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SOME SUMS VIA EULER'S TRANSFORM

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Abstract. In this paper, we give some sums involving the generalized harmonic numbers $H_n^r(\sigma)$ and the (q,r)-binomial coefficient $\binom{L}{k}_{q,r}$ by using Euler's transform. For example, for $(c,r) \in \mathbb{Z}^+ \times \mathbb{R}^+$,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n+r}{n-k}\frac{c^{n+1}H_k^{r-1}(\sigma)}{(n+1)(1+c)^{n+1}} = -\left(c+\frac{1}{\sigma}\right)\ln(1+c\sigma)+c,$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\binom{L}{k}_{2,r} = \sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^k\binom{j-k+2L+r}{j-k}\binom{r}{n-j}\binom{L}{k}_2,$$

where σ is appropriate parameter, $H_n^r(\sigma)$ is the generalized hyperharmonic number of order r and $\binom{L}{k}_q$ is the q-binomial coefficient.

1. Introduction

For $m \in \mathbb{Z}$, the polylogarithm([1,2]) is defined by

$$Li_m(t) = \sum_{n=0}^{\infty}\frac{t^n}{n^m}. \tag{1}$$

The harmonic numbers, denoted by H_n, are defined by

$$H_0 = 0 \text{ and } H_n = \sum_{k=1}^{n}\frac{1}{k} \text{ for } n \geq 1,$$

and their generating function is given as

$$\sum_{n=0}^{\infty}H_nt^n = \frac{-\ln(1-t)}{1-t}.$$

As known, harmonic numbers are interesting research objects([9, 14, 20, 22, 25, 27]). Recently, these numbers have been generalized by several authors.

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There are a lot of works involving harmonic numbers and generalized of them ([10–12, 17, 23, 24]).

For instance, Guo and Cha [17] defined the generalized harmonic numbers by

$$H_0(\sigma) = 0 \text{ and } H_n(\sigma) = \sum_{k=1}^{n} \frac{\sigma^k}{k} \text{ for } n \geq 1, \tag{2}$$

where σ is appropriate parameter, and their generating function is

$$\sum_{n=0}^{\infty} H_n(\sigma)t^n = \frac{-\ln(1-\sigma t)}{1-t}. \tag{3}$$

When $\sigma = 1/\alpha$ for $\alpha \in \mathbb{R}^+$, $H_n(1/\alpha) := \sum_{k=1}^{n} \frac{1}{k\alpha^k}$ are called the generalized harmonic numbers given by Genčev [16].

In [25], Ömür et al. defined the generalized hyperharmonic numbers of order r, $H_n^r(\sigma)$, as follows: $H_n^r(\sigma) = 0$, for $r < 0$ or $n \leq 0$,

$$H_n^0(\sigma) = \frac{\sigma^n}{n} \text{ and } H_n^r(\sigma) = \sum_{i=1}^{n} H_i^{r-1}(\sigma) \text{ for } n, r \geq 1,$$

where σ is as above, and their generating function is

$$\sum_{n=0}^{\infty} H_n^r(\sigma)t^n = \frac{-\ln(1-\sigma t)}{(1-t)^r}. \tag{4}$$

In [20], Koparal et al. defined the generalized harmonic numbers, $H_{n,m}(\sigma)$ by

$$H_{0,m}(\sigma) = 0 \text{ and } H_{n,m}(\sigma) = \sum_{k=1}^{n} \frac{\sigma^k}{k^m} \text{ for } n, m \geq 1, \tag{5}$$

where σ is as above.
For $m = 1$, (5) reduced to (2) as $H_{n,1}(\sigma) = H_n(\sigma)$. Their generating function is

$$\sum_{n=0}^{\infty} H_{n,m}(\sigma)t^n = \frac{Li_m(\sigma t)}{1-t}. \tag{6}$$

For $a, b \in \mathbb{Z}^+$, it is known that

$$\sum_{n=0}^{\infty} \binom{a+n-b}{n-b} t^n = \frac{t^b}{(1-t)^{a+1}}. \tag{7}$$

