Honam Mathematical J. **46** (2024), No. 3, pp. 428–451
https://doi.org/10.5831/HMJ.2024.46.3.428

ON ROUGH LACUNARY STATISTICAL CONVERGENCE FOR DOUBLE SEQUENCES IN NEUTROSOPHIC NORMED SPACE

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Abstract. Within the neutrosophic normed space ($\mathfrak{NNS}$), we present the notion of rough lacunary statistical convergence of double sequences in this study. Additionally, we delve into the exploration of rough lacunary statistical cluster points for double sequences in $\mathfrak{NNS}$ and scrutinize the correlation between this set of cluster points and the set of rough lacunary statistical limit points associated with the mentioned convergence.

1. Introduction

In mathematics, the concept of sequence convergence has undergone diverse generalizations with the introduction of various summability methods. Statistical convergence, introduced independently by Steinhaus [29] and Fast [6], is one such notion that generalizes the ordinary convergence of sequences comprising real and complex numbers. Fridy and Orhan [7] initially explored lacunary statistical convergence. Subsequently, Çakan et al. [5] delved into the examination of lacunary statistical convergence using a double sequence. See [4, 9, 10, 11, 12, 22, 25, 26, 27] for the fundamental characteristics and details of these novel ideas.

The idea of rough convergence for sequences in a finite-dimensional normed linear space was first introduced by Phu [23], who also introduced the idea of roughness degree. This idea was later extended to an infinite-dimensional normed linear space [24]. In addition to exploring rough convergence, Phu investigated analytical properties such as convexity and the closeness of the set of rough limits. Aytar [3] extended the concept of rough convergence to rough statistical convergence, utilizing natural density, and examined the relationship between the set of statistical cluster points and the set of rough statistical limit points for a sequence. Building on the concept of rough convergence,

Received January 05, 2024. Revised July 6, 2024. Accepted July 12, 2024.
2020 Mathematics Subject Classification. 40A05, 40A30, 40A35, 40H05.
Key words and phrases. rough convergence, lacunary statistical convergence, neutrosophic normed space.
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various authors explored rough convergence and statistical rough convergence for sequences of different forms. The exploration extended to the study of rough convergence, rough statistical convergence for double sequences in [17, 19, 20].

Zadeh [30] introduced the Theory of Fuzzy Sets ($\mathfrak{FS}$), which had a profound impact on various scientific fields. However, *FS* sometimes struggle with the challenge of handling uncertain membership degrees. To address this, Atanassov [2] extended the theory to Intuitionistic Fuzzy Sets ($\mathfrak{IFS}$). Kramosil and Michalek [18] explored Fuzzy Metric Spaces ($\mathfrak{FMS}$) using fuzzy and probabilistic metric space concepts. By treating the distance between two points as a non-negative fuzzy number, Kaleva and Seikkala [13] examined $\mathfrak{FMS}$. Qualifications for $\mathfrak{FMS}$ were specified by George and Veeramani [8]. $\mathfrak{FMS}$ attracted attention due to its useful applications in fixed-point theory, medical imaging, and decision-making.

Smarandache [28] investigated the 'Neutrosophic set' ($\mathfrak{NS}$) as a generalization of $\mathfrak{FS}$, $\mathfrak{IFS}$ to address uncertainty in practical problem-solving. The membership functions of falsehood (F), indeterminacy (I), and truth (T) comprise the $\mathfrak{NS}$. Neutrosophy implies impartial knowledge of thought, distinguishing $\mathfrak{NS}$ from fuzzy, neutral, logic, and intuitive fuzzy sets.

In $\mathfrak{NS}$, uncertainty is independent of T and F values, making $\mathfrak{NS}$ more general than $\mathfrak{IFS}$ due to the lack of limitations among the degrees of T, F, and I. Neutrosophy signifies impartial knowledge and neutral describes the fundamental difference from neutral, fuzzy, intuitive fuzzy sets, and logic.

Menger [21] introduced Triangular Norms (t-norms) ($\mathfrak{TN}$) as a generalization of the probability distribution with the triangle inequality in metric space terms. Triangular Conorms (t-conorms) ($\mathfrak{TC}$), known as dual operations of $\mathfrak{TN}$, play a crucial role in fuzzy operations such as intersections and unions. $\mathfrak{TN}$ and $\mathfrak{TC}$ are essential components for handling fuzzy operations in the context of metric spaces.

Neutrosophic metric space, defined by continuous t-norms and continuous t-conorms, was first proposed by Kirişci and Şimşek [15]. Additionally, Kirişci and Şimşek [16] extended their investigation to $\mathfrak{NMS}$ and explored statistical convergence within the framework of $\mathfrak{NMS}$.

Antal et al. [1] proposed the idea of rough statistical convergence for sequences. The authors in the study [14] proposed a modification to the definition of $\mathfrak{NMS}$, as originally introduced in [15]. The study introduces the concept of rough lacunary statistical convergence of sequences within this adapted space.

At times, the precise values of terms in a convergent sequence (ω_{uv}) become challenging to ascertain for large enough u, v. In these situations, an approximation error is produced by using a different sequence (υ_{uv}) for the approximation. The idea of rough convergence was born out of these circumstances.

The objective of our research is to expand the concept of convergence to double sequences within $\mathfrak{NMS}$ and investigate various algebraic and topological properties. We investigated this novel convergence, in which the limit might appear as a set instead of a single point, by closely examining the topological (closedness) and geometric properties of the limit set. Moreover, for a given roughness degree $r > 0$, examples were provided to show that the set of all rough lacunary statistical convergent sequences is not a linear space. Furthermore, we established the connection between the cluster point set and the limit set under rough lacunary statistical convergence by introducing the idea of a rough lacunary statistical cluster point in $\mathfrak{NNS}$.

2. Auxiliary Definitions

A few necessary definitions are provided in this section.

Definition 2.1. *Assuming $\mathcal{F}$ is a linear space over the field $\mathcal{V}$ and $\Diamond$ and $*$ are $\mathfrak{TN}$ and $\mathfrak{TC}$, respectively. Let Θ, Ω and Ψ be single valued fuzzy sets on $\mathcal{F} \times (0, \infty)$. We designate the 6-tuple $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ as a $\mathfrak{NNS}$ if, for all $\omega, \gamma \in \mathcal{F}$ and $\tau, \kappa > 0$, the following conditions are satisfied:*
(i) $\Theta(\omega,\tau) + \Omega(\omega,\tau) + \Psi(\omega,\tau) \leq 3$,
(ii) $\Theta(\omega,\tau) = 1$, $\Omega(\omega,\tau) = 0$ *and* $\Psi(\omega,\tau) = 0$ *iff* $\omega = 0$,
(iii) $\Theta(\beta\omega,\tau) = \Theta\left(\omega, \frac{\tau}{|\beta|}\right)$, $\Omega(\beta\omega,\tau) = \Omega\left(\omega, \frac{\tau}{|\beta|}\right)$ *and* $\Psi(\beta\omega,\tau) = \Psi\left(\omega, \frac{\tau}{|\beta|}\right)$ *for any* $0 \neq \beta \in \mathcal{F}$,
(iv) $\Theta(\omega+\gamma,\tau+\kappa) \geq \Theta(\omega,\tau)\Diamond\Theta(\gamma,\kappa)$, $\Omega(\omega+\gamma,\tau+\kappa) \leq \Omega(\omega,\tau) * \Omega(\gamma,\kappa)$ *and* $\Psi(\omega+\gamma,\tau+\kappa) \leq \Psi(\omega,\tau) * \Psi(\gamma,\kappa)$,
(v) $\Theta(\omega,.)$, $\Omega(\omega,.)$ *and* $\Psi(\omega,.)$ *are continuous on* $(0,\infty)$,
(vi) $\lim_{\tau\to\infty}\Theta(\omega,\tau) = 1$, $\lim_{\tau\to\infty}\Omega(\omega,\tau) = 0$ *and* $\lim_{\mu\to\infty}\Psi(\omega,\tau) = 0$,
(vii) $\lim_{\tau\to 0}\Theta(\omega,\tau) = 0$, $\lim_{\mu\to 0}\Omega(\omega,\tau) = 1$ *and* $\lim_{\mu\to 0}\Psi(\omega,\tau) = 1$.
In this scenario, we denote the 3-tuple (Θ, Ω, Ψ) as a neutrosophic norm (shortly, $\mathfrak{NN}$) on $\mathcal{F}$.

Example 2.2. *Let $(\mathcal{F}, \|\|.)$ be a normed space. Consider $\gamma_1 \Diamond \gamma_2 = \gamma_1 \cdot \gamma_2$ and $\gamma_1 * \gamma_2 = \min\{\gamma_1 + \gamma_2, 1\}$, $\forall \gamma_1, \gamma_2 \in [0,1]$. Additionally, define Θ, Ω, and Ψ as follows:*

$$\Theta(\omega,\tau) = \frac{\tau}{\tau + \|\omega\|},\ \psi(u,\mu) = \frac{\|\omega\|}{\tau + \|\omega\|} \text{ and } \Psi(\omega,\mu) = \frac{2\|\omega\|}{\tau + 2\|\omega\|}$$

for all $\omega \in \mathcal{F}$ and $\tau > 0$. Then $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ is a $\mathfrak{NNS}$.

Definition 2.3. *Consider a $\mathfrak{NNS}$ $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ and let $\omega \in \mathcal{F}$. For a given $r > 0$ and $\tau \in (0,1)$, the set*

$$\mathcal{B}_\omega^{(\Theta,\Omega,\Psi)}(r,\tau) = \{\upsilon \in \mathcal{F} : \Theta(\omega-\upsilon, r) > 1-\tau,\ \Omega(\omega-\upsilon, r) < \tau \text{ and } \Psi(\omega-\upsilon, r) < \tau\}$$

defines an open ball with centered at ω and radius r w.r.t $\tau \in (0,1)$.
Define

$$\Im_{(\Theta,\Omega,\Psi)}(\mathcal{F}) = \left\{\mathcal{A} \subset \mathcal{F} : \text{ for all } \omega \in \mathcal{A},\ \exists r > 0 \text{ and } \tau \in (0,1) : \mathcal{B}_\omega^{(\Theta,\Omega,\Psi)}(r,\tau) \subset \mathcal{A}\right\}.$$

Then $\Im_{(\Theta,\Omega,\Psi)}(\mathcal{F})$ *defines a topology on* $\mathcal{F}$*, which is induced by* NN (Θ,Ω,Ψ)*. Since*

$$\left\{\upsilon\in\mathcal{F}:\Theta\left(\omega-\upsilon,\frac{1}{s}\right)>1-\frac{1}{s},\ \Omega\left(\omega-\upsilon,\frac{1}{s}\right)<\frac{1}{s}\text{ and }\Psi\left(\omega-\upsilon,\frac{1}{s}\right)<\frac{1}{s}\right\}$$

is a local base at $\omega\in\mathcal{F}$*, the topology* $\Im_{(\Theta,\Omega,\Psi)}(\mathcal{F})$ *on* $\mathcal{F}$ *is first countable.*

Definition 2.4. *Let* $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$ *be a* $\mathfrak{NNS}$*. A sequence* (ω_u) *in* $\mathcal{F}$ *converges to* ω *w.r.t* NN (Θ,Ω,Ψ)*, if*

$$\Theta(\omega_u-\omega,\tau)\to 1,\ \Omega(\omega_u-\omega,\tau)\to 0\text{ and }\Psi(\omega_u-\omega,\tau)\to 0\text{ as }u\to\infty,$$

supplies for each $\tau>0$*. We write the limit as* $(\Theta,\Omega,\Psi)-\lim\omega_u=\omega$*.*

