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CERTAIN INTEGRAL TRANSFORMS OF EXTENDED BESSEL-MAITLAND FUNCTION ASSOCIATED WITH BETA FUNCTION

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Abstract. This paper deals with a new extension of the generalized Bessel-Maitland function (EGBMF) associated with the beta function. We evaluated integral representations, recurrence relation and integral transforms such as Mellin transform, Laplace transform, Euler transform, K-transform and Whittaker transform. Furthermore, the Riemann-Liouville fractional integrals are also discussed.

1. Introduction

The special functions of mathematical sciences have been discovered to be of great assistance in solving the initial value problems (IVP) and the boundary value problems (BVP) that are connected to partial differential equations and fractional differential equations respectively. Many researchers in the field of integral transformation have produced a sizable number of papers. Some of them have researched several integral transforms involving various special functions of mathematical physics; these transforms are crucial in a wide range of physics and engineering domains. Mathematicians have become far more interested in studying certain special functions, such as Bessel functions, generalized Bessel functions, Bessel-Maitland functions, Mittag-Leffler functions, Whittaker functions, and so forth, because of the growing applications of integral transform and special functions together in the fields of applied mathematics, mathematical physics, and engineering.

Differential equation theory and the science of the Bessel function are closely related fields of study. Watson [27] provides an in-depth description of the Bessel function's use. In recent years, numerous integral transforms involving a variety of special functions of mathematical physics have been established and devolved by many researchers ([1],[2],[3],[5],[6],[7],[9],[10],[11],[12],[13],[15],[17],

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[23],[26]). Integral transforms involving Bessel-Maitland function play a crucial role in many problems of physics and applied mathematics. Due to the importance of such types of transforms, we developed a new class of (p,q) beta-type integral operators that include generalized Bessel-Maitland function and obtained certain integral representations, recurrence relation, Mellin transform, Laplace transform, Euler transform, K-transform and Whittaker transform involving Bessel-Maitland function. We have mentioned several previously discovered as well as newly discovered results as a major finding of our results. For the seek of this investigation, let's begin by reviewing the definition of well-known functions as generalized Bessel function and Bessel-Maitland function.

Edward Maitland Wright [29] introduced Bessel function which is generalized to Bessel-Maitland function. It is given by

$$J_s^r(z) = \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma(rn+s+1)n!}, \ \Re(r) > 0, \Re(s) > -1, z \in \mathbb{C}. \tag{1}$$

Applied sciences, engineering, biological, chemical, and physical sciences are among the fields in which the Bessel-Maitland function is used [27]. M. Singh [23] further explored and analyzed the generalized Bessel-Maitland function, which is defined as

$$J_{s,q}^{r,\delta}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{qn}(-z)^n}{\Gamma(rn+s+1)n!}, \tag{2}$$

where $s, r, \delta \in \mathbb{C}; \Re(r) \geq 0, \Re(s) \geq -1, \Re(\delta) \geq 0$ and $q \in \mathbb{N} \cup (0, 1)$.

By selecting the specific values of the parameters r, s, δ, q, we were able to disclose a few unique connections of the generalized Bessel-Maitland function given by equation (2) in terms of the other special functions:

If s is interchanged by $s+\alpha$ and put $q = 1, \delta = 1$ and also replace z by $\frac{z^2}{4}$ in (2), we have

$$J_{s+\alpha,1}^{r,1}\left(\frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^{-s-2\alpha} \sum_{n=0}^{\infty} \Gamma(\alpha+n+1) J_{s,\alpha}^r(z), \tag{3}$$

where $J_{s,\alpha}^r(z)$ represent the Bessel-Maitland function given by Pathak [19]. Also, if we take $r = 1$ and $\alpha = \frac{1}{2}$ in (3), we get

$$J_{s+\frac{1}{2},1}^{1,1}\left(\frac{z^2}{4}\right) = \left(\frac{z}{2}\right)^{-s-1} \sum_{n=0}^{\infty} \Gamma(\frac{3}{2}+n) H_s(z), \tag{4}$$

where $H_s(z)$ represents the Struve's function [27].

If $q = 0$, (2) reduces to

$$J_{s,0}^{r,\delta}(z) = J_s^r(z), \tag{5}$$

where $J_s^r(z)$ is the generalized Bessel function given (1).