Let $q \geq 1$ and $L \geq 0$ be two integers. The $q-$binomial coefficient $\binom{L}{k}_q$ defined by

$$\sum_{n=0}^{\infty} \binom{L}{n}_q t^n = \left(1 + t + t^2 + \cdots + t^q\right)^L$$

is a natural extension of the binomial coefficient. For an appropriate introduction of these numbers see Smith and Hogatt [26], Bollinger [6] and Andrews and Baxter [3].
Messahel et al. [21] defined the $r-$ binomial coefficient $\binom{L}{n}_{q,r}$ by

$$\sum_{n=0}^{\infty}(-1)^n\binom{L}{n}_{q,r}t^n=(1+t+...+t^q)^L\left(1+2t+...+qt^{q-1}\right)^r. \tag{8}$$

They established a connection between these coefficients and the partial $r-$Bell polynomials, and many combinatorial properties of these new coefficients.

In [7], let $f(x)$ be a function defined and integrable on $(-r,\lambda]$ for some $r>0$, $\lambda>0$. Also, let $f(x)$ be analytic in a neighborhood of the origin with Taylor series $f(x)=\sum_{n=0}^{\infty}f_nx^n$. Boyadzhiev gave that

$$\int\limits_{x=0}^{\lambda}f(x)\,dx=\sum_{n=0}^{\infty}\left(\frac{\lambda}{\lambda+1}\right)^{n+1}\sum_{k=0}^{n}\binom{n}{k}\frac{f_k}{k+1}. \tag{9}$$

By using Euler's transform for power series, some authors deal with various binomial identities with harmonic numbers([1, 2, 5, 7, 8, 14, 15, 18, 19, 25]).

Boyadzhiev [8] studied some binomial sums with harmonic numbers by using the Euler transform. The author proved the identity as follows: For $n\in\mathbb{N}$ and $\lambda,\mu\in\mathbb{C}$,

$$\sum_{k=1}^{n}\binom{n}{k}\mu^k\lambda^{n-k}H_k=(\lambda+\mu)^nH_n-\left(\lambda(\lambda+\mu)^{n-1}+\frac{\lambda^2}{2}(\lambda+\mu)^{n-2}+...+\frac{\lambda^n}{n}\right).$$

The author also gave the following expansion in a neighborhood of zero,

$$\sum_{n=1}^{\infty}\beta^nH_n\left(\frac{\sigma}{\beta}\right)t^n=\frac{-\ln(1-\sigma t)}{1-\beta t}, \tag{10}$$

where σ,β are appropriate parameters. In particular $\beta=1$, (10) reduces to (3). Also there is the generating function given by

$$\sum_{n=1}^{\infty}\left(\beta\sum_{k=0}^{n-1}(\sigma+\beta)^{n-k-1}\sigma^kH_k+\sigma^nH_n\right)t^n=\frac{-\ln(1-\sigma t)}{1-(\sigma+\beta)t}. \tag{11}$$

In [15], Frontczak proved a new expression for binomial sums with harmonic numbers. His derivation is based on elementary analysis of the Euler's transform of these sums. The author discovered some known identities involving skew-harmonic, and Fibonacci and Lucas numbers. For example, for $n\in\mathbb{Z}^+$,

$$\begin{aligned}\sum_{k=0}^{n}\binom{n}{k}\mu^k\lambda^{n-k}H_k^- &= (\mu-\lambda)^nH_n^-+\lambda^nH_n+\mu\sum_{k=0}^{n-1}(\mu+\lambda)^k\lambda^{n-k-1}H_{n-k-1}\\ &\quad +2\lambda\sum_{k=0}^{n-1}(\mu+\lambda)^k(\mu-\lambda)^{n-k-1}H_{n-k-1}^-,\end{aligned}$$

where $H_k^- = \sum_{k=1}^{n} \frac{(-1)^{k+1}}{k}$ with $H_0^- = 0$ are skew-harmonic numbers.