Definition 2.5. *A sequence* (ω_u) *in* $\mathcal{F}$ *is statistically convergent to* $\omega\in\mathcal{F}$ *w.r.t* $\mathfrak{NN}$ (Θ,Ω,Ψ)*, if for all* $\gamma\in(0,1)$ *and* $\tau>0$*,*

$$\lim_{t\to\infty}\frac{1}{t}\left|\{u\le t:\Theta(\omega_u-\omega,\tau)\le 1-\gamma\text{ or }\Omega(\omega_u-\omega,\tau)\ge\gamma\text{ or }\Psi(\omega_u-\omega,\tau)\ge\gamma\}\right|=0.$$

We represent the limit as $(\Theta,\Omega,\Psi)_{st}-\lim\omega_u=\omega$*.*

Definition 2.6. *A sequence* (ω_u) *in* $\mathcal{F}$ *is said to be rough convergent to* $\omega\in\mathcal{F}$ *w.r.t* $\mathfrak{NN}$ (Θ,Ω,Ψ) *for some* $r\ge 0$*, if for each* $\gamma\in(0,1)$ *and* $\tau>0$*, there exists* $u_0\in\mathbb{N}$ *such that*

$$\Theta(\omega_u-\omega,r+\tau)>1-\gamma,\ \Omega(\omega_u-\omega,r+\tau)<\gamma\text{ and }\Psi(\omega_u-\omega,r+\tau)<\gamma,\ \forall u\ge u_0.$$

We represent the limit as $(\Theta,\Omega,\Psi)^r-\lim\omega_u=\omega$*.*

Definition 2.7. *A double sequence* (ω_{uv}) *is considered rough convergent (r-convergent) to* ω *with the roughness degree* r*, denoted by* $\omega_{uv}\xrightarrow{r}\omega$*, if for every* $\varepsilon>0$*,* $\exists k_\varepsilon\in\mathbb{N}$ *such that for all* $u,v\ge k_\varepsilon$*, the condition* $\|\omega_{uv}-\omega\|<r+\varepsilon$ *holds. Equivalently, it can be expressed as* $\limsup\|\omega_{uv}-\omega\|\le r$*.*

Definition 2.8. *A double sequence* $\theta_2=\theta_{st}=\{(k_s,l_t)\}$ *is called double lacunary sequence if there exist two increasing sequences of integers* (k_u) *and* (l_s) *such that*

$$k_0=0,\ h_s=k_s-k_{s-1}\to\infty\text{ and }l_0=0,\ \overline{h}_t=l_t-l_{t-1}\to\infty,\ s,t\to\infty.$$

We use the notation $k_{st}:=k_sl_t$*,* $h_{st}:=h_s\overline{h}_t$ *and* θ_{st} *is determined by*

$$\mathcal{I}_{st}:=\{(s,t):k_{s-1}<k\le k_s\text{ and }l_{t-1}<l\le l_t\},$$

$$q_s:=\frac{k_s}{k_{s-1}},\ \overline{q}_t:=\frac{l_t}{l_{t-1}}\text{ and }q_{st}:=q_s\overline{q}_t.$$

3. Main Results

Within the context of $\mathfrak{NNS}$ $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$, we introduce the concepts of rough convergence and rough lacunary statistical convergence for double sequences in this section.

Definition 3.1. *We define a double sequence (ω_{uv}) in $\mathcal{F}$ as rough convergent to $\omega\in\mathcal{F}$ w.r.t $\mathfrak{NN}$ (Θ,Ω,Ψ) for some $r\geq 0$ if, for any $\gamma\in(0,1)$ and $\tau>0$, there exist u_0, $v_0\in\mathbb{N}$ such that*

$$\Theta\left(\omega_{uv}-\omega,r+\tau\right)>1-\gamma,\ \Omega\left(\omega_{uv}-\omega,r+\tau\right)<\gamma \text{ and } \Psi\left(\omega_{uv}-\omega,r+\tau\right)<\gamma,$$

for all $u\geq u_0$, $v\geq v_0$.
The convergence of the sequence (ω_{uv}) is characterized by the limit expressed as $(\Theta,\Omega,\Psi)^r-\lim\omega_{uv}=\omega$. The roughness degree of convergence of the sequence (ω_{uv}) is represented by r in this context.

Definition 3.2. *We say that a double sequence (ω_{uv}) in $\mathcal{F}$ is rough lacunary statistically convergent to $\omega\in\mathcal{F}$ w.r.t $\mathfrak{NN}$ (Θ,Ω,Ψ) for a few $r\geq 0$, if for each $\gamma\in(0,1)$ and $\tau>0$,*

$$\begin{aligned}\lim_{s,t\to\infty}\tfrac{1}{h_{st}}\big|\{(u,v)\in\mathcal{I}_{st}:\ &\Theta\left(\omega_{uv}-\omega,r+\tau\right)\leq 1-\gamma,\\ &\Omega\left(\omega_{uv}-\omega,r+\tau\right)\geq\gamma \text{ or } \Psi\left(\omega_{uv}-\omega,r+\tau\right)\geq\gamma\}\big|=0\end{aligned}$$

supplies. We demonstrate the limit as $(\Theta,\Omega,\Psi)^r_{S_{\theta_2}}-\lim\omega_{uv}=\omega$.

Remark 3.3. *The ordinary convergence of (ω_{uv}) w.r.t. $\mathfrak{NN}$ (Θ,Ω,Ψ) is equal the rough convergence of a sequence (ω_{uv}) in $\mathcal{F}$ when $r=0$. The lacunary statistical convergence of (ω_{uv}) w.r.t $\mathfrak{NN}$ (Θ,Ω,Ψ) is what we refer to in this case as the rough lacunary statistical convergence of (ω_{uv}), and denote the limit as $(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\lim\omega_{uv}=\omega$.*

Let $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$ be an $\mathfrak{NNS}$ and $(\omega_{uv})\in\mathcal{F}$. In this context, both $(\Theta,\Omega,\Psi)^r-\lim\omega_{uv}$ and $(\Theta,\Omega,\Psi)^r_{S_{\theta_2}}-\lim\omega_{uv}$ may not be unique. Therefore, we use

$$(\Theta,\Omega,\Psi)-\text{LIM}^r\left(\omega_{uv}\right)=\left\{\omega\in\mathcal{F}:(\Theta,\Omega,\Psi)^r-\lim\omega_{uv}=\omega\right\},$$

and

$$(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\text{LIM}^r\left(\omega_{uv}\right)=\left\{\omega\in\mathcal{F}:(\Theta,\Omega,\Psi)^r_{S_{\theta_2}}-\lim\omega_{uv}=\omega\right\}$$

to demonstrate the set of all $(\Theta,\Omega,\Psi)^r-\lim\omega_{uv}$ and the set of all $(\Theta,\Omega,\Psi)^r_{S_{\theta_2}}-\lim\omega_{uv}$ of the double sequence (ω_{uv}), respectively. We define the sequence (ω_{uv}) as rough convergent w.r.t $\mathfrak{NN}$ (Θ,Ω,Ψ) if $(\Theta,\Omega,\Psi)-\text{LIM}^r\left(\omega_{uv}\right)\neq\emptyset$ and as rough lacunary statistically convergent w.r.t $\mathfrak{NN}$ (Θ,Ω,Ψ) if $(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\text{LIM}^r\left(\omega_{uv}\right)\neq\emptyset$ for some $r\geq 0$. Certainly, it is evident from the definitions that $0\leq r_1\leq r_2$, then

$$(\Theta,\Omega,\Psi)-\text{LIM}^{r_1}\left(\omega_{uv}\right)\subset(\Theta,\Omega,\Psi)-\text{LIM}^{r_2}\left(\omega_{uv}\right),$$

and

$$(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^{r_1}(\omega_{uv})\subset(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^{r_2}(\omega_{uv})$$

for a sequence (ω_{uv}) in $\mathcal{F}$.

Example 3.4. *Take* $\mathfrak{NNS}$ $(\mathbb{R},\Theta,\Omega,\Psi,\Diamond,*)$ *as described in Example 2.2. The sequence* (ω_{uv}) *in* $\mathbb{R}$ *is defined as follows:*

$$\omega_{uv}=\begin{cases}-1, & \text{if } u,v=3s-2\\ 4, & \text{if } u,v=3s-1\\ 1, & \text{if not}\end{cases},\ s\in\mathbb{N}.$$

Then

$$(\Theta,\Omega,\Psi)-\mathrm{LIM}^{r}(\omega_{uv})=\left\{\begin{array}{ll}[4-r,r-1], & \text{if } r\geq\frac{5}{2}\\ \emptyset, & \text{if not.}\end{array}\right\}$$

Assuming $\theta_2=(k_{st})$ *is a double lacunary sequence such that* $\liminf_s\frac{k_s}{k_{s-1}}>1$, $\liminf_t\frac{l_t}{l_{t-1}}>1$ *and consider*

$$\upsilon_{uv}=\begin{cases}uv, & \text{if } u=2^s,\ v=2^t\\ -2, & \text{if } u=2s,\ v=2t\\ 2, & \text{if not}\end{cases},\ s,t\in\mathbb{N}.$$

Then

$$(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^{r}(\upsilon_{uv})=\begin{cases}[2-r,r-2], & \text{if } r\geq 2\\ \emptyset, & \text{if not.}\end{cases}$$

In relation to $\mathfrak{NN}$ (Θ,Ω,Ψ), *none of the sequences* (ω_{uv}) *nor* (υ_{uv}) *converge in the ordinary sense. On the other hand, they are, respectively, rough lacunary statistical convergent and rough convergent. Moreover, there is no* $(\Theta,\Omega,\Psi)^r-\lim\upsilon_{uv}$.

Remark 3.5. *The convergence of a subsequence in a* $\mathfrak{NNS}$ *implies that it converges to the same limit in the corresponding* NN (Θ,Ω,Ψ). *In line with this characteristic, for any subsequence* $\left(\omega_{u_jv_k}\right)$ *of a sequence* (ω_{uv}) *in a* $\mathfrak{NNS}$ *and* $r\geq 0$, *we have*

$$(\Theta,\Omega,\Psi)-\mathrm{LIM}^{r}(\omega_{uv})\subset(\Theta,\Omega,\Psi)-\mathrm{LIM}^{r}\,\omega_{u_jv_k}.$$

This inclusion is valid when $(\Theta,\Omega,\Psi)^r-\lim\omega_{uv}$ *exists. However, it is important to note that this does not hold true in the context of rough lacunary statistical convergence. To illustrate and support the aforementioned statement, consider the following example.*

Example 3.6. *Let* $(\mathbb{R},\|\cdot\|)$ *be the usual normed space. Explain* $\gamma_1\star\gamma_2=\min\{\gamma_1,\gamma_2\}$ *and* $\gamma_1\circ\gamma_2=\max\{\gamma_1,\gamma_2\}$, $\forall\gamma_1,\gamma_2\in[0,1]$. *If*

$$\Theta(\omega,\tau)=\frac{\tau}{\tau+\|\omega\|},\ \Omega(\omega,\tau)=\frac{\|\omega\|}{\tau+\|\omega\|}\ \text{and}\ \Psi(\omega,\tau)=\frac{q\|\omega\|}{\tau+q\|\omega\|},\ q\in\mathbb{R}^+,$$

then $(\mathbb{R},\Theta,\Omega,\Psi,\lozenge,*)$ *is a* $\mathfrak{NNS}$. *Assume that* $\theta_2=(k_u l_v)$ *be a double lacunary sequence such that* $\liminf_s \frac{k_s}{k_{s-1}}>1$, $\liminf_t \frac{l_t}{l_{t-1}}>1$ *and consider*

$$\omega_{uv}=\begin{cases} u^2v^2, & \text{if } u=2^s,\ v=2^t \\ 1, & \text{if not}\end{cases},\ s,t\in\mathbb{N}.$$

Then $(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^r(\omega_{uv})=[1-r,1+r]$ *for all* $r\geq 0$, *but* $(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^r\,\omega_{2^s2^t}=\emptyset$.