If $q=1, r=1$, s is interchanged by $s-1$ and z is interchanged by $-z$, (2) becomes

$$J^{1,\delta}_{s-1,1}(-z)=\frac{1}{\Gamma(s)}\phi(\delta,s,z), \tag{6}$$

where $\phi(\delta,s,z)$ is called as the confluent hypergeometric function [25].

If z is interchanged by $\frac{z^2}{4}$ and put $q=0$ and $r=1$, (2) reduces to

$$J^{1,\delta}_{s,0}\left(\frac{z^2}{4}\right)=\left(\frac{z}{2}\right)^{-s}J_s(z), \tag{7}$$

where $J_s(z)$ is the ordinary Bessel function [21].
If s is interchanged by $s-1$ and z is interchanged by $-z$, (2) becomes generalized Mittag-Leffler function [22]

$$J^{r,\delta}_{s-1,q}(-z)=E^{\delta,q}_{r,s}(z), \tag{8}$$

where $r,s,\delta\in\mathbb{C},\Re(r)>0,\Re(s)>0,\Re(\delta)>0,q\in(0,1)\cup\mathbb{N}$.
If s is interchanged by $s-1$, z is interchanged by $-z$ and $q=1$, (2) reduces to

$$J^{r,\delta}_{s-1,1}(-z)=E^{\delta}_{r,s}(z). \tag{9}$$

If s is interchanged by $s-1$, z is interchanged by $-z$ and $q=1,\delta=1$, (2) reduces to

$$J^{r,1}_{s-1,1}(-z)=E_{r,s}(z),$$

where $r,s\in\mathbb{C},\Re(r)>0$ and $E_{r,s}(z)$ given by Wiman [28].

If z is interchanged by $-z$ and $s=0, q=1=\delta$, (2) reduces to

$$J^{r,1}_{0,1}(-z)=E_r(z), \tag{10}$$

where $r\in\mathbb{C},\Re(r)>0$ and $E_r(z)$ is the Mittag-Leffler function named after the great Swedish mathematician Gösta Magnus Mittag-Leffler (1846–1927).

If $r=k\in\mathbb{N}$ and $q\in\mathbb{N}$, (2) can be written as

$$J^{k,\delta}_{s,q}(z)=\frac{1}{\Gamma(s+1)}{}_qF_k\left[\begin{array}{cc}\triangle(q,\delta); & \\ & ;\ \dfrac{-q^q z}{k^k}\\ \triangle(k,s+1); & \end{array}\right], \tag{11}$$

where $J^{k,\delta}_{s,q}(z)$ is another representation of the Bessel-Maitland function given in [23], ${}_qF_k(.)$ is the generalized hypergeometric function and the symbol $\triangle(q,\delta)$ is a q tuple given by $\frac{\delta}{k},\frac{\delta+1}{k},......,\frac{\delta+q-1}{k}$.

In 1997, Chaudhry et al. [20] give the extension of the Euler-Beta function. In 2018, Shadab et al. [8] further generalized the Euler-Beta function in terms of Mittag-Leffler function. Recently, Khan et al. [14] also introduced an extension of the beta function defined as

$$B^{\alpha,\beta,p}_{\mu,\nu}(x,y)=\int_0^1 u^{x-1}(1-u)^{y-1}E_{\alpha,\beta}\left(-\frac{p}{u^\mu(1-u)^\nu}\right)du, \tag{12}$$

$$\left(\Re(x),\Re(y)>0;\Re(p)\geq 0;\mu,\nu,\beta,\delta\in\mathbb{R}^+\right).$$

Motivated by the above mention work in the present paper, we unify and generalize Bessel-Maitland function $J_{s,q}^{r,\delta}(z)$ in terms of extended beta function and investigate some useful properties such as integral representations and integral transform. we also acquire the Riemann-Liouville fractional integral. The generalized Bessel-Maitland function $J_{s,q}^{r,\delta}(z)$ are expressed in terms of Struve's function, Mittag-Leffler function,Fox-H function, confluent hypergeometric function and hypergeometric function etc. Therefore, the result presented in this paper are easily converted in terms of similar type of new and interesting result with different arguments after some suitable parametric replacement.