In [4], Batır and Sofo obtained some general combinatorics formulas. Applying these formulas, they obtained some new identities and gave some known identities included in the works of Frontczak and Boyadzhiev. For example, for $n \in \mathbb{N}$ and $\lambda \in \mathbb{C}$,

$$\sum_{k=0}^{n} \binom{n}{k} \lambda^k H_k^2 = (1+\lambda)^n \left(H_n^2 - \sum_{k=1}^{n} \frac{H_n - 2H_k + H_{n-k}}{k(1+\lambda)^k} - 2\sum_{k=1}^{n} \frac{1}{k^2(1+\lambda)^k} \right).$$

In [25], Ömür et al. gave that for $r \geq 2$,

$$\begin{aligned} &\sum_{k=0}^{n} \binom{n+r-1}{n-k} \left(\frac{\mu}{\lambda}\right)^k H_k^r(\sigma) \\ = &\sum_{k=0}^{n} \sum_{i=0}^{n-k} \binom{k+r-2}{k} \binom{n-k}{i} \left(\frac{\mu}{\lambda}\right)^i \left(\frac{\lambda+\mu}{\lambda}\right)^k H_i(\sigma), \end{aligned}$$

and

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} \beta^n \binom{r-2+k}{k} H_{n-k}(\sigma/\beta)\, t^n = \frac{-\ln(1-\sigma t)}{(1-\beta t)^r}, \tag{12}$$

where σ, β are appropriate parameters.

The generalized derangement numbers $d_{n,r}$ are introduced by Munarini [22] as

$$d_{n,r} = \sum_{k=0}^{n} (-1)^k \binom{r+n-k}{n-k} \frac{n!}{k!}$$

and can be generated by

$$\sum_{n=0}^{\infty} d_{n,r} \frac{t^n}{n!} = \frac{e^{-t}}{(1-t)^{r+1}}. \tag{13}$$

It is clear that $d_{n,0} = d_n$. The first few generalized derangement numbers $d_{n,r}$ are

$$d_{0,r} = 1, \quad d_{1,r} = r, \quad d_{2,r} = r^2 + r + 1, \qquad d_{3,r} = r^3 + 3r^2 + 5r + 2.$$

In [13], Dağlı and Qi obtained identities involving generalized derangement numbers and generalized harmonic numbers. For example, for $n, r \in \mathbb{N}$,

$$\sum_{k=0}^{n} \frac{(-1)^{n-k}}{(n-k)!} H_k^{r+1}\left(\frac{1}{\alpha}\right) = \sum_{k=0}^{n} \frac{d_{n-k,r-1}}{(n-k)!} H_k\left(\frac{1}{\alpha}\right).$$

The Stirling numbers of the second kind $S_2(n,k)$ are defined by

$$x^n = \sum_{k=0}^{n} S_2(n,k) x^{\underline{k}},$$

where $x^{\underline{n}}$ stands for the falling factorial defined by $x^{\underline{0}} = 1$ and $x^{\underline{n}} = x(x-1)...(x-n+1)$. Their generating function is given by

$$\sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!} = \frac{1}{k!}\left(e^t - 1\right)^k \text{ for } k \geq 0. \tag{14}$$

2. Main Results

In this section, we will give some sums involving the generalized harmonic numbers $H_{n,m}(\sigma)$ and $H_n^r(\sigma)$, and the $(q,r)-$ binomial coefficient $\binom{L}{k}_{q,r}$ by using Euler's transform.

Lemma 2.1. *[8] Let a function analytical on the unit disk be $f(t) = \sum_{n=0}^{\infty} f_n t^n$. The Euler's transform can be given as*

$$\frac{1}{(1-\lambda t)^m} f\left(\frac{\mu t}{1-\lambda t}\right) = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{m-1+n}{n-k}\mu^k \lambda^{n-k} f_k t^n,$$

where λ, μ are appropriate parameters.