We can now give our auxiliary theorem, which plays an important role in the proofs of the following results.

Lemma 3.7. *Assume that* $(\mathcal{F},\Theta,\Omega,\Psi,\lozenge,*)$ *is a* $\mathfrak{NNS}$ *and* (ω_{uv}) *is a sequence in* $\mathcal{F}$. *For each* $\gamma\in(0,1)$ *and* $\tau>0$, *the corresponding propositions hold:*
(i) $(\Theta,\Omega,\Psi)^r_{S_{\theta_2}}-\lim\omega_{uv}=\omega$.
(ii)

$$\begin{aligned}&\lim_{s,t\to\infty}\frac{1}{h_{st}}\left|\left\{(u,v)\in\mathcal{I}_{st}:\Theta(\omega_{uv}-\omega,r+\tau)\leq 1-\gamma\right\}\right|,\\ &=\lim_{s,t\to\infty}\frac{1}{h_{st}}\left|\left\{(u,v)\in\mathcal{I}_{st}:\Omega(\omega_{uv}-\omega,r+\tau)\geq\gamma\right\}\right|,\\ &=\lim_{s,t\to\infty}\frac{1}{h_{st}}\left|\left\{(u,v)\in\mathcal{I}_{st}:\Psi(\omega_{uv}-\omega,r+\tau)\geq\gamma\right\}\right|=0.\end{aligned}$$

(iii)

$$\begin{aligned}&\lim_{s,t\to\infty}\frac{1}{h_{st}}\left|\{(u,v)\in\mathcal{I}_{st}:\Theta(\omega_{uv}-\omega,r+\tau)>1-\gamma\right.\\ &\left.\Omega(\omega_{uv}-\omega,r+\tau)<\gamma,\Psi(\omega_{uv}-\omega,r+\tau)<\gamma\}\right|=1.\end{aligned}$$

(iv)

$$\begin{aligned}&\lim_{s,t\to\infty}\frac{1}{h_{st}}\left|\left\{(u,v)\in\mathcal{I}_{st}:\Theta(\omega_{uv}-\omega,r+\tau)>1-\gamma\right\}\right|\\ &=\lim_{s,t\to\infty}\frac{1}{h_{st}}\left|\left\{(u,v)\in\mathcal{I}_{st}:\Omega(\omega_{uv}-\omega,r+\tau)<\gamma\right\}\right|\\ &=\lim_{s,t\to\infty}\frac{1}{h_{st}}\left|\left\{(u,v)\in\mathcal{I}_{st}:\Psi(\omega_{uv}-\omega,r+\tau)<\gamma\right\}\right|=1.\end{aligned}$$

Proof. The results are self-evident, and therefore, the proof is omitted. □

Theorem 3.8. *Suppose that* $(\mathcal{F},\Theta,\Omega,\Psi,\lozenge,*)$ *be a* $\mathfrak{NNS}$ *and take* $(\omega_{uv})\in\mathcal{F}$. *Then, if* $(\Theta,\Omega,\Psi)^r-\lim\omega_{uv}$ *exists,*

$$(\Theta,\Omega,\Psi)-\mathrm{LIM}^r(\omega_{uv})\subset(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^r(\omega_{uv}).$$

Proof. Let $\omega\in(\Theta,\Omega,\Psi)-\mathrm{LIM}^r(\omega_{uv})$. For all $\gamma\in(0,1)$ and $\tau>0$, $\exists u_0$, $v_0\in\mathbb{N}$ such that

$$\begin{aligned}&\Theta(\omega_{uv}-\omega,r+\tau)>1-\gamma,\ \Omega(\omega_{uv}-\omega,r+\tau)<\gamma \text{ and}\\ &\Psi(\omega_{uv}-\omega,r+\tau)<\gamma,\ \forall u\geq u_0,\ v\geq v_0.\end{aligned}$$

Thus

$$\begin{aligned}&\{(u,v)\in\mathbb{N}\times\mathbb{N}:\Theta(\omega_{uv}-\omega,r+\tau)\leq 1-\gamma\\ &\text{or }\Omega(\omega_{uv}-\omega,r+\tau)\geq\gamma\text{ or }\Psi(\omega_{uv}-\omega,r+\tau)\geq\gamma\}\\ &\subset\{(1,1),(2,2),\ldots,(u_0-1,v_0-1)\}.\end{aligned}$$

Since

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} \left|\left\{(u,v) \in \mathcal{I}_{st} : (u,v) \in \{(1,1),(2,2),\ldots,(u_0-1,v_0-1)\}\right\}\right| = 0,$$

we get

$$\begin{aligned}&\lim_{s,t\to\infty} \tfrac{1}{h_{st}} \mid \{(u,v) \in \mathcal{I}_{st} : \Theta(\omega_{uv}-\omega, r+\tau) \le 1-\gamma \\ &\Omega(\omega_{uv}-\omega, r+\tau) \ge \gamma \text{ or } \Psi(\omega_{uv}-\omega, r+\tau) \ge \gamma\}|\} = 0.\end{aligned}$$

Hence, $\omega \in (\Theta,\Omega,\Psi)_{S_{\theta_2}} - \mathrm{LIM}^r(\omega_{uv})$. The conclusion is thereby established. □

Based on the theorem mentioned above, it is evident that rough lacunary statistical convergence is a broader concept compared to rough convergence in $\mathfrak{NNS}$. However, it's essential to note that the reverse inclusion relation, as demonstrated in Example 3.4, does not hold true.

For a certain roughness degree $r > 0$, an IFNS states that the sum of two rough statistically convergent sequences and the scalar multiplication of a rough statistically convergent sequence are both rough statistically convergent. It is important to remember, nevertheless, that this analogous assertion does not hold in general when discussing rough lacunary statistical convergence in neutrosophic normed spaces ($\mathfrak{NNS}$). The following claims are supported by the given instances and statements.

Proposition 3.9. *Consider the* $\mathfrak{NNS}$ $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$ *Consider two sequences* (ω_{uv}) *and* (ϑ_{uv}) *in* $\mathcal{F}$. *If* $(\Theta,\Omega,\Psi)^{r_1}_{S_{\theta_2}} - \lim \omega_{uv} = \omega$ *and* $(\Theta,\Omega,\Psi)^{r_2}_{S_{\theta_2}} - \lim \vartheta_{uv} = \vartheta$ *for some* $r_1, r_2 \ge 0$, *then*

$$(\Theta,\Omega,\Psi)^{(r_1+r_2)}_{S_{\theta_2}} - \lim\left[\omega_{uv} + \vartheta_{uv}\right] = \omega + \vartheta.$$

Proof. Given $\gamma \in (0,1)$, there exists $\gamma_1 \in (0,1)$ such that $(1-\gamma_1)\Diamond(1-\gamma_1) > 1-\gamma$ and $\gamma_1 * \gamma_1 < \gamma$. Suppose $(\Theta,\Omega,\Psi)^{r_1}_{S_{\theta_2}} - \lim \omega_{uv} = \omega$ and $(\Theta,\Omega,\Psi)^{r_2}_{S_{\theta_2}} - \lim \vartheta_{uv} = \vartheta$ for a certain $r_1, r_2 \ge 0$. For any $\tau > 0$, take into

$$\begin{aligned}&P = \left\{(u,v) \in \mathbb{N}\times\mathbb{N} : \Theta\left(\omega_{uv}-\omega, r_1+\tfrac{\tau}{2}\right) > 1-\gamma_1,\right. \\ &\left.\Omega\left(\omega_{uv}-\omega, r_1+\tfrac{\tau}{2}\right) < \gamma_1 \text{ and } \Psi\left(\omega_{uv}-\omega, r_1+\tfrac{\tau}{2}\right) < \gamma_1\right\},\end{aligned}$$

and

$$\begin{aligned}&Q = \left\{(u,v) \in \mathbb{N}\times\mathbb{N} : \Theta\left(\vartheta_{uv}-\vartheta, r_2+\tfrac{\tau}{2}\right) > 1-\gamma_1,\right. \\ &\left.\Omega\left(\vartheta_{uv}-\vartheta, r_2+\tfrac{\tau}{2}\right) < \gamma_1 \text{ and } \Psi\left(\vartheta_{uv}-\vartheta, r_2+\tfrac{\tau}{2}\right) < \gamma_1\right\}.\end{aligned}$$

Then, by Lemma 3.7, we deduce

$$\begin{aligned}&\lim_{s,t\to\infty} \tfrac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in P\}\right| \\ &= \lim_{s,t\to\infty} \tfrac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in Q\}\right| = 1.\end{aligned}$$

It is evident that $P \cap Q \neq \emptyset$ and

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in P \cap Q\}\right| = 1.$$

Consider $(u,v) \in P \cap Q$. Then

$$\begin{aligned}
&\Theta\left((\omega_{uv}+\vartheta_{uv})-(\omega+\vartheta), r_1+r_2+\tau\right)\\
&\geq \Theta\left(\omega_{uv}-\omega, r_1+\tfrac{\tau}{2}\right) \Diamond \Theta\left(\vartheta_{uv}-\vartheta, r_2+\tfrac{\tau}{2}\right)\\
&> (1-\gamma_1) \Diamond (1-\gamma_1)\\
&> 1-\gamma,
\end{aligned}$$

$$\begin{aligned}
&\Omega\left((\omega_{uv}+\vartheta_{uv})-(\omega+\vartheta), r_1+r_2+\tau\right)\\
&\leq \Omega\left(\omega_{uv}-\omega, r_1+\tfrac{\tau}{2}\right) * \Omega\left(\vartheta_{uv}-\vartheta, r_2+\tfrac{\tau}{2}\right)\\
&< \gamma_1 * \gamma_1\\
&< \gamma,
\end{aligned}$$

and

$$\begin{aligned}
&\Psi\left((\omega_{uv}+\vartheta_{uv})-(\omega+\vartheta), r_1+r_2+\tau\right)\\
&\leq \Psi\left(\omega_{uv}-\omega, r_1+\tfrac{\tau}{2}\right) * \Psi\left(\vartheta_{uv}-\vartheta, r_2+\tfrac{\tau}{2}\right)\\
&< \gamma_1 * \gamma_1\\
&< \gamma.
\end{aligned}$$

Hence

$$\begin{aligned}
&P \cap Q \subseteq \{(u,v) \in \mathbb{N}\times\mathbb{N} : \Theta\left((\omega_{uv}+\vartheta_{uv})-(\omega+\vartheta), r_1+r_2+\tau\right) > 1-\gamma,\\
&\Omega\left((\omega_{uv}+\vartheta_{uv})-(\omega+\vartheta), r_1+r_2+\tau\right) < \gamma\\
&\text{and } \Psi\left((\omega_{uv}+\vartheta_{uv})-(\omega+\vartheta), r_1+r_2+\tau\right) < \gamma\}.
\end{aligned}$$

