, g_1, J) onto an r-lightlike manifold M_2. To prove that Δ defines a totally geodesic foliation, it is sufficient to prove that $\hat{\nabla}_W Y \in \Gamma(\Delta)$, for all $W, Y \in \Gamma(\Delta)$. Since ∇ is a metric connection on M_1, therefore for any $W, Y \in \Gamma(\Delta)$ and $Z \in \Gamma(S(ker\ \phi_*))$, we have

$$\begin{aligned} g_1(\hat{\nabla}_W Y, Z) &= g_1(\nabla_W Y - \mathcal{T}_W Y, Z) = g_1(\nabla_W Y, Z) = -g_1(Y, \nabla_W Z) \\ &= -g_1(JY, J\nabla_W Z) = -g_1(\nabla_W JZ, JY) \\ &= -g_1(\nabla_W(JQ_2 Z + \omega Q_3 Z + JQ_4 Z + fQ_5 Z + \omega Q_5 Z), JY) \\ &= -g_1(\nabla_W JQ_2 Z + \nabla_W JQ_4 Z + \nabla_W fQ_5 Z, JY) \\ &\quad -g_1(\nabla_W \omega Q_3 Z + \nabla_W \omega Q_5 Z, JY), \end{aligned}$$

further using Eq. (12), we get

$$\begin{aligned} g_1(\hat{\nabla}_W Y, Z) = &-g_1(\nabla_W JQ_2 Z + \nabla_W JQ_4 Z + \nabla_W fQ_5 Z, JY) \\ &- g_1(\mathcal{T}_W \omega Q_3 Z + \mathcal{T}_W \omega Q_5 Z, JY). \end{aligned} \tag{50}$$

Then from Eq. (50), we conclude that the distribution Δ determines a totally geodesic foliation if and only if $g_1(\hat{\nabla}_W Y, Z) = 0$, that is, if and only if

$$\begin{aligned} &g_1(\hat{\nabla}_W JQ_2 Z + \hat{\nabla}_W JQ_4 Z + \hat{\nabla}_W JQ_5 Z, JY) \\ &= -g_1(\mathcal{T}_W \omega Q_3 Z + \mathcal{T}_W \omega Q_5 Z, JY), \end{aligned} \tag{51}$$

which completes the proof. □

Theorem 4.2. *Consider a semi-slant lightlike submersion $\phi : M_1 \to M_2$ from an indefinite Kaehler manifold (M_1, g_1, J) onto an r-lightlike manifold M_2. Then D_1 defines a totally geodesic foliation if and only if*

(i) $g_1(\hat{\nabla}_W fZ, JY) = -g_1(\mathcal{T}_W \omega Z, JY)$,
(ii) $\nabla_W JN$ *and* $\mathcal{T}_W JX$ *have no components in* D_1,

for all $W, Y \in \Gamma(D_1), Z \in \Gamma(D_2), X \in \Gamma(Jltr(ker\ \phi_*)), N \in \Gamma(ltr(ker\ \phi_*))$.

Proof. Since ∇ is a metric connection on M_1, therefore for any $W, Y \in \Gamma(D_1)$ and $Z \in \Gamma(D_2)$, we have

$$\begin{aligned} g_1(\hat{\nabla}_W Y, Z) &= g_1(\nabla_W Y - \mathcal{T}_W Y, Z) = g_1(\nabla_W Y, Z) = -g_1(Y, \nabla_W Z) \\ &= -g_1(JY, J\nabla_W Z) = -g_1(\nabla_W JZ, JY) \\ &= -g_1(\nabla_W fZ, JY) - g_1(\nabla_W \omega Z, JY) \\ &= -g_1(\hat{\nabla}_W fZ, JY) - g_1(\mathcal{T}_W \omega Z, JY). \end{aligned} \tag{52}$$

Now for $W, Y \in \Gamma(D_1)$ and $N \in \Gamma(ltr(ker\ \phi_*))$, we have

$$\begin{aligned} g_1(\hat{\nabla}_W Y, N) &= g_1(\nabla_W Y, N) = -g_1(Y, \nabla_W N) = -g_1(JY, J\nabla_W Y) \\ &= -g_1(JY, \nabla_W JN) = -g_1(JY, \hat{\nabla}_W JN + \mathcal{T}_W JN) \\ &= -g_1(JY, \hat{\nabla}_W JN). \end{aligned} \tag{53}$$