Theorem 2.2. *Let a function analytical on the unit disk be $f(t) = \sum_{n=0}^{\infty} f_n t^n$. For positive integer c, we have*

$$\int_{x=0}^{c} (1+x)^{r-1} f(x)\,dx = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+r}{n-k}\frac{c^{n+1} f_k}{(n+1)(c+1)^{n+1}}, \tag{15}$$

and

$$\int_{x=0}^{\infty} (1+x)^{r-1} f(x)\,dx = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+r}{n-k}\left(\frac{1}{n+1}\right) f_k. \tag{16}$$

Proof. Using Lemma 2.1, with the substitution $x = \frac{t}{1-t}$, we have

$$\int_{x=0}^{c} (1+x)^{r-1} f(x)\,dx = \int_{t=0}^{c/(1+c)} \frac{1}{(1-t)^{r+1}} f\left(\frac{t}{1-t}\right) dt$$

$$= \int_{t=0}^{c/(1+c)} \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+r}{n-k} f_k t^n dt = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+r}{n-k}\frac{c^{n+1} f_k}{(n+1)(c+1)^{n+1}}.$$

The proof of (16) is similar to the proof of (15). □

If we take $r = 1$ in (15), we have (9).

Corollary 2.3. *For positive integer c, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+r}{n-k}\frac{(-1)^{k}d_{k,r-2}c^{n+1}}{k!\,(n+1)\,(c+1)^{n+1}}=e^{c}-1.$$

Proof. Taking $f(x)=\frac{e^{x}}{(1+x)^{r-1}}$ in (15), by (13), we have

$$\int_{x=0}^{c}(1+x)^{r-1}f(x)\,dx=\int_{x=0}^{c}e^{x}dx=e^{c}-1,$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+r}{n-k}\frac{c^{n+1}f_{k}}{(n+1)\,(1+c)^{n+1}}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+r}{n-k}\frac{(-1)^{k}d_{k,r-2}c^{n+1}}{k!\,(n+1)\,(c+1)^{n+1}}.$$

From here, we have the proof. □

Corollary 2.4. *For $(c,r)\in\mathbb{Z}^{+}\times\mathbb{R}^{+}$, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{k}\binom{n+r}{n-k}\frac{c^{n+1}H_{k}^{r-1}(\sigma)}{(n+1)\,(1+c)^{n+1}}=-\left(c+\frac{1}{\sigma}\right)\ln(1+c\sigma)+c,$$

where σ is as above.

Proof. From (4), taking

$$f(x)=-\frac{\ln(1+\sigma x)}{(1+x)^{r-1}}=\sum_{n=1}^{\infty}(-1)^{n}H_{n}^{r-1}(\sigma)\,x^{n}$$

in (15), we have

$$\begin{aligned}\int_{x=0}^{c}(1+x)^{r-1}f(x)\,dx&=-\int_{x=0}^{c}\ln(1+\sigma x)\,dx\\&=-\left(x\ln(1+\sigma x)\right)\Big|_{x=0}^{c}+\int_{x=0}^{c}\left(1-\frac{1}{1+\sigma x}\right)dx\\&=-\left(x+\frac{1}{\sigma}\right)\ln(1+\sigma x)+x\Big|_{x=0}^{c}\\&=-\left(c+\frac{1}{\sigma}\right)\ln(1+c\sigma)+c,\qquad(17)\end{aligned}$$

and

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n+r}{n-k}\frac{c^{n+1}f_{k}}{(n+1)\,(c+1)^{n+1}}\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^{k}\binom{n+r}{n-k}\frac{c^{n+1}H_{k}^{r-1}(\sigma)}{(n+1)\,(c+1)^{n+1}}.\qquad(18)\end{aligned}$$

Thus, from (17) and (18), the proof is complete. □

Note that for $\sigma = 1$, Corollary 2.4 becomes

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n+r}{n-k}\frac{c^{n+1}H_k^{r-1}}{(n+1)(c+1)^{n+1}} = -(c+1)\ln(c+1)+c \tag{19}$$

and when $c = 1$ and $e-1$ in (19), respectively,

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}(-1)^k\binom{n+r}{n-k}\frac{H_k^{r-1}}{(n+1)2^{n+1}} = -2\ln 2+1,$$

and

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{n+1}(-1)^{k+i}\binom{n+r}{k+r}\binom{n+1}{i}\frac{H_k^{r-1}}{n+1}e^{-i} = -1.$$

Now, using (12) in (15) and partial integration, we have the following result.