The generalized Bessel-Maitland function associated with extended beta function defined as :

$$J_{s,q}^{r,\delta;l}(z;p,\alpha) = \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-z)^n}{B(\delta, l-\delta)\Gamma(rn+s+1)n!}, \tag{13}$$

$$(\Re(l) > \Re(\delta) > 0; p > 0, q \in \mathbb{N}).$$

The Pochhammer symbol $(l)_{qn}$ is expressed as

$$(l)_{qn} = \frac{\Gamma(l+qn)}{\Gamma(l)},$$

and a new generalization of the beta function defined by

$$B_{p,q}^{\alpha}(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1} exp\left(\frac{-p}{u^{\alpha}} - \frac{q}{(1-u)^{\alpha}}\right) du, \tag{14}$$

where $(\Re(p) > 0, \Re(q) > 0, \Re(x) > 0, \Re(y) > 0)$.

For $p = 0, q = 0$, we get beta function [4]

$$B(y,z) = \int_0^1 p^{y-1}(1-p)^{z-1} dp = \frac{\Gamma(y)\Gamma(z)}{\Gamma(y+z)}. \tag{15}$$

The Mellin transform [24] of the function $f(z)$ is defined as

$$M[f(z);\eta] = \int_0^{\infty} u^{\eta-1} f(u) du = f^*(\eta),\ \Re(\eta) > 0. \tag{16}$$

The Laplace transform of a function $f(z)$ is defined by

$$\mathfrak{L}\{f(z);\eta\} = \int_0^{\infty} e^{-\eta t} f(t) dt,\ \Re(\eta) > 0 \in \mathbb{C}. \tag{17}$$

The Euler-Beta [24] transform of the function $f(z)$ is defined as

$$B[f(z) : c, d] = \int_0^1 v^{c-1}(1-v)^{d-1} f(v) dv. \tag{18}$$

Riemann-Liouville fractional integral operators are given by [16]

$$(I_{0+}^{\mu} f)(x) = \frac{1}{\Gamma(\mu)} \int_0^x \frac{f(u)}{(x-t)^{1-\mu}} du \tag{19}$$

$$(I_{-}^{\mu} f)(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty \frac{f(u)}{(t-x)^{1-\mu}} du, \tag{20}$$

which are the left and right sided operators respectively.

2. Characterization of EGBMF as integral

In this section, we express an integral representation of the EGBMF followed by recurrence relation.

Theorem 2.1. *Let, $p > 0, q \in \mathbb{N}, \Re(l) > \Re(\delta) > 0, \Re(s) > 0, \Re(r) > -1$, the EGBMF has the following integral representation*

$$\begin{aligned} &J_{s,q}^{r,\delta;l}(z;p,\alpha) \\ &= \frac{1}{B(\delta, l-\delta)} \int_0^1 u^{\delta-1}(1-u)^{l-\delta-1} exp\left(\frac{-p}{u^\alpha} - \frac{q}{(1-u)^\alpha}\right) J_{s,q}^{r,l}(u^q z) du. \end{aligned} \tag{21}$$

Proof. Using (14) in (13), we acquire

$$\begin{aligned} &J_{s,q}^{r,\delta;l}(z;p,\alpha) \\ &= \sum_{n=0}^{\infty} \int_0^1 u^{\delta+qn-1}(1-u)^{l-\delta-1} exp\left(\frac{-p}{u^\alpha} - \frac{q}{(1-u)^\alpha}\right) du \frac{(l)_{qn}(-z)^n}{B(\delta, l-\delta)\Gamma(rn+s+1)n!}. \end{aligned}$$

Reciprocate the order of summation and integration, that is sure under the presumption given in the description of theorem, we acquire

$$\begin{aligned} &J_{s,q}^{r,\delta;l}(z;p,\alpha) \\ &= \frac{1}{B(\delta, l-\delta)} \int_0^1 u^{\delta-1}(1-u)^{l-\delta-1} exp\left(\frac{-p}{u^\alpha} - \frac{q}{(1-u)^\alpha}\right) \sum_{n=0}^{\infty} \frac{(l)_{qn}(-u^q z)^n}{\Gamma(rn+s+1)n!} du. \end{aligned}$$

employing (2), we get desired result. □

Corollary 2.2. *Consider $u = \frac{t}{1+t}$ in Theorem 2.1, we acquire*

$$\begin{aligned} J_{s,q}^{r,\delta;l}(z;p,\alpha) = \frac{1}{B(\delta, l-\delta)} \int_0^\infty \frac{t^{\delta-1}}{(1+t)^l} \\ exp\left\{-(1+t)^\alpha\left(\frac{p}{t^\alpha} + q\right)\right\} J_{s,q}^{r,l}\left(\frac{t^q z}{(1+t)^q}\right) dt. \end{aligned} \tag{22}$$