For $W, Y \in \Gamma(D_1)$ and $X \in \Gamma(Jltr(ker\ \phi_*)$, we have

$$\begin{aligned} g_1(\hat{\nabla}_W Y, X) &= g_1(\nabla_W Y - \mathcal{T}_W Y, X) = g_1(\nabla_W Y, X) = -g_1(Y, \nabla_W X) \\ &= -g_1(JY, J\nabla_W X) = -g_1(JY, \nabla_W JX) \\ &= -g_1(JY, \mathcal{T}_W JX) \end{aligned} \tag{54}$$

Thus the proof follows from Eqs. (52), (53) and (54). □

Theorem 4.3. *If $\phi : M_1 \to M_2$ is a semi-slant lightlike submersion from an indefinite Kaehler manifold (M_1, g_1, J) onto an r-lightlike manifold M_2, then D_2 defines a totally geodesic foliation if and only if*

(i) $g_1(\hat{\nabla}_X JZ, fY) = -g_1(P_2 T_X JZ, \omega Y)$,
(ii) $g_1(fY, \hat{\nabla}_X JK) = -g_1(\omega Y, P_2 \mathcal{T}_X JN)$,
(iii) $g_1(fY, \mathcal{T}_X JW) = -g_1(\omega Y, \mathcal{H}(\nabla_X JW))$,

for all $X, Y \in \Gamma(D_2), Z \in \Gamma(D_1), N \in \Gamma(ltr(ker\ \phi_*))$ *and*

$$W \in \Gamma(Jltr(ker\ \phi_*)).$$

Proof. As ∇ is a metric connection on M_1, therefore for any $X, Y \in \Gamma(D_2)$ and $Z \in \Gamma(D_1)$, we have

$$\begin{aligned} g_1(\hat{\nabla}_X Y, Z) &= g_1(\nabla_X Y - \mathcal{T}_X Y, Z) = g_1(\nabla_X Y, Z) = -g_1(Y, \nabla_X Z) \\ &= -g_1(JY, J\nabla_X Z) = -g_1(\nabla_X JZ, JY) \\ &= -g_1(\hat{\nabla}_X JZ + \mathcal{T}_X JZ, fY + \omega Y) \\ &= -g_1(\hat{\nabla}_X JZ, fY) - g_1(\mathcal{T}_X JZ, \omega Y) \\ &= -g_1(\hat{\nabla}_X JZ, fY) - g_1(P_2\mathcal{T}_X JZ, \omega Y). \end{aligned} \tag{55}$$

For $X, Y \in \Gamma(D_2)$ and $N \in \Gamma(ltr(ker\ \phi_*)$, we have

$$\begin{aligned} g_1(\hat{\nabla}_X Y, N) &= -g_1(Y, \nabla_X N) = -g_1(JY, J\nabla_X Y) \\ &= -g_1(JY, \nabla_X JN) = -g_1(fY + \omega Y, \hat{\nabla}_X JN + \mathcal{T}_X JN) \\ &= -g_1(fY, \hat{\nabla}_X JN) - g_1(\omega Y, \mathcal{T}_X JN) \\ &= -g_1(fY, \hat{\nabla}_X JN) - g_1(\omega Y, P_2\mathcal{T}_X JN) \end{aligned} \tag{56}$$

Also for any $X, Y \in \Gamma(D_2)$ and $W \in \Gamma(Jltr(ker\ \phi_*)$, we have

$$\begin{aligned} g_1(\hat{\nabla}_X Y, W) &= g_1(\nabla_X Y - \mathcal{T}_X Y, W) = g_1(\nabla_X Y, W) = -g_1(Y, \nabla_X W) \\ &= -g_1(JY, J\nabla_X W) = -g_1(JY, \nabla_X JW) \\ &= -g_1(fY + \omega X, \mathcal{H}\nabla_X JW + \mathcal{T}_X JW) \\ &= -g_1(fY, \mathcal{T}_X JW) - g_1(\omega Y, \mathcal{H}(\nabla_X JW)). \end{aligned} \tag{57}$$

Hence the proof follows from Eqs. (55), (56) and (57). □

Theorem 4.4. *Let $\phi : M_1 \to M_2$ be a semi-slant lightlike submersion from an indefinite Kaehler manifold (M_1, g_1, J) onto an r-lightlike manifold M_2. Then ϕ is a totally geodesic map if and only if*

(i) $\omega(\hat{\nabla}_W fy + \mathcal{T}_W \omega Y) + C(P_2 T_W fY + P_2\mathcal{H}(\nabla_W \omega Y) = 0,$
(ii) $\omega(\hat{\nabla}_W JP_1U + \hat{\nabla}_W BP_2U + \mathcal{T}_W CP_2U) + C(P_2\mathcal{T}_W JP_1U + P_2\mathcal{T}_W BP_2U$
$+P_2\mathcal{H}(\nabla_W CP_2U)) = 0,$

for each $W, Y \in \Gamma(ker\ \phi_*)$ and $U \in \Gamma(ker\ \phi_*)^\perp$.