Corollary 2.5. *For positive integer c and integers m, r such that $m \neq r$ and $r \geq 2$, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(-1)^k\binom{m-2+i}{i}\binom{n+r}{n-k}\frac{c^{n+1}}{(n+1)(c+1)^{n+1}}H_{k-i}$$
$$= \frac{(c+1)^{r-m}-1}{(r-m)^2} - \frac{(c+1)^{r-m}}{r-m}\ln(c+1).$$

For example, for $r = 3$, $m = 2$ and $c = e-1$, then

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}\sum_{j=0}^{n+1}(-1)^{k+j}\frac{1}{n+1}\binom{n+1}{j}\binom{n+3}{n-k}e^{-j}H_{k-i} = -1.$$

Theorem 2.6. *Let n and m be any positive integers. Then*

$$\sum_{k=0}^{n}(-1)^k\binom{m-1+n}{n-k}H_{k,m}(\sigma) = \sum_{j=0}^{n}\sum_{i=0}^{j}(-1)^i\frac{\sigma^i}{i^m}\binom{j-1}{i-1}\binom{n-j+m-2}{n-j},$$

where σ is as above.

Proof. For (6), by applying Lemma 2.1, the left hand side is

$$\frac{1}{(1+t)^m}f\left(\frac{t}{1+t}\right) = \frac{1}{(1+t)^m}\frac{Li_m\left(\frac{\sigma t}{1+t}\right)}{1-\frac{t}{1+t}} = \frac{1}{(1+t)^{m-1}}Li_m\left(\frac{\sigma t}{1+t}\right)$$

From (1) and (7), we have

$$\begin{aligned}
&\frac{1}{(1+t)^{m}} f\left(\frac{t}{1+t}\right) \\
&= \frac{1}{(1+t)^{m-1}} \sum_{i=0}^{\infty} \frac{\sigma^i t^i}{i^m} \frac{1}{(1+t)^i} \\
&= \sum_{k=0}^{\infty} (-1)^k \binom{k+m-2}{k} t^k \sum_{i=0}^{\infty} \frac{\sigma^i t^i}{i^m} \sum_{k=0}^{\infty} (-1)^k \binom{k+i-1}{k} t^k \\
&= \sum_{k=0}^{\infty} (-1)^k \binom{k+m-2}{k} t^k \sum_{i=0}^{\infty} \sum_{k=i}^{\infty} \frac{\sigma^i}{i^m} (-1)^{k-i} \binom{k-1}{k-i} t^k \\
&= \sum_{k=0}^{\infty} (-1)^k \binom{k+m-2}{k} t^k \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{\sigma^i}{i^m} (-1)^{k-i} \binom{k-1}{k-i} t^k \\
(20) \quad &= \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sum_{i=0}^{j} (-1)^{n-i} \frac{\sigma^i}{i^m} \binom{j-1}{i-1} \binom{n-j+m-2}{n-j} t^n.
\end{aligned}$$

At the same time, by using Euler's transform, the right hand side is

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^{n-k} \binom{m-1+n}{n-k} f_k t^n \\
(21) \quad &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^{n-k} \binom{m-1+n}{n-k} H_{k,m}(\sigma) t^n.
\end{aligned}$$

Thus, comparing coefficients in (20) and (21), the proof is obtained. □

Corollary 2.7. *Let n and m be any positive integers. Then*

$$\begin{aligned}
&\sum_{i=0}^{n} \sum_{k=0}^{i} (-1)^{n+i-k} \binom{m-1+i}{i-k} i! H_{k,m}(\sigma) S_2(n,i) \\
&= \sum_{i=1}^{n} \sum_{k=1}^{i} (-1)^k \binom{n}{i} \frac{\sigma^k (m-1)^{n-i}}{k^m} k! S_2(i,k),
\end{aligned}$$

where σ is as above.