This means that

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in P \cap Q\}\right|$$

$$\leq \lim_{s,t\to\infty} \frac{1}{h_{st}} \mid \{(u,v) \in \mathcal{I}_{st} : \Theta\left((\omega_{uv}+\vartheta_{uv})-(\omega+\vartheta), r+\tau\right) > 1-\gamma,$$

$$\Omega\left((\omega_{uv}+\vartheta_{uv})-(\omega+\vartheta), r+\tau\right) < \gamma \text{ and } \Psi\left((\omega_{uv}+\vartheta_{uv})-(\omega+\vartheta), r+\tau\right) < \gamma\}|.$$

Thus,

$$\begin{aligned}
&\lim_{s,t\to\infty} \frac{1}{h_{st}} \mid \{(u,v) \in \mathcal{I}_{st} : \Theta\left((\omega_{uv}+\vartheta_{uv})-(\omega+\vartheta), r_1+r_2+\tau\right) > 1-\gamma,\\
&\quad \Omega\left((\omega_{uv}+\vartheta_{uv})-(\omega+\vartheta), r_1+r_2+\mu\right) < \gamma\\
&\quad \text{and } \Psi\left((\omega_{uv}+\vartheta_{uv})-(\omega+\vartheta), r_1+r_2+\mu\right) < \gamma\}| = 1
\end{aligned}$$

is what we have. Consequently, $(\Theta,\Omega,\Psi)^{(r_1+r_2)}_{S_{\theta_2}} - \lim\left[\omega_{uv}+\vartheta_{uv}\right] = \omega+\vartheta$. □

Remark 3.10. *Proposition 3.9 is not valid for $0 < t < r_1+r_2$ when at least one of r_1 and r_2 is non-zero.*

Example 3.11. *Take into consideration the $\mathfrak{NNS}$ $(\mathbb{R},\Theta,\Omega,\Psi,\Diamond,*)$ as defined in Example 2.2. Establish*

$$\omega_{uv} = \begin{cases} 0, & \text{if } u = 3^s,\ v = 3^t \\ -1, & \text{if } u = 2s,\ v = 2t \\ 1, & \text{if not} \end{cases}, \quad s,t \in \mathbb{N}$$

and

$$\vartheta_{uv} = \begin{cases} 1, & \text{if } u = 3^s,\ v = 3^t \\ -2, & \text{if } u = 2s,\ v = 2t \\ 2, & \text{if not} \end{cases},\quad s,t \in \mathbb{N}.$$

It is evident that

$$(\Theta,\Omega,\Psi) - \mathrm{LIM}^{r_1}(\omega_{uv}) = \begin{cases} [1-r_1, r_1-1], & \text{if } r_1 \geq 1 \\ \emptyset, & \text{if not,} \end{cases}$$

and

$$(\Theta,\Omega,\Psi) - \mathrm{LIM}^{r_2}(\vartheta_{uv}) = \begin{cases} [2-r_2, r_2-2], & \text{if } r_2 \geq 2 \\ \emptyset, & \text{if not.} \end{cases}$$

$$\omega_{uv} + \vartheta_{uv} = \begin{cases} 1, & \text{if } u = 3^s,\ v = 3^t \\ -3, & \text{if } u = 2s,\ v = 2t \\ 3, & \text{if not} \end{cases},\quad s,t \in \mathbb{N}.$$

Then

$$(\Theta,\Omega,\Psi) - \mathrm{LIM}^{t}[\omega_{uv} + \vartheta_{uv}] = \begin{cases} [3-t, t-3], & \text{if } t \geq 3 \\ \emptyset, & \text{if not.} \end{cases}$$

$(\Theta,\Omega,\Psi)^{r_1} - \lim \omega_{uv}$ *and* $(\Theta,\Omega,\Psi)^{r_2} - \lim \vartheta_{uv}$ *are equal to* 0 *if* $r_1 = 1$ *and* $r_2 = 2$. *We obtain* $(\Theta,\Omega,\Psi) - \mathrm{LIM}^{t}[\omega_{uv} + \vartheta_{uv}] = \emptyset$ *for* $0 < t < r_1 + r_2 = 3$.

Proposition 3.12. *Suppose that* $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$ *be a* $\mathfrak{NNS}$ *and* (ω_{uv}) *be a sequence in* $\mathcal{F}$. *If* $(\Theta,\Omega,\Psi)^{r}_{S_{\theta_2}} - \lim \omega_{uv} = \omega$ *for some* $r \geq 0$, *then* $(\Theta,\Omega,\Psi)^{|c|r}_{S_{\theta_2}} - \lim c\omega_{uv} = c\omega$ *for any* $c \in \mathcal{F}$.

Proof. When $0 = c \in \mathcal{F}$, the outcome is clear. Let $0 \neq c \in \mathcal{F}$. For given $\gamma \in (0,1)$, one has $\gamma_2 \in (0,1)$ such that $1-\gamma_2 > 1-\gamma$. Since $(\Theta,\Omega,\Psi)^{r}_{S_{\theta_2}} - \lim \omega_{uv} = \omega$, we can consider the set

$$U = \Big\{(u,v) \in \mathbb{N}\times\mathbb{N} : \Theta\left(\omega_{uv} - \omega, r + \tfrac{\tau}{2|c|}\right) > 1-\gamma_2,$$
$$\Omega\left(\omega_{uv} - \omega, r + \tfrac{\tau}{2|c|}\right) < \gamma_2 \text{ and } \Psi\left(\omega_{uv} - \omega, r + \tfrac{\tau}{2|c|}\right) < \gamma_2\Big\}$$

with

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in U\}\right| = 1.$$

Consider $(u,v) \in U$. Then

$$\begin{aligned}\Theta(c\omega_{uv} - c\omega, |c|r + \tau) &= \Theta\left(\omega_{uv} - \omega, r + \tfrac{\tau}{|c|}\right) \\ &\geq \Theta\left(\omega_{uv} - \omega, r + \tfrac{\tau}{2|c|}\right) \\ &> 1-\gamma_2 > 1-\gamma,\end{aligned}$$

$$\begin{aligned}\Omega(c\omega_{uv} - c\omega, |c|r + \tau) &= \Omega\left(\omega_{uv} - \omega, r + \tfrac{\tau}{|c|}\right) \\ &\leq \Omega\left(\omega_{uv} - \omega,, r + \tfrac{\tau}{2|c|}\right) \\ &< \gamma_2 < \gamma,\end{aligned}$$

and

$$\begin{aligned}\Psi\left(c\omega_{uv}-c\omega,|c|r+\tau\right) &= \Psi\left(\omega_{uv}-\omega, r+\tfrac{\tau}{|c|}\right)\\ &\le \Psi\left(\omega_{uv}-\omega, r+\tfrac{\tau}{2|c|}\right)\\ &< \gamma_2<\gamma.\end{aligned}$$

Consequently,

$$\left\{\begin{array}{ll} U & \subset \{(u,v)\in\mathbb{N}\times\mathbb{N}:\Theta\left(c\omega_{uv}-c\omega,|c|r+\tau\right)>1-\gamma, \\ & \Omega\left(c\omega_{uv}-c\omega,|c|r+\tau\right)<\gamma \text{ and } \Psi\left(c\omega_{uv}-c\omega,|c|r+\tau\right)<\gamma\}.\end{array}\right.$$

Therefore,

$$\begin{array}{l}\lim_{s,t\to\infty}\frac{1}{h_{st}}\left|(u,v)\in\mathcal{I}_{st}:\Theta\left(c\omega_{uv}-c\omega,|c|r+\tau\right)>1-\gamma\right.\\ \left.\Omega\left(c\omega_{uv}-c\omega,|c|r+\tau\right)<\gamma \text{ and } \Psi\left(c\omega_{uv}-c\omega,|c|r+\tau\right)<\gamma\}\right|=1.\end{array}$$

Consequently, $(\Theta,\Omega,\Psi)_{S_{\theta_2}}^{|c|r}-\lim c\omega_{uv}=c\omega$. □

Remark 3.13. *When $0<t<|c|r$, Proposition 3.12 is invalid for a positive real number r.*

Example 3.14. *Take a look at Example 3.11 and assume $c=2$.*

$$2\omega_{uv}=\left\{\begin{array}{ll}0, & \text{if } u=3^s,\ v=3^t\\ -2, & \text{if } u=2s,\ v=2t\\ 2, & \text{if not,}\end{array}\right.,\quad s,t\in\mathbb{N}$$

and

$$(\Theta,\Omega,\Psi)-LIM^p\left(2\omega_{uv}\right)=\left\{\begin{array}{ll}[2-p,p-2], & \text{if } p\ge 2\\ \emptyset, & \text{if not}\end{array}\right.$$

is evident. If $r=3$, then $(\Theta,\Omega,\Psi)-\mathrm{LIM}^3\left(\omega_{uv}\right)=[-2,2]$ and $(\Theta,\Omega,\Psi)-\mathrm{LIM}^{(2\times 3)}\left(2\omega_{uv}\right)=[-4,4]=2[-2,2]$. However, for any $2\le p<6$, we obtain $(\Theta,\Omega,\Psi)-\mathrm{LIM}^p\left(2\omega_{uv}\right)=[2-p,p-2]\ne 2[-2,2]$.

Remarks 3.10 and 3.13 make it abundantly evident that the set of rough lacunary statistically convergent sequences does not constitute a linear space for any fixed $r>0$, in contrast to ordinary convergent sequences. Let's now discuss how a S_{θ_2}-bounded sequence is defined in $\mathfrak{NNS}$.