Proof. Since ϕ is a semi-slant lightlike submersion, we have

$$(\nabla\phi_*)(U_1, U_2) = \nabla^\phi_{U_1}\phi_*(U_2) - \phi_*(\nabla_{U_1}U_2) = 0 \tag{58}$$

for all $U_1, U_2 \in \Gamma(ker\ \phi_*)^\perp$. Now for $W, Y \in \Gamma(ker\ \phi_*)$, we have

$$
\begin{aligned}
(\nabla\phi_*)(W,Y) &= -\phi_*(\nabla_W Y) = \phi_*(\nabla_W J^2Y) = \phi_*(J\nabla_W JY) \\
&= \phi_*(J\nabla_W(fY+\omega Y)) = \phi_*(J\nabla_W fY + J\nabla_W \omega Y) \\
&= \phi_*(J(\mathcal{T}_W fY + \hat{\nabla}_W fY) + J(\mathcal{H}(\nabla_W \omega Y) + \mathcal{T}_W \omega Y)) \\
&= \phi_*(J(P_1\mathcal{T}_W fY + P_2\mathcal{T}_W fY) + f\hat{\nabla}_W fY + \omega\hat{\nabla}_W fY \\
&\quad + JP_1\mathcal{H}(\nabla_W \omega Y) + JP_2\mathcal{H}(\nabla_W \omega Y) + fT_W\omega Y + \omega T_W \omega Y) \\
&= \phi_*(JP_1\mathcal{T}_W fY + BP_2\mathcal{T}_W fY + CP_2\mathcal{T}_W fY + \omega\hat{\nabla}_W fY \\
&\quad + f\hat{\nabla}_W fY + JP_1\mathcal{H}(\nabla_W \omega Y) + BP_2\mathcal{H}(\nabla_W \omega Y) \\
&\quad + CP_2\mathcal{H}(\nabla_W \omega Y) + f\mathcal{T}_W \omega Y + \omega\mathcal{T}_W \omega Y)). \qquad (59)
\end{aligned}
$$

Now for $W \in \Gamma(ker\ \phi_*)$ and $U \in \Gamma((ker\ \phi_*)^\perp)$, since $(\nabla\phi_*)(W,U) = (\nabla\phi_*)(U,W)$, therefore we have

$$
\begin{aligned}
(\nabla\phi_*)(W,U) &= -\phi_*(\nabla_W U) = \phi_*(J\nabla_W JU) \\
&= \phi_*(J\nabla_W(JP_1U + BP_2U + CP_2U)) \\
&= \phi_*(J(\mathcal{T}_W JP_1U + \hat{\nabla}_W JP_1U + \mathcal{T}_W BP_2U + \hat{\nabla}_W BP_2U \\
&\quad + \mathcal{H}(\nabla_W CP_2U) + \mathcal{T}_W CP_2U)) \\
&= \phi_*(JP_1\mathcal{T}_W JP_1U + BP_2\mathcal{T}_W JP_1U + CP_2T_W JP_1U \\
&\quad + f\hat{\nabla}_W JP_1U + \omega\hat{\nabla}_W JP_1U + JP_1\mathcal{T}_W BP_2U \\
&\quad + BP_2\mathcal{T}_W BP_2U + CP_2\mathcal{T}_W BP_2U + f\hat{\nabla}_W BP_2U \\
&\quad + \omega\hat{\nabla}_W BP_2U + JP_1\mathcal{H}(\nabla_W CP_2U) \\
&\quad + BP_2\mathcal{H}(\nabla_W CP_2U) + CP_2\mathcal{H}(\nabla_W CP_2U) \\
&\quad + f\mathcal{T}_W CP_2U + \omega\mathcal{T}_W CP_2U). \qquad (60)
\end{aligned}
$$