Proof. From (6), by applying Lemma 2.1, we have

$$\frac{1}{(1+t)^{m-1}} Li_m\left(\frac{\sigma t}{1+t}\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (-1)^{n-k} \binom{m-1+n}{n-k} H_{k,m}(\sigma) t^n.$$

From here, when $t = e^{-x} - 1$, by (1) and (14), the left hand side is

$$\begin{aligned} & e^{x(m-1)} Li_m \left(\sigma \left(1 - e^{x}\right)\right) \\ = & \sum_{k=0}^{\infty} \frac{(m-1)^k}{k!} x^k \sum_{k=1}^{\infty} (-1)^k \frac{\sigma^k k!}{k^m} \frac{(e^x - 1)^k}{k!} \\ = & \sum_{k=0}^{\infty} \frac{(m-1)^k}{k!} x^k \sum_{n=1}^{\infty} \sum_{k=1}^{n} (-1)^k \frac{\sigma^k}{k^m} \frac{k!}{n!} S_2(n,k) x^n \\ = & \sum_{n=1}^{\infty} \sum_{i=1}^{n} \sum_{k=1}^{i} (-1)^k \binom{n}{i} \frac{\sigma^k (m-1)^{n-i}}{k^m} \frac{k!}{n!} S_2(i,k) x^n. \end{aligned} \tag{22}$$

At the same time, by using Euler's transform and by (14), the right hand side is

$$\begin{aligned} & \sum_{i=0}^{\infty} \sum_{k=0}^{i} (-1)^{i-k} \binom{m-1+i}{i-k} H_{k,m}(\sigma) \left(e^{-x} - 1\right)^i \\ = & \sum_{i=0}^{\infty} \sum_{k=0}^{i} (-1)^{i-k} \binom{m-1+i}{i-k} i! H_{k,m}(\sigma) \frac{(e^{-x} - 1)^i}{i!} \\ = & \sum_{n=0}^{\infty} \sum_{i=0}^{n} \sum_{k=0}^{i} (-1)^{n+i-k} \binom{m-1+i}{i-k} i! H_{k,m}(\sigma) S_2(n,i) \frac{x^n}{n!}. \end{aligned} \tag{23}$$

Thus, comparing coefficients in (22) and (23), the proof is obtained. □

Theorem 2.8. *Let n and r be any positive integers. Then*

$$\sum_{k=0}^{n} \binom{n}{k} \binom{L}{k}_{2,r} = \sum_{j=0}^{n} \sum_{k=0}^{j} (-1)^k \binom{j-k+2L+r}{j-k} \binom{r}{n-j} \binom{L}{k}_2, \tag{24}$$

and

$$\begin{aligned} & \sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^{i+k} \binom{n}{k} \binom{r}{k-i} \binom{L}{i}_2 \\ = & (-2)^n \sum_{k=0}^{n} \sum_{i=0}^{k} (-1)^{i+k} 2^{-k} \binom{k-i+2L+r}{k-i} \binom{r}{n-k} \binom{L}{i}_2. \end{aligned} \tag{25}$$

Proof. For

$$f(t) = \sum_{n=0}^{\infty} \binom{L}{n}_{2,r} t^n = \left(1 + t + t^2\right)^L (1 + 2t)^r,$$

by applying Lemma 2.1, the left hand side is

$$\frac{1}{1-t}f\left(\frac{t}{1-t}\right) = \frac{1}{1-t}\left(\frac{1-t+t^2}{(1-t)^2}\right)^L\left(\frac{1+t}{1-t}\right)^r$$
$$= \frac{1}{(1-t)^{2L+r+1}}\left(1-t+t^2\right)^L(1+t)^r.$$

From Binomial Theorem, (7) and (8), we have

$$\frac{1}{1-t}f\left(\frac{t}{1-t}\right)$$
$$= \sum_{k=0}^{\infty}\binom{k+2L+r}{k}t^k\sum_{k=0}^{\infty}(-1)^k\binom{L}{k}_2 t^k\sum_{k=0}^{\infty}\binom{r}{k}t^k$$
$$= \sum_{i=0}^{\infty}\sum_{k=0}^{i}(-1)^k\binom{L}{k}_2\binom{i-k+2L+r}{i-k}t^i\sum_{k=0}^{\infty}\binom{r}{k}t^k$$
$$(26)\qquad = \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^k\binom{L}{k}_2\binom{j-k+2L+r}{j-k}\binom{r}{n-j}t^n.$$