Definition 3.15. *We say that a sequence (ω_{uv}) in $\mathcal{F}$ is S_{θ_2}-bounded w.r.t. $\mathfrak{NN}$ (Θ,Ω,Ψ), if for all $\gamma\in(0,1)$, $\exists\alpha>0$ such that*

$$\lim_{s,t\to\infty}\frac{1}{h_{st}}\left|\{(u,v)\in\mathcal{I}_{st}:\Theta\left(\omega_{uv},\alpha\right)\le 1-\gamma \text{ or } \Omega\left(\omega_{uv},\alpha\right)\ge\gamma,\ \Psi\left(\omega_{uv},\alpha\right)\ge\gamma\}\right|=0.$$

Theorem 3.16. *Let (ω_{uv}) be a sequence in $\mathcal{F}$ and $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$ be a $\mathfrak{NNS}$. In such case, for some $r\ge 0$, (ω_{uv}) is S_{θ_2}-bounded iff $(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^r\left(\omega_{uv}\right)\ne\emptyset$.*

Proof. Assume that (ω_{uv}) is S_{θ_2}-bounded. For each $\gamma \in (0,1)$,there exists $\alpha > 0$ such that the set

$$\mathcal{K} = \{(u,v) \in \mathbb{N} \times \mathbb{N} : \Theta\left(\omega_{uv}, \alpha\right) \leq 1-\gamma \text{ or } \Omega\left(\omega_{uv}, \alpha\right) \geq \gamma,\ \Psi\left(\omega_{uv}, \alpha\right) \geq \gamma\}$$

has

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in \mathcal{K}\}\right| = 0.$$

Thus, utilizing Lemma 3.7, we obtain

$$\mathcal{K}^c = \{(u,v) \in \mathbb{N} \times \mathbb{N} : \Theta\left(\omega_{uv}, \alpha\right) > 1-\gamma \text{ and } \Omega\left(\omega_{uv}, \alpha\right) < \gamma,\ \Psi\left(\omega_{uv}, \alpha\right) < \gamma\}$$

and

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in \mathcal{K}^c\}\right| = 1.$$

Consider $(u,v) \in \mathcal{K}^c$. For each $\tau > 0$, we have

$$\begin{aligned}\Theta\left(\omega_{uv}, \alpha+\tau\right) &\geq \Theta\left(\omega_{uv}, \alpha\right) \Diamond \Theta(0,\tau)\\ &> (1-\gamma)\Diamond 1\\ &= 1-\gamma,\end{aligned}$$

$$\begin{aligned}\Omega\left(\omega_{uv}, \alpha+\tau\right) &\leq \Omega\left(\omega_{uv}, \alpha\right) * \Omega(0,\tau)\\ &< \gamma * 0,\\ &= \gamma,\end{aligned}$$

and

$$\begin{aligned}\Psi\left(\omega_{uv}, \alpha+\tau\right) &\leq \Psi\left(\omega_{uv}, \alpha\right) * \Psi(0,\tau)\\ &< \gamma * 0,\\ &= \gamma.\end{aligned}$$

So, we obtain

$$\begin{aligned}\mathcal{K}^c \subset \{(u,v) \in \mathbb{N} \times \mathbb{N} :\ & \Theta\left(\omega_{uv}, \alpha+\tau\right) > 1-\gamma,\\ & \Omega\left(\omega_{uv}, \alpha+\tau\right) < \gamma \text{ and } \Psi\left(\omega_{uv}, \alpha+\tau\right) < \gamma\}\end{aligned}$$

and

$$\begin{aligned}\lim_{s,t\to\infty} \tfrac{1}{h_{st}} \mid \{(u,v) \in \mathcal{I}_{st} :\ & \Theta\left(\omega_{uv}, \alpha+\tau\right) > 1-\gamma,\\ & \Omega\left(\omega_{uv}, \alpha+\tau\right) < \gamma \text{ and } \Psi\left(\omega_{uv}, \alpha+\tau\right) < \gamma\} \mid = 1.\end{aligned}$$

Consequently, we have $0 \in (\Theta,\Omega,\Psi)_{S_{\theta_2}} - \mathrm{LIM}^{\alpha}\, \omega_{uv}$. Consequently, $(\Theta,\Omega,\Psi)_{S_{\theta_2}} - \mathrm{LIM}^{\alpha}\, \omega_{uv} \neq \emptyset$.
On the contrary, assume $(\Theta,\Omega,\Psi)_{S_{\theta_2}} - \mathrm{LIM}^{r}\, \omega_{uv} \neq \emptyset$ for some $r \geq 0$. So, there exist $\omega \in \mathcal{F}$ such that $\omega \in (\Theta,\Omega,\Psi)_{S_{\theta_2}} - \mathrm{LIM}^{r}\, \omega_{uv}$. Therefore, for each $\gamma \in (0,1)$ and $\tau > 0$, we get

$$\begin{aligned}\mathcal{L} = \{(u,v) \in \mathbb{N} \times \mathbb{N} :\ & \Theta\left(\omega_{uv} - \omega, r+\tau\right) > 1-\gamma\\ & \Omega\left(\omega_{uv} - \omega, r+\tau\right) < \gamma \text{ and } \Psi\left(\omega_{uv} - \omega, r+\tau\right) < \gamma\}\end{aligned}$$

with

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in \mathcal{L}\}\right| = 1.$$

Select an $T > 0$ large enough such that $W = T - (r + \tau) > 0$, $\Theta(\omega, W) = 1$ and $\Omega(\omega, W) = \Psi(\omega, W) = 0$. Let $(u, v) \in \mathcal{L}$. Then

$$\begin{aligned}\Theta(\omega_{uv}, T) &\geq \Theta(\omega_{uv} - \omega, r + \tau)\lozenge\Theta(\omega, W)\\ &> (1-\gamma)\lozenge 1\\ &= 1-\gamma,\\ \Omega(\omega_{uv}, T) &\leq \Omega(\omega_{uv} - \omega, r + \tau) * \Omega(\omega, W)\\ &< \gamma * 0\\ &= \gamma.\end{aligned}$$

Likewise, we obtain $\Psi(\omega_{uv}, T) < \gamma$. Thus,

$$\mathcal{L} \subset \{(u, v) \in \mathbb{N} \times \mathbb{N} : \Theta(\omega_{uv}, T) > 1 - \gamma, \Omega(\omega_{uv}, T) < \gamma, \Psi(\omega_{uv}, T) < \gamma\}.$$

and so, we get

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} |\{(u, v) \in \mathcal{I}_{st} : \Theta(\omega_{uv}, T) > 1 - \tau, \Omega(\omega_{uv}, T) < \tau, \Psi(\omega_{uv}, T) < \tau\}| = 1.$$

As a result, the sequence (ω_{uv}) is S_{θ_2}-bounded. □

In contrast to ordinary convergence, Theorem 3.16 makes clear that the S_{θ_2}-boundedness of a sequence in a $\mathfrak{NNS}$ ensues the presence of a rough lacunary statistical limit. The limits of rough lacunary statistical convergence and rough convergence for a sequence are seen as sets, but the convergence limit in a $\mathfrak{NNS}$ is unique. Consequently, our focus lies on understanding the topological and geometrical characteristics of these limit sets.

Theorem 3.17. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \lozenge, *)$ be a $\mathfrak{NNS}$ and (ω_{uv}) be a sequence in $\mathcal{F}$. After that, for each $r \geq 0$, $(\Theta, \Omega, \Psi)_{S_{\theta_2}} - \mathrm{LIM}^r(\omega_{uv})$ is closed.*

Proof. Select $\gamma_1 \in (0, 1)$ for $\gamma \in (0, 1)$ such that $(1 - \gamma_1)\lozenge(1 - \gamma_1) > (1-\gamma)$ and $\gamma_1 * \gamma_1 < \gamma$. Suppose $\omega \in \overline{(\Theta, \Omega, \Psi)_{S_{\theta_2}} - \mathrm{LIM}^r(\omega_{uv})}$. Then, $\exists$ a sequence (ψ_{uv}) of members of $(\Theta, \Omega, \Psi)_{S_{\theta_2}} - \mathrm{LIM}^r(\omega_{uv})$ such that $(\Theta, \Omega, \Psi) - \lim \psi_{uv} = \omega$. So, for all $\tau > 0$, $\exists\ u_0, v_0 \in \mathbb{N}$ such that

$$\Theta\left(\psi_{uv} - \omega, \frac{\tau}{2}\right) > 1 - \tau_1,\ \Omega\left(\psi_{uv} - \omega, \frac{\tau}{2}\right) < \tau_1 \text{ and } \Psi\left(\psi_{uv} - \omega, \frac{\tau}{2}\right) < \tau_1$$

for all $u \geq u_0$, $v \geq v_0$. Adjust $s, t > m_0$ so that $\psi_{st} \in (\Theta, \Omega, \Psi)_{S_{\theta_2}} - \mathrm{LIM}^r(\omega_{uv})$. According to Lemma 3.7, we write

$$\begin{aligned}\mathcal{K} = \big\{(u, v) \in \mathbb{N} \times \mathbb{N} : \Theta\left(\omega_{uv} - \psi_{st}, r + \tfrac{\tau}{2}\right) > 1 - \tau_1,\\ \Omega\left(\omega_{uv} - \psi_{st}, r + \tfrac{\tau}{2}\right) < \tau_1 \text{ and } \Psi\left(\omega_{uv} - \psi_{st}, r + \tfrac{\tau}{2}\right) < \tau_1\big\}\end{aligned}$$

with

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} |\{(u, v) \in \mathcal{I}_{st} : (u, v) \in \mathcal{K}\}| = 1.$$

For $(u, v) \in \mathcal{K}$, we obtain

$$\begin{aligned}\Theta(\omega_{uv} - w, r + \tau) &\geq \Theta\left(\omega_{uv} - \psi_{st}, r + \tfrac{\tau}{2}\right)\lozenge\Theta\left(\psi_{st} - w, \tfrac{\tau}{2}\right)\\ &> (1 - \gamma_1)\lozenge(1 - \gamma_1) > 1 - \gamma,\end{aligned}$$

$$\begin{aligned}\Omega\left(\omega_{uv}-w,r+\tau\right) &\leq \Omega\left(\omega_{uv}-\psi_{st},r+\tfrac{\tau}{2}\right)*\Omega\left(\psi_{st}-w,\tfrac{\tau}{2}\right)\\ &< \gamma_1*\gamma_1<\gamma,\end{aligned}$$

and

$$\begin{aligned}\Psi\left(\omega_{uv}-w,r+\tau\right) &\leq \Psi\left(\omega_{uv}-\psi_{st},r+\tfrac{\tau}{2}\right)*\Psi\left(\psi_{st}-w,\tfrac{\tau}{2}\right)\\ &< \gamma_1*\gamma_1<\gamma.\end{aligned}$$

for $(u,v)\in\mathcal{K}$. It follows that

$$\begin{aligned}&\lim_{s,t\to\infty}\tfrac{1}{h_{st}}\left|\{(u,v)\in\mathcal{I}_{st}:\Theta\left(\omega_{uv}-w,r+\tau\right)>1-\gamma,\right.\\ &\left.\psi\left(\omega_{uv}-w,r+\tau\right)<\gamma \text{ and } \Psi\left(\omega_{uv}-w,r+\tau\right)<\gamma\}\right|=1,\end{aligned}$$

or $w\in(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^r\left(\omega_{uv}\right)$. Hence, the outcome guarantees. □

For a value of $r=0$, the lacunary statistical convergence in $\mathfrak{NNS}$ becomes the rough version. Consequently, $(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^r\left(\omega_{uv}\right)$ forms a singleton set, making it closed.