Thus the proof follows from Eqs. (58), (59) and (60). □

5. Semi-Slant Lightlike Submersions with Totally Umbilical Fibres

Let ϕ be a Riemannian submersion from a Riemannian manifold (M_1, g_1) onto a Riemannian manifold (M_2, g_2). Then ϕ is said to be a Riemannian submersion with totally umbilical fibres if there exists a mean curvature vector field H of the fibres such that $\mathcal{T}_X Y = g_1(X,Y)H$, for all $X, Y \in \Gamma(ker\ \phi_*)$. Since, we know that P_1 and P_2 respectively, denote the projections of $tr(ker\ \phi_*)$ on $ltr(ker\ \phi_*)$ and $S(ker\ \phi_*)^\perp$, then taking into account the decomposition of $tr(ker\ \phi_*)$ as $tr(ker\ \phi_*) = ltr(ker\ \phi_*) \perp S(ker\ \phi_*)^\perp$, we have

$$\mathcal{T}_X Y = P_1\mathcal{T}_X Y + P_2\mathcal{T}_X Y, \qquad (61)$$

where $P_1\mathcal{T}_XY \in \Gamma(ltr(ker\ \phi_*))$ and $P_2\mathcal{T}_XY \in \Gamma(S(ker\ \phi_*)^\perp)$. Let ϕ be a lightlike submersion defined from an indefinite Kaehler manifold (M_1, g_1, J) on to an r-lightlike manifold (M_2, g_2). Then ϕ is said to be lightlike submersion with totally umbilical fibres if and only if on each coordinate neighbourhood U there exist smooth vector field $H^{P_1} \in \Gamma(ltr(ker\ \phi_*))$ and $H^{P_2} \in \Gamma(S(ker\ \phi_*)^\perp)$, such that

$$P_1\mathcal{T}_XY = g_1(X,Y)H^{P_1}, P_2\mathcal{T}_XY = g_1(X,Y)H^{P_2}, \tag{62}$$

for all $X, Y \in \Gamma(ker\ \phi_*)$.

Theorem 5.1. *Consider a semi-slant lightlike submersion* $\phi : M_1 \to M_2$ *with totally umbilical fibres from an indefinite Kaehler manifold* (M_1, g_1, J) *onto an r-lightlike manifold* (M_2, g_2)*. Then we have* $H^{P_2} \in \Gamma(\phi(D_2))$*.*

Proof. For $X, Y \in \Gamma(D_1)$ and $W \in \Gamma(\eta)$, from Eqs. (2), (7), (6), (9) and (12)-(15), we obtain

$$\begin{aligned} \mathcal{T}_XJY + \hat{\nabla}_XY &= JP_1\mathcal{T}_XY + BP_2\mathcal{T}_XY + CP_2\mathcal{T}_XY \\ &\quad + f\hat{\nabla}_XY + \omega\hat{\nabla}_XY, \end{aligned} \tag{63}$$

On comparing the horizontal and transversal components in Eq. (63), we get

$$\mathcal{T}_XJY = CP_2\mathcal{T}_XY + \omega\hat{\nabla}_XY \tag{64}$$

and

$$\hat{\nabla}_XY = JP_1\mathcal{T}_XY + BP_2\mathcal{T}_XY + f\hat{\nabla}_XY. \tag{65}$$

Now for $X, Y \in \Gamma(D_1), W \in \Gamma(\eta)$ and using Eq. (64), we have

$$g_1(\mathcal{T}_XJY, W) = g_1(CP_2\mathcal{T}_XY, W) = -g_1(P_2\mathcal{T}_XY, JW). \tag{66}$$

Also from Eq. (61), we have

$$g_1(\mathcal{T}_XJY, W) = g_1(P_1\mathcal{T}_XJY + P_2\mathcal{T}_XJY, W) = g_1(P_2\mathcal{T}_XJY, W). \tag{67}$$