At the same time, by using Euler's transform, the right hand side is

$$(27)\qquad \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}f_k t^n = \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\binom{L}{k}_{2,r}t^n.$$

Thus, comparing coefficients in (26) and (27), the proof of (24) is obtained. Similarly, using

$$f(t) = \left(1+t+t^2\right)^L(1-t)^r = \sum_{n=0}^{\infty}\sum_{i=0}^{n}(-1)^{n-i}\binom{L}{i}_2\binom{r}{n-i}t^n,$$

the proof of (25) is similar to the proof of (24). □

Corollary 2.9. *Let n and r be any positive integers. Then*

$$\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^{k+i}i!\binom{L}{k}_2\binom{j-k+2L+r}{j-k}\binom{r}{i-j}S_2(n,i)$$
$$= \sum_{i=0}^{n}\sum_{k=0}^{i}(-1)^i i!\binom{i}{k}\binom{L}{k}_{2,r}S_2(n,i).$$

Proof. When $t = 1-e^{-x}$ in (26) and (27), we have

$$(28)\qquad \sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^k\binom{L}{k}_2\binom{j-k+2L+r}{j-k}\binom{r}{n-j}\left(1-e^{-x}\right)^n$$
$$= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n}{k}\binom{L}{k}_{2,r}\left(1-e^{-x}\right)^n.$$

From (14), we have

$$\sum_{n=0}^{\infty}\sum_{j=0}^{n}\sum_{k=0}^{j}(-1)^k\binom{L}{k}_2\binom{j-k+2L+r}{j-k}\binom{r}{n-j}\left(1-e^{-x}\right)^n$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\sum_{k=0}^{j}(-1)^{k+i}\,i!\binom{L}{k}_2\binom{j-k+2L+r}{j-k}\binom{r}{i-j}\sum_{n=i}^{\infty}(-1)^n S_2(n,i)\frac{x^n}{n!}$$

$$(29) = \sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{j} i!\binom{L}{k}_2\binom{j-k+2L+r}{j-k}\binom{r}{i-j}S_2(n,i)\frac{(-1)^{n+k+i}x^n}{n!},$$

and

$$\sum_{i=0}^{\infty}\sum_{k=0}^{i}\binom{i}{k}\binom{L}{k}_{2,r}\left(1-e^{-x}\right)^i$$
$$=\sum_{i=0}^{\infty}\sum_{k=0}^{i}(-1)^i\,i!\binom{i}{k}\binom{L}{k}_{2,r}\sum_{n=i}^{\infty}(-1)^n S_2(n,i)\frac{x^n}{n!}$$
$$(30) \qquad =\sum_{n=0}^{\infty}\sum_{i=0}^{n}\sum_{k=0}^{i}(-1)^i\,i!\binom{i}{k}\binom{L}{k}_{2,r}(-1)^n S_2(n,i)\frac{x^n}{n!}.$$

Thus, by (28), comparing coefficients in (29) and (30), the proof is obtained. □

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References

[1] M. Abouzahra and L. Lewin, *The polylogarithm in algebraic number fields*, J. Number Theory **21** (1985), no. 2, 214–244.
[2] A. Adelberg, *Kummer congruences for universal Bernoulli numbers and related congruences for poly-Bernouylli numbers*, Int. Math. J. **1** (2002), no. 1, 53–63.
[3] G. E. Andrews and J. Baxter, *Lattice gas generalization of the hard hexagon model III q-trinomials coefficients*, J. Stat. Phys. **47** (1987), 297–330.
[4] N. Batır and A. Sofo, *A unified treatment of certain classes of combinatorial identities*, J. Integer Seq. **24** (2021), Article 21.3.2.
[5] N. Batır, *Combinatorial identities involving harmonic numbers*, Integers **20** (2020), A25.
[6] R. C. Bollinger, *A note on Pascal T−triangles, Multinomial coefficients and Pascal pyramids*, Fibonacci Quart. **24** (1986), 140–144.
[7] K. N. Boyadzhiev, *A binomial formula for evaluating integrals*, Discrete Math. Lett. **10** (2022), 51–55.
[8] K. N. Boyadzhiev, *Harmonic number identities via Euler's transform*, J. Integer Seq. **12** (2009), Article 09.6.1.