Theorem 3.18. *Let $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$ be a $\mathfrak{NNS}$ and (ω_{uv}) be a sequence in $\mathcal{F}$. If $r\geq 0$, then $(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^r\left(\omega_{uv}\right)$ is a convex set.*

Proof. Assume that $\gamma\in(0,1)$ and $w_1,w_2\in(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^r\left(\omega_{uv}\right)$. Then, there exists $\gamma_1\in(0,1)$ such that $(1-\gamma_1)\Diamond(1-\gamma_1)>1-\gamma$ and $\gamma_1*\gamma_1<\gamma$. We show that

$$\beta w_1+(1-\beta)w_2\in(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^r\left(\omega_{uv}\right)$$

for any $\beta\in[0,1]$. The proof is straightforward when $\beta=0$ and $\beta=1$. Consider $\beta\in(0,1)$. For any $\tau>0$, we define

$$\begin{aligned}T=\Big\{&(u,v)\in\mathbb{N}\times\mathbb{N}:\Theta\left(\omega_{uv}-w_1,r+\tfrac{\tau}{2\beta}\right)>1-\gamma_1\\ &\Omega\left(\omega_{uv}-w_1,r+\tfrac{\tau}{2\beta}\right)<\gamma_1 \text{ and } \Psi\left(\omega_{uv}-w_1,r+\tfrac{\tau}{2\beta}\right)<\gamma_1\Big\},\end{aligned}$$

and

$$\begin{aligned}V=\Big\{&(u,v)\in\mathbb{N}\times\mathbb{N}:\Theta\left(\omega_{uv}-w_2,r+\tfrac{\tau}{2(1-\beta)}\right)>1-\gamma_1,\\ &\Omega\left(\omega_{uv}-w_2,r+\tfrac{\tau}{2(1-\beta)}\right)<\gamma_1 \text{ and } \Psi\left(\omega_{uv}-w_2,r+\tfrac{\tau}{2(1-\beta)}\right)<\gamma_1\Big\}.\end{aligned}$$

Since $w_1,w_2\in(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^r\left(\omega_{uv}\right)$, we get

$$\begin{aligned}&\lim_{s,t\to\infty}\tfrac{1}{h_{st}}\left|\{(u,v)\in\mathcal{I}_{st}:(u,v)\in T\}\right|\\ &=\lim_{s,t\to\infty}\tfrac{1}{h_{st}}\left|\{(u,v)\in\mathcal{I}_{st}:(u,v)\in V\}\right|=1.\end{aligned}$$

So, $G\cap H\neq\emptyset$ and

$$\lim_{s,t\to\infty}\frac{1}{h_{st}}\left|\{(u,v)\in\mathcal{I}_{st}:(u,v)\in T\cap V\}\right|=1.$$

Consider $(u,v)\in T\cap V$. Next,

$$\begin{aligned}
&\Theta\left(\omega_{uv}-[\beta w_1+(1-\beta)w_2],r+\tau\right)\\
&=\Theta\left((1-\beta)\left(\omega_{uv}-w_2\right)+\beta\left(\omega_{uv}-w_1\right),(1-\beta)r+\beta r+\tau\right)\\
&\geq\Theta\left((1-\beta)\left(\omega_{uv}-w_2\right),(1-\beta)r+\tfrac{\tau}{2}\right)\Diamond\Theta\left(\beta\left(\omega_{uv}-w_1\right),\beta r+\tfrac{\tau}{2}\right)\\
&=\Theta\left(\omega_{uv}-w_2,r+\tfrac{\tau}{2(1-\beta)}\right)\Diamond\Theta\left(\omega_{uv}-w_1,r+\tfrac{\tau}{2\beta}\right)\\
&>(1-\gamma_1)\Diamond(1-\gamma_1)>1-\gamma,
\end{aligned}$$

and

$$\begin{aligned}
&\Omega\left(\omega_{uv}-[\beta w_1+(1-\beta)w_2],r+\tau\right)\\
&=\Omega\left((1-\beta)\left(\omega_{uv}-w_2\right)+\beta\left(\omega_{uv}-w_1\right),(1-\beta)r+\beta r+\tau\right)\\
&\leq\Omega\left((1-\beta)\left(\omega_{uv}-w_2\right),(1-\beta)r+\tfrac{\tau}{2}\right)*\Omega\left(\beta\left(\omega_{uv}-w_1\right),\beta r+\tfrac{\tau}{2}\right)\\
&=\Omega\left(\omega_{uv}-w_2,r+\tfrac{\tau}{2(1-\beta)}\right)*\Omega\left(\omega_{uv}-w_1,r+\tfrac{\tau}{2\beta}\right)\\
&<\gamma_1*\gamma_1<\gamma.
\end{aligned}$$

In a similar vein

$$\Psi\left(\omega_{uv}-[\beta w_1+(1-\beta)w_2],r+\tau\right)<\gamma.$$

This indicates that the set

$$\begin{gathered}
\{(u,v)\in\mathbb{N}\times\mathbb{N}:\Theta\left(\omega_{uv}-[\beta w_1+(1-\beta)w_2],r+\tau\right)>1-\gamma,\\
\Omega\left(\omega_{uv}-[\beta w_1+(1-\beta)w_2],r+\tau\right)<\gamma\\
\text{and }\Psi\left(\omega_{uv}-[\beta w_1+(1-\beta)w_2],r+\tau\right)<\gamma\}
\end{gathered}$$

contains $T\cap V$ as a subset. Consequently, we have

$$\begin{gathered}
\lim_{s,t\to\infty}\tfrac{1}{h_{st}}\mid\{(u,v)\in\mathcal{I}_{st}:\Theta\left(\omega_{uv}-[\beta w_1+(1-\beta)w_2],r+\tau\right)>1-\gamma,\\
\Omega\left(\omega_{uv}-[\beta w_1+(1-\beta)w_2],r+\tau\right)<\gamma\\
\text{and }\Psi\left(\omega_{uv}-[\beta w_1+(1-\beta)w_2],r+\tau\right)<\gamma\}|=1.
\end{gathered}$$

Thus, $\beta w_1+(1-\beta)w_2\in(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^r\left(\omega_{uv}\right)$. □

When r is equal to 0, rough lacunary statistical convergence becomes equivalent to lacunary statistical convergence in $\mathfrak{NMS}$. Consequently, $(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^r\left(\omega_{uv}\right)$ forms a singleton set, making it inherently convex.

Theorem 3.19. *Let $(\mathcal{F},\Theta,\Omega,\Psi,\Diamond,*)$ be a $\mathfrak{NMS}$ and (ω_{uv}) be a sequence in $\mathcal{F}$. If there is a sequence (υ_{uv}) in $\mathcal{F}$ with $(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\lim\upsilon_{uv}=\omega$ such that for each $\gamma\in(0,1)$ we have $\Theta\left(\omega_{uv}-\upsilon_{uv},r\right)>1-\gamma$, $\Omega\left(\omega_{uv}-\upsilon_{uv},r\right)<\gamma$ and $\Psi\left(\omega_{uv}-\upsilon_{uv},r\right)<\gamma$ for all $(u,v)\in\mathbb{N}\times\mathbb{N}$, then $(\Theta,\Omega,\Psi)_{S_{\theta_2}}-\mathrm{LIM}^r\left(\omega_{uv}\right)=\omega$ for some $r\geq0$.*

Proof. For given $\gamma \in (0,1)$, choose $\gamma_1 \in (0,1)$ such that $(1-\gamma_1) \lozenge (1-\gamma_1) > 1-\gamma$ and $\gamma_1 * \gamma_1 < \gamma$. Assume $(\Theta, \Omega, \Psi)_{S_{\theta_2}} - \lim \upsilon_{uv} = \omega$ and

$$\Theta(\omega_{uv} - \upsilon_{uv}, r) > 1-\gamma,\ \Omega(\omega_{uv} - \upsilon_{uv}, r) < \gamma \text{ and } \Psi(\omega_{uv} - \upsilon_{uv}, r) < \gamma$$

for each $\gamma \in (0,1)$ and for every $(u,v) \in \mathbb{N} \times \mathbb{N}$. For all $\tau > 0$ and the sets

$$\begin{array}{c} U = \{(u,v) \in \mathbb{N} \times \mathbb{N} : \Theta(\upsilon_{uv} - \omega, \tau) \leq 1-\gamma_1 \text{ or} \\ \text{or } \Omega(\upsilon_{uv} - \omega, \tau) \geq \gamma_1 \text{ or } \Psi(\upsilon_{uv} - \omega, \tau) \geq \gamma_1\}, \end{array}$$

and

$$\begin{array}{c} V = \{(u,v) \in \mathbb{N} \times \mathbb{N} : \Theta(\omega_{uv} - \upsilon_{uv}, r) \leq 1-\gamma_1 \text{ or} \\ \text{or } \Omega(\omega_{uv} - \upsilon_{uv}, r) \geq \gamma_1 \text{ or } \Psi(\omega_{uv} - \upsilon_{uv}, r) \geq \gamma_1\} \end{array}$$

we get

$$\lim_{s,t\to\infty} \tfrac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in U\}\right|$$

$$= \lim_{s,t\to\infty} \tfrac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in V\}\right| = 0.$$

$$\begin{array}{l} \Longrightarrow \lim_{s,t\to\infty} \frac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in U^c\}\right| \\ = \lim_{s,t\to\infty} \frac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in V^c\}\right| = 1. \end{array}$$

Evidently, $U^c \cap V^c \neq \emptyset$ and

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in U^c \cap V^c\}\right| = 1.$$

Consider $(u,v) \in U^c \cap V^c$. Then

$$\begin{aligned} \Theta(\omega_{uv} - \omega, r+\tau) &\geq \Theta(\omega_{uv} - \upsilon_{uv}, r) \lozenge \Theta(\upsilon_{uv} - \omega, \tau) \\ &> (1-\gamma_1) \lozenge (1-\gamma_1) > 1-\gamma, \\ \Omega(\omega_{uv} - \omega, r+\tau) &\leq \Omega(\omega_{uv} - \upsilon_{uv}, r) * \Omega(\upsilon_{uv} - \omega, \tau) \\ &< \gamma_1 * \gamma_1 < \gamma, \end{aligned}$$

and

$$\Psi(\omega_{uv} - \omega, r+\tau) \leq \Psi(\omega_{uv} - \upsilon_{uv}, r) * \Psi(\upsilon_{uv} - \omega, \tau) < \gamma_1 * \gamma_1 < \gamma.$$

So,

$$\begin{aligned} U^c \cap V^c &\subset \{(u,v) \in \mathbb{N} \times \mathbb{N} : \Theta(\omega_{uv} - \omega, r+\tau) > 1-\gamma, \\ &\Omega(\omega_{uv} - \omega, r+\tau) < \gamma \text{ and } \Psi(\omega_{uv} - \omega, r+\tau) < \gamma\}. \end{aligned}$$

This implies

$$\begin{array}{l} \lim_{s,t\to\infty} \frac{1}{h_{st}} \mid \{(u,v) \in \mathcal{I}_{st} : \Theta(\omega_{uv} - \omega, r+\tau) > 1-\gamma, \\ \Omega(\omega_{uv} - \omega, r+\tau) < \gamma \text{ and } \Psi(\omega_{uv} - \omega, r+\tau) < \gamma\} \mid = 1. \end{array}$$

Thus $(\Theta, \Omega, \Psi)_{S_{\theta_2}} - \mathrm{LIM}^r(\omega_{uv}) = \omega$. □

As per Theorem 3.19, every rough lacunary statistically convergent sequence (ω_{uv}) within an $\mathfrak{NNG}$ may be approximated by a lacunary statistically convergent sequence (υ_{uv}) with an approximation error of "r" given a certain roughness degree "r". Moreover, the lacunary statistical limit of (υ_{uv}) aligns with that of (ω_{uv}).

Considering that the limit of a statistically convergent rough lacunary sequence in a $\mathfrak{NNG}$ could contain several points, the following obvious question emerges: "What is $(\Theta, \Omega, \Psi)_{S_{\theta_2}} - \mathrm{LIM}^r(\omega_{uv})$'s diameter?" Addressing this query, we introduce the ensuing theorem.