Now from Eqs. (66) and (67), we get

$$-g_1(P_2\mathcal{T}_XY, JW) = g_1(P_2\mathcal{T}_XJY, W). \tag{68}$$

Next using Eq. (62) in Eq. (68), we obtain

$$-g_1(X, JY).g_1(H^{P_2}, W) = g_1(X,Y).g_1(H^{P_2}, JW). \tag{69}$$

On interchanging the role of X and Y in the above equation, we get

$$-g_1(Y, JX).g_1(H^{P_2}, W) = g_1(Y,X).g_1(H^{P_2}, JW),$$

which further gives

$$g_1(X, JY).g_1(H^{P_2}, W) = g_1(X,Y).g_1(H^{P_2}, JW). \tag{70}$$

Now adding Eqs. (68) and (70), we get

$$g_1(X,Y).g_1(H^{P_2}, JW) = 0,$$

this further gives

$$g_1(H^{P_2}, JW) = 0.$$

As $J\eta = \eta$, thus we conclude that $H^{P_2} \in \Gamma(\phi(D_2))$. □

Corollary 5.2. *Let ϕ be a semi-slant lightlike submersion with totally umbilical fibres from an indefinite Kaehler manifold (M_1, g_1, J) onto an r-lightlike manifold (M_2, g_2). If $H^{P_2} \in \Gamma(\eta)$. Then $H^{P_2} = 0$.*

Theorem 5.3. *Suppose that $\phi : M_1 \to M_2$ is a proper semi-slant lightlike submersion with totally umbilical fibres from an indefinite Kaehler manifold (M_1, g_1, J) onto an r-lightlike manifold (M_2, g_2). Then $H^{P_1} = 0$.*

Proof. For $Z \in \Gamma(D_2)$ and using Eqs. (2), (7), (6), (12)-(15) and Lemma (3.8), we get

$$\begin{aligned} & cos\theta(Z)(\mathcal{T}_Z Z^* + \hat{\nabla}_Z Z^*) + \mathcal{H}\nabla_Z \omega Z + \mathcal{T}_Z \omega Z \\ & = CP_2\mathcal{T}_Z Z + \omega\hat{\nabla}_Z Z + JP_1\mathcal{T}_Z Z + BP_2\mathcal{T}_Z Z + f\hat{\nabla}_Z Z. \end{aligned} \tag{71}$$

On comparing the transversal components on both sides of Eq. (71), we get

$$cos\theta(Z)\hat{\nabla}_Z Z^* + \mathcal{T}_Z \omega Z = JP_1\mathcal{T}_Z Z + BP_2\mathcal{T}_Z Z + f\hat{\nabla}_Z Z.$$

Further taking an inner product of the above equation with $J\xi \in \Gamma(J\Delta)$, we get

$$cos\theta(Z) g_1(\hat{\nabla}_Z Z^*, J\xi) + g_1(\mathcal{T}_z \omega Z, J\xi) = g_1(JP_1\mathcal{T}_Z Z, J\xi), \tag{72}$$

using Eq. (12-(15) and (2), we have

$$\begin{aligned} g_1(\hat{\nabla}_Z Z^*, J\xi) &= g_1(\nabla_Z Z^*, J\xi) = -g_1(J\nabla_Z Z^*, \xi) = -g_1(\nabla_Z JZ^*, \xi) \\ &= g_1(JZ^*, \nabla_Z \xi) = g_1(\omega Z^*, \mathcal{T}_Z \xi), \end{aligned}$$

since fibres are totally umbilical, therefore the above equation reduces to

$$g_1(\hat{\nabla}_Z Z^*, J\xi) = g_1(\omega Z^*, H) g_1(Z, \xi) = 0. \tag{73}$$

Also using Eqs. (12-(15) together with the totally umbilical property of fibres, we have

$$\begin{aligned} g_1(\mathcal{T}_z \omega Z, J\xi) &= g_1(\nabla_Z \omega Z, J\xi) = -g_1(\omega Z, \nabla_Z J\xi) \\ &= -g_1(\omega Z, \mathcal{T}_z J\xi) = g_1(Z, J\xi) g_1(\omega Z, H^{P_2}) \\ &= 0. \end{aligned} \tag{74}$$

From Eqs. (73) and (74) in (72), we obtain

$$g_1(JP_1\mathcal{T}_Z Z, J\xi) = 0,$$

which further yields

$$g_1(P_1\mathcal{T}_Z Z, \xi) = 0.$$

In view of Eq. (62), the above equation reduces to

$$g_1(Z, Z) g_1(H^{P_1}, Z) = 0.$$

By non-degeneracy of D_2, we conclude that $H^{P_1} = 0$. This completes the proof. □

Theorem 5.4. *Assume that*

$$\phi : M_1 \to M_2$$

is a proper semi-slant lightlike submersion with totally umbilical fibres from an indefinite Kaehler manifold (M_1, g_1, J) *onto an r-lightlike manifold* (M_2, g_2) *such that*

$$H^{P_2} \in \Gamma(\eta).$$

Then the fibres are always totally geodesics.

Proof. On using the Corollary (5.2) and Theorem (5.3), the result follows. □

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