[9] K. N. Boyadzhiev, *Power series with skew-harmonic numbers, dilogarithms, and double integrals*, Tatra Mt. Math. Publ. **56** (2013), 93–108.

[10] J. L. Cereceda, *An introduction to hyperharmonic numbers (classroom note)*, Int. J. Math. Educ. Sci. Technol. **46** (2015), no. 3, 461–469.

[11] J. Choi and H. M. Srivastava, *Some summation formulas involving harmonic numbers and generalized harmonic numbers*, Math. Comput. Model. **54** (2011), 2220–2234.

[12] M. C. Dağlı, *A new generalization of hyperharmonic numbers*, Ann. Pol. Math. **129** (2022), 17–24.

[13] M. C. Dağlı and F. Qi, *Several recurrence relations and identities on generalized derangement numbers*, Results Nonlinear Anal. **5** (2022), no. 2, 185–190.

[14] R. Frontczak, *Harmonic Sums via Euler's Transform: Complementing the Approach of Boyadzhiev*, J. Integer Seq. **23** (2020), Article 20.3.2.

[15] R. Frontczak, *Binomial sums with skew-harmonic numbers*, Palest. J. Math. **10** (2021), no. 2, 756–763.

[16] M. Gencev, *Binomial sums involving harmonic numbers*, Math. Slovaca **61** (2011), no. 2, 215–226.

[17] D. Guo and W. Chu, *Summation formulae involving multiple harmonic numbers*, Appl. Anal. Discrete Math. **15** (2021), no. 1, 201–212.

[18] E. Kılıc, Y. T. Ulutas, and N. Ömür, *Sums of products of the terms of the generalized Lucas sequence V_{kn}*, Hacet. J. Math. Stat. **40** (2011), no. 2, 147–161.

[19] E. Kılıc, N. Ömür and Y.T.Ulutas, *Some finite sums involving generalized Fibonacci and Lucas numbers*, Discrete Dyn. Nat. Soc. **284261** (2011), 1–11.

[20] S. Koparal, N. Ömür, and L. Elkhiri, *On generalized hyperharmonic numbers of order r, $H^r_{n,m}(\sigma)$*,, Notes on Number Theory and Discrete Mathematics **29** (2023), no. 4, 804–812.

[21] A. Messahel, L. Elkhiri, and M. Mihoubi, *The r-bi^{q}nomial coefficient and some properties*, in submitted.

[22] E. Munarini, *Riordan matrices and sums of harmonic numbers*, Appl. Anal. Discrete Math. **5** (2011), 176–200.

[23] N. Ömür and S. Koparal, *On the matrices with the generalized hyperharmonic numbers of order r*, Asian-Eur. J. Math. , **11** (2018), no. 3, 1850045.

[24] N. Ömür and G. Bilgin, *Some applications of the generalized hyperharmonic numbers of order r, $H^r_n(\alpha)$*,, Adv. Appl. Math. Sci. **17** (2018), no. 9, 617–627.

[25] N. Ömür, S. Koparal, and L. Elkhiri, *On sums with generalized harmonic numbers via Euler's Transform*, Notes Numb. Thy. Disc. Math. **29** (2023), no. 4, 695–704.

[26] C. Smith and V. E. Hogatt, *Generating functions of central values of generalized Pascal triangles*, Fibonacci Quart. **17** (1979), 58–67.

[27] J. Spieß, *Some identities involving harmonic numbers*, Math. Comp. **55** (1990), no. 132, 839–863.

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