Theorem 3.20. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ be a $\mathfrak{NNG}$ and (ω_{uv}) be a sequence in $\mathcal{F}$. It follows that for every $r > 0$ and $\gamma \in (0,1)$, there is no pair of elements $w_1, w_2 \in (\Theta, \Omega, \Psi)_{S_{\theta_2}} - \mathrm{LIM}^r(\omega_{uv}) = \omega$ such that $\Theta(w_1 - w_2, qr) \leq 1 - \gamma$ or $\Omega(w_1 - w_2, qr) \geq \gamma$ or $\Psi(w_1 - w_2, qr) \geq \gamma$ for $q > 2$.*

Proof. If one possesses $\gamma_1 \in (0,1)$ then $(1-\gamma_1) \Diamond (1-\gamma_1) > 1-\gamma$ and $\gamma_1 * \gamma_1 < \gamma$ for a given $\gamma \in (0,1)$. Whenever feasible, allow $w_1, w_2 \in (\Theta, \Omega, \Psi)_{S_{\theta_2}} - \mathrm{LIM}^r(\omega_{uv}) = \omega$ such that

$$\Theta(w_1 - w_2, qr) \leq 1 - \gamma \text{ or } \Omega(w_1 - w_2, qr) \geq \gamma \text{ or } \Psi(w_1 - w_2, qr) \geq \gamma \text{ for } q > 2.$$

For each $\tau > 0$ and establish the following sets

$$\begin{aligned} K &= \left\{(u,v) \in \mathbb{N} \times \mathbb{N} : \Theta\left(\omega_{uv} - w_1, r + \tfrac{\tau}{2}\right) > 1 - \gamma_1, \right.\\ &\left. \Omega\left(\omega_{uv} - w_1, r + \tfrac{\tau}{2}\right) < \gamma_1 \text{ and } \Psi\left(\omega_{uv} - w_1, r + \tfrac{\tau}{2}\right) < \gamma_1\right\}, \end{aligned}$$

and

$$\begin{aligned} L = & \left\{(u,v) \in \mathbb{N} \times \mathbb{N} : \Theta\left(\omega_{uv} - w_2, r + \tfrac{\tau}{2}\right) > 1 - \gamma_1, \right.\\ &\left. \Omega\left(\omega_{uv} - w_2, r + \tfrac{\tau}{2}\right) < \gamma_1 \text{ and } \Psi\left(\omega_{uv} - w_2, r + \tfrac{\tau}{2}\right) < \gamma_1\right\}, \end{aligned}$$

we have

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in K\}\right| = \lim_{s,t\to\infty} \frac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in L\}\right| = 1.$$

Then, $K \cap L \neq \emptyset$ and

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in K \cap L\}\right| = 1$$

are evident. Let $(u,v) \in K \cap L$. Assume $\Theta(w_1 - w_2, qr) \leq 1 - \gamma$ for $q > 2$. Then,

$$\begin{aligned} 1 - \gamma &\geq \Theta(w_1 - w_2, 2r + \tau) \\ &\geq \Theta\left(\omega_{uv} - w_1, r + \tfrac{\tau}{2}\right) \Diamond \Theta\left(\omega_{uv} - w_2, r + \tfrac{\tau}{2}\right) \\ &> (1 - \gamma_1) \Diamond (1 - \gamma_1) > 1 - \gamma, \end{aligned}$$

which is ridiculous. It is ludicrous to assume

$$\begin{aligned}\gamma &\leq \Omega(w_1 - w_2, 2r + \tau) \\ &\leq \Omega\left(\omega_{uv} - w_1, r + \tfrac{\tau}{2}\right) * \Omega\left(\omega_{uv} - w_2, r + \tfrac{\tau}{2}\right) \\ &< \gamma_1 * \gamma_1 < \gamma\end{aligned}$$

if $\Omega(w_1 - w_2, qr) \geq \gamma$ for $q > 2$.

$$\begin{aligned}\gamma &\leq \Psi(w_1 - w_2, 2r + \tau) \\ &\leq \Psi\left(\omega_{uv} - w_1, r + \tfrac{\tau}{2}\right) * \Psi\left(\omega_{uv} - w_2, r + \tfrac{\tau}{2}\right) \\ &< \gamma_1 * \gamma_1 < \gamma\end{aligned}$$

if $\Psi(w_1 - w_2, qr) \geq \gamma$ for $q > 2$, which is ludicrous once more. The theorem's proof is therefore, completed in every case by the nonsensical result that follows. □

The diameter of the limit set $(\Theta, \Omega, \Psi)_{S_{\theta_2}} - \mathrm{LIM}^r(\omega_{uv})$ cannot be larger than $2r$, according to Theorem 3.20

Theorem 3.21. *In the event that* $(\Theta, \Omega, \Psi)_{S_{\theta_2}} - \mathrm{LIM}(\omega_{uv}) = \omega$, $\tau \in (0,1)$ *occurs such that, for some* $r > 0$, $\overline{\mathcal{B}_\omega^{(\Theta,\Omega,\Psi)}(r, \gamma_1)} \subset (\Theta, \Omega, \Psi)_{S_{\theta_2}} - \mathrm{LIM}^r(\omega_{uv})$.

Proof. If $\gamma \in (0,1)$ is known, find $\exists \gamma_1 \in (0,1)$ such that $(1-\gamma_1) \lozenge (1-\gamma_1) > 1 - \gamma$ and $\gamma_1 * \gamma_1 < \gamma$. Assume that $(\Theta, \Omega, \Psi)_{S_{\theta_2}} - \mathrm{LIM}\, \omega_{uv} = \omega$. For every $\tau > 0$ and consider the set

$$\begin{aligned}\mathcal{L} = \{&(u,v) \in \mathbb{N} \times \mathbb{N} : \Theta(\omega_{uv} - \omega, \tau) > 1 - \gamma_1 \\ &\Omega(\omega_{uv} - \omega, \tau) < \gamma_1 \text{ and } \Psi(\omega_{uv} - \omega, \tau) < \gamma_1\}.\end{aligned}$$

Then, we get

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} |\{(u,v) \in \mathcal{I}_{st} : (u,v) \in \mathcal{L}\}| = 1.$$

Select p such that $p \in \overline{\mathcal{B}_\omega^{(\Theta,\Omega,\Psi)}(r, \gamma_1)}$, $r > 0$.

$$\Theta(\omega - p, r) \geq 1 - \gamma_1,\ \Omega(\omega - p, r) \leq \gamma_1 \text{ and } \Psi(\omega - p, r) \leq \gamma_1$$

in such case. Likewise, for $(u,v) \in \mathcal{L}$, we get

$$\Theta(\omega_{uv} - p, r + \tau) > 1 - \gamma,\ \Omega(\omega_{uv} - p, r + \tau) < \gamma \text{ and } \Psi(\omega_{uv} - p, r + \tau) < \gamma.$$

Consequently,

$$\begin{aligned}\mathcal{L} \subset \{&(u,v) \in \mathbb{N} \times \mathbb{N} : \Theta(\omega_{uv} - p, r + \tau) > 1 - \gamma \\ &\psi(\omega_{uv} - p, r + \tau) < \gamma \text{ and } \eta(\omega_{uv} - p, r + \tau) < \gamma\}.\end{aligned}$$

So, we obtain

$$\begin{aligned}&\lim_{s,t\to\infty} \tfrac{1}{h_{st}} |\{(u,v) \in \mathcal{I}_{st} : \Theta(\omega_{uv} - p, r + \tau) > 1 - \gamma, \\ &\psi(\omega_{uv} - p, r + \tau) < \gamma \text{ and } \eta(\omega_{uv} - p, r + \tau) < \gamma\}| = 1.\end{aligned}$$

Thus, $p \in (\Theta, \Omega, \Psi)_{S_{\theta_2}} - \mathrm{LIM}^r (\omega_{uv})$. This gives that $\overline{\mathcal{B}_{\omega}^{(\Theta,\Omega,\Psi)} (r, \gamma_1)} \subset (\Theta, \Omega, \Psi)_{S_{\theta_2}} - \mathrm{LIM}^r (\omega_{uv})$. □

Now, let's introduce and explore the concept of a rough lacunary statistical cluster point in a $\mathfrak{NNS}$ as stated below:

Definition 3.22. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ be a $\mathfrak{NNS}$ and consider (ω_{uv}) as a sequence in $\mathcal{F}$. For each $r \geq 0$ designated as $(\Theta, \Omega, \Psi)_{S_{\theta_2}}$-cluster point, we define $p \in \mathcal{F}$ as a rough lacunary statistical cluster point of (ω_{uv}) w.r.t. $\mathfrak{NN}$ (Θ, Ω, Ψ) if*

$$\lim_{s,t\to\infty} \tfrac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : \Theta(\omega_{uv} - p, r + \tau) > 1 - \gamma,\right.$$
$$\left.\psi(\omega_{uv} - p, r + \tau) < \gamma \text{ and } \eta(\omega_{uv} - p, r + \tau) < \gamma\}\right| \neq 0$$

holds for every $\tau > 0$ and $\gamma \in (0,1)$. We use $\Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$ to represent the collection of all $(\Theta, \Omega, \Psi)^r_{S_{\theta_2}}$-cluster points of the sequence (ω_{uv}).

When r equals 0, we refer to the rough lacunary statistical cluster point of a sequence (ω_{uv}) in $\mathcal{F}$ as the lacunary statistical cluster point of (ω_{uv}) w.r.t $\mathfrak{NN}$ (Θ, Ω, Ψ), denoted as $(\Theta, \Omega, \Psi)_{S_{\theta_2}}$-cluster point. In this scenario, we represent the collection of all $(\Theta, \Omega, \Psi)_{S_{\theta_2}}$-cluster points of (ω_{uv}) by $\Gamma_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$.

This is how we now display the set $\Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$'s topological property:

Theorem 3.23. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ be a $\mathfrak{NNS}$ and consider (ω_{uv}) as a sequence in $\mathcal{F}$. It follows that for any $r \geq 0$, $\Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$ is a closed set.*

Proof. Assume that $\gamma \in (0,1)$. Then, there exists $\gamma_1 \in (0,1)$ such that $(1-\gamma_1) \Diamond (1-\gamma_1) > 1-\gamma$ and $\gamma_1 * \gamma_1 < \gamma$. Suppose $p \in \overline{\Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})}$. Then, there is a sequence (p_{uv}) of members in $\Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$ such that $(\Theta, \Omega, \Psi) - \lim p_{uv} = p$. Thus, for each $\tau > 0$, $\exists u_0, v_0 \in \mathbb{N}$ such that

$$\Theta\left(p_{uv} - p, \frac{\tau}{2}\right) > 1 - \gamma_1, \ \Omega\left(p_{uv} - p, \frac{\tau}{2}\right) < \gamma_1 \text{ and } \Psi\left(p_{uv} - p, \frac{\tau}{2}\right) < \gamma_1$$

for all $u \geq u_0$, $v \geq v_0$. Assign $s \geq u_0$ and $t \geq v_0$. Next,

$$\Theta\left(p_{st} - p, \frac{\tau}{2}\right) > 1 - \gamma_1, \ \Omega\left(p_{st} - p, \frac{\tau}{2}\right) < \gamma_1 \text{ and } \Psi\left(p_{st} - p, \frac{\tau}{2}\right) < \gamma_1.$$

Also, we have

$$W = \left\{(u,v) \in \mathbb{N}\times\mathbb{N} : \Theta\left(\omega_{uv} - p_{st}, r + \tfrac{\tau}{2}\right) > 1 - \gamma_1,\right.$$
$$\left.\Omega\left(\omega_{uv} - p_{st}, r + \tfrac{\tau}{2}\right) < \gamma_1 \text{ and } \Psi\left(\omega_{uv} - p_{st}, r + \tfrac{\tau}{2}\right) < \gamma_1\right\}$$

with

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} \left|\{(u,v) \in \mathcal{I}_{st} : (u,v) \in W\}\right| \neq 0.$$

If $(u,v) \in W$, then we get

$$\begin{aligned} \Theta(\omega_{uv} - p, r+\tau) &\geq \Theta\left(\omega_{uv} - p_{st}, r+\tfrac{\tau}{2}\right) \Diamond \Theta\left(p_{st} - p, \tfrac{\tau}{2}\right) \\ &> (1-\gamma_1) \Diamond (1-\gamma_1) > 1-\gamma, \\ \Omega(\omega_{uv} - p, r+\tau) &\leq \Omega\left(\omega_{uv} - p_{st}, r+\tfrac{\tau}{2}\right) * \Omega\left(p_{st} - p, \tfrac{\tau}{2}\right) \\ &< \gamma_1 * \gamma_1 < \gamma, \end{aligned}$$

and

$$\begin{aligned} \Psi(\omega_{uv} - p, r+\tau) &\leq \Psi\left(\omega_{uv} - p_{st}, r+\tfrac{\tau}{2}\right) * \Psi\left(p_{st} - p, \tfrac{\tau}{2}\right) \\ &< \gamma_1 * \gamma_1 < \gamma. \end{aligned}$$

Thus, we get

$$\begin{aligned} W \quad &\subset \{(u,v) \in \mathbb{N}\times\mathbb{N} : \Theta(\omega_{uv} - p, r+\tau) > 1-\gamma \\ &\Omega(\omega_{uv} - p, r+\tau) < \gamma \text{ and } \Psi(\omega_{uv} - p, r+\tau) < \gamma\}. \\ \Longrightarrow \quad &\lim_{s,t\to\infty} \tfrac{1}{h_{st}} |\{(u,v) \in \mathcal{I}_{st} : \Theta(\omega_{uv} - p, r+\tau) > 1-\gamma \\ &\Omega(\omega_{uv} - p, r+\tau) < \gamma \text{ and } \Psi(\omega_{uv} - p, r+\tau) < \gamma\}| \neq 0. \end{aligned}$$

$p \in \Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$ as a result, and $\Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$ is closed. □

Theorem 3.24. *Let $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ be a $\mathfrak{NMG}$ and let (ω_{uv}) be a sequence in $\mathcal{F}$. Assume $q \in \Gamma_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$. If, for each $\gamma \in (0,1)$,*

$$\Theta(p-q, r) > 1-\gamma,\ \Omega(p-q, r) < \gamma \text{ and } \Psi(p-q, r) < \gamma.$$

hold for some $r \geq 0$, then $p \in \Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$.

Proof. For given $\gamma \in (0,1)$, $\exists \gamma_1 \in (0,1)$ such that $(1-\gamma_1) \Diamond (1-\gamma_1) > 1-\gamma$ and $\gamma_1 * \gamma_1 < \gamma$. Suppose that $q \in \Gamma_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$. Then for every $\tau > 0$, the set

$$\begin{aligned} T \quad &= \{(u,v) \in \mathbb{N}\times\mathbb{N} : \Theta(\omega_{uv} - q, \tau) > 1-\gamma_1, \\ &\Omega(\omega_{uv} - q, \tau) < \gamma_1 \text{ and } \Psi(\omega_{uv} - q, \tau) < \gamma_1\} \end{aligned}$$

has

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} |\{(u,v) \in \mathcal{I}_{st} : (u,v) \in T\}| \neq 0.$$

Consider $p \in \mathcal{F}$ such that

$$\Theta(p-q, r) > 1-\gamma_1,\ \Omega(p-q, r) < \gamma_1 \text{ and } \Psi(p-q, r) < \gamma_1$$

for some $r \geq 0$. For any pair $(u,v) \in T$, following a similar approach as mentioned above, we derive

$$\Theta(\omega_{uv} - p, r+\tau) > 1-\gamma,\ \Omega(\omega_{uv} - p, r+\tau) < \gamma \text{ and } \Psi(\omega_{uv} - p, r+\tau) < \gamma$$

Therefore,

$$\begin{aligned} T \quad & \subset \{(u,v) \in \mathbb{N}\times\mathbb{N} : \Theta(\omega_{uv} - p, r+\tau) > 1-\gamma, \\ & \Omega(\omega_{uv} - p, r+\tau) < \gamma \text{ and } \Psi(\omega_{uv} - p, r+\tau) < \gamma\}. \\ \Longrightarrow \quad & \lim_{s,t\to\infty} \tfrac{1}{h_{st}} |\{(u,v) \in \mathcal{I}_{st} : \Theta(\omega_{uv} - p, r+\tau) > 1-\gamma, \\ & \Omega(\omega_{uv} - p, r+\tau) < \gamma \text{ and } \Psi(\omega_{uv} - p, r+\tau) < \gamma\}| \neq 0. \end{aligned}$$

So, $p \in \Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$. □

The aforementioned theorem makes it abundantly evident that there is a corresponding rough lacunary statistical cluster point for each lacunary statistical cluster point in a sequence in a $\mathfrak{NNS}$. The following theorem is presented in view of this fact.

Theorem 3.25.

$$\Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv}) = \bigcup_{\omega\in\Gamma_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})} \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{\omega}(r,\gamma)}$$

exists for some $r > 0$ *and* $\gamma \in (0,1)$.

Proof. Suppose $\gamma \in (0,1)$ is given. So, $\exists\gamma_1 \in (0,1)$ such that $(1-\gamma_1)\lozenge(1-\gamma_1) > 1-\gamma$ and $\gamma_1 * \gamma_1 < \gamma$. For some $r > 0$, let

$$p \in \bigcup_{\omega\in\Gamma_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})} \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{\omega}(r,\gamma)}.$$

Then, $\exists\omega \in \Gamma_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$ such that $p \in \overline{\mathcal{B}^{(\Theta,\Omega,\Psi)}_{\omega}(r,\gamma_1)}$, that is,

$$\Theta(\omega - p, r) > 1-\gamma_1,\ \Omega(\omega - p, r) < \gamma_1 \text{ and } \Psi(\omega - p, r) < \gamma_1.$$

By $\omega \in \Gamma_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$, for each $\tau > 0$ and the set

$$\begin{aligned} H = \{(u,v) \in \mathbb{N}\times\mathbb{N} : \Theta(\omega_{uv} - \omega, \tau) > 1-\gamma_1, \\ \Omega(\omega_{uv} - \omega, \tau) \leq \gamma_1 \text{ and } \Psi(\omega_{uv} - \omega, \tau) \leq \gamma_1\}, \end{aligned}$$

we get

$$\lim_{s,t\to\infty} \frac{1}{h_{st}} |\{(u,v) \in \mathcal{I}_{st} : (u,v) \in H\}| \neq 0.$$

Consider $(u,v) \in H$. In a similar vein, we get

$$\Theta(\omega_{uv} - p, r+\tau) > 1-\gamma,\ \Omega(\omega_{uv} - p, r+\tau) < \gamma \text{ and } \Psi(\omega_{uv} - p, r+\tau) < \gamma,$$

as mentioned earlier. Thus,

$$\begin{aligned} H \quad & \subset \{(u,v) \in \mathbb{N}\times\mathbb{N} : \Theta(\omega_{uv} - p, r+\tau) > 1-\gamma, \\ & \Omega(\omega_{uv} - p, r+\tau) < \gamma \text{ and } \Psi(\omega_{uv} - p, r+\tau) < \gamma\} \\ \Rightarrow \quad & \lim_{s,t\to\infty} \tfrac{1}{h_{st}} |\{(u,v) \in \mathcal{I}_{st} : \Theta(\omega_{uv} - p, r+\tau) > 1-\gamma, \\ & \Omega(\omega_{uv} - p, r+\tau) < \gamma \text{ and } \Psi(\omega_{uv} - p, r+\tau) < \gamma\}| \neq 0, \end{aligned}$$

that is, $p \in \Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$. So, we have

$$(1) \qquad \bigcup_{\omega \in \Gamma_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})} \overline{\mathcal{B}_{\omega}^{(\Theta,\Omega,\Psi)}(r,\gamma_1)} \subset \Gamma^r_{(\varphi,\psi,\eta)_{S_{\theta_2}}}(\omega_{uv}).$$

Conversely, if $p \in \Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$, then let on contrary

$$p \notin \bigcup_{\omega \in \Gamma_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})} \overline{\mathcal{B}_{\omega}^{(\Theta,\Omega,\Psi)}(r,\gamma_1)}.$$

Then, for every $\omega \in \Gamma_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$, we obtain $p \notin \overline{\mathcal{B}_{\omega}^{(\Theta,\Omega,\Psi)}(r,\gamma_1)}$, i.e.,

$$\Theta(\omega - p, r) < 1 - \gamma_1 \text{ or } \Omega(\omega - p, r) > \gamma_1 \text{ or } \Psi(\omega - p, r) > \gamma_1.$$

Therefore, by Theorem 3.24, we have $p \notin \Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$, which goes against what we assumed. Thus

$$(2) \qquad \Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv}) \subset \bigcup_{\omega \in \Gamma_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})} \overline{\mathcal{B}_{\omega}^{(\Theta,\Omega,\Psi)}(r,\gamma_1)}$$

By combining (1) and (2), the result follows. □

Theorem 3.26. *Suppose* $(\mathcal{F}, \Theta, \Omega, \Psi, \Diamond, *)$ *is a* $\mathfrak{NNS}$. *Suppose* (ω_{uv}) *be sequence in* $\mathcal{F}$ *such that* $(\Theta,\Omega,\Psi)_{S_{\theta_2}} - \lim \omega_{uv} = \omega$. *Then* $\Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv}) \subset (\Theta,\Omega,\Psi)_{S_{\theta_2}} - \text{LIM}^r(\omega_{uv})$ *for some* $r > 0$.

Proof. Assume $(\Theta,\Omega,\Psi)_{S_{\theta_2}} - \lim \omega_{uv} = \omega$. Thus $\omega \in \Gamma_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv})$. By Theorem 3.25, for some $r > 0$ and $\gamma \in (0,1)$,

$$(3) \qquad \Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv}) = \overline{\mathcal{B}_{\omega}^{(\Theta,\Omega,\Psi)}(r,\gamma)}.$$

Also, by Theorem 3.21,

$$(4) \qquad \overline{\mathcal{B}_{\omega}^{(\Theta,\Omega,\Psi)}(r,\gamma_1)} \subset (\Theta,\Omega,\Psi)_{S_{\theta_2}} - \text{LIM}^r(\omega_{uv}).$$

Hence by (3) and (4), we have

$$\Gamma^r_{(\Theta,\Omega,\Psi)_{S_{\theta_2}}}(\omega_{uv}) \subset (\Theta,\Omega,\Psi)_{S_{\theta_2}} - \text{LIM}^r(\omega_{uv}).$$

□

4. Conclusion

When a convergent double sequence (w_{uv}) comprises terms that are difficult to estimate for sufficiently big u, v, another double sequence (υ_{uv}) must be used to approximate the value of the terms, which introduces approximation error. Rough convergence was introduced as a result of this. A growing number of mathematicians are investigating the connection between statistical convergence and the notions of convergence in neutrosophic normed space. Nevertheless, the more general idea in this theory has not yet been investigated by taking the Pringsheim limit into account. By extending neutrosophic theory, this study has significantly contributed to the body of literature. Two additions to the subject of neutrosophic theory are made by this study: For double sequences in $\mathfrak{NNS}$, (i) a rough lacunary statistical convergence of a sort; (ii) a rough lacunary statistical limit and cluster points. These concepts and conclusions may be utilized as theoretical tools to examine optimum approaches of turnpike theory in a fuzzy environment.

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