Corollary 2.3. *Consider $u = \sin^2\theta$ in Theorem 2.1, we acquire*

(23)

$$J_{s,q}^{r,\delta;l}(z;p,\alpha) = \frac{2}{B(\delta, l-\delta)} \int_0^{\frac{\pi}{2}} (\sin\theta)^{2\delta-1} (\cos\theta)^{2l-2\delta-1} exp\left\{\frac{-p}{(sin\theta)^{2\alpha}} - \frac{q}{(cos\theta)^{2\alpha}}\right\} J_{s,q}^{r,l}(z\sin^{2q}\theta)d\theta.$$

Corollary 2.4. *The recurrence relation for the extended generalized Bessel-Maitland function(EGBMF) defined as*

$$J_{s,q}^{r,\delta;l}(z;p,\alpha) = (s+1)J_{s+1,q}^{r,\delta;l}(z;p,\alpha) + rz\frac{d}{dz}J_{s+1,q}^{r,\delta;l}(z;p,\alpha) \tag{24}$$

$$(\Re(l) > \Re(\delta) > 0, \Re(\mu) > 0, \Re(\nu) > -1; p > 0, q \in \mathbb{N}).$$

Proof. Taking R.H.S of (24) and employing (13), we obtain

$$(s+1)J_{s+1,q}^{r,\delta;l}(z;p,\alpha) + rz\frac{d}{dz}J_{s+1,q}^{r,\delta;l}(z;p,\alpha)$$

$$= (s+1)\sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-z)^n}{B(\delta, l-\delta)\Gamma(rn+s+2)n!} + rz\frac{d}{dz}\sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-z)^n}{B(\delta, l-\delta)\Gamma(rn+s+2)n!}$$

$$= (s+1)\sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-z)^n}{B(\delta, l-\delta)\Gamma(rn+s+2)n!} + r\sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}n(-z)^n}{B(\delta, l-\delta)\Gamma(rn+s+2)n!}$$

$$= \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-z)^n}{B(\delta, l-\delta)\Gamma(rn+s+2)n!}(rn+s+1)$$

$$= \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-z)^n}{B(\delta, l-\delta)\Gamma(rn+s+1)n!}$$

$$= J_{s,q}^{r,\delta;l}(z;p,\alpha).$$

□

3. Properties of the EGBMF with derivative

Theorem 3.1. *The subsequent derivative formula of the extended generalized Bessel-Maitland function is*

$$\frac{d^n}{dz^n}J_{s,q}^{r,\delta;l}(z;p,\alpha) = (-1)^n(l)_q(l+q)_q\cdots(l+(n-1)q)_q J_{s+nr}^{r,s+nq;l+nq}(z;p,\alpha) \tag{25}$$

$$(\Re(l) > \Re(\delta) > 0, \Re(s) > 0, \Re(r) > -1; p > 0, q \in \mathbb{N}).$$

Proof. Taking differentiation w.r.t 'z' in (13), we obtain

$$\frac{d}{dz}J_{s,q}^{r,\delta;l}(z;p,\alpha) = (-1)(l)_q J_{s+r}^{r,s+q;l+q}(z;p,\alpha). \tag{26}$$

Again taking the derivative in (26), we get

$$\frac{d^2}{dz^2} J_{s,q}^{r,\delta;l}(z;p,\alpha) = (-1)^2 (l)_q (l+q)_q J_{s+2r}^{r,s+2q;l+2q}(z;p,\alpha).$$

In a similar manner, the n-times differentiation w.r.t. 'z' completes the proof of the theorem. □

Theorem 3.2. *The extended generalized Bessel-Maitland function(EGBMF) holds the following relation for* $\Re(l) > \Re(\delta) > 0, \Re(s) > 0, \Re(r) > -1; p > 0, q \in \mathbb{N}$.

$$\frac{d^n}{dz^n}[z^s J_{s,q}^{r,\delta;l}(\lambda z^r;p,\alpha)] = z^{s-n} J_{s-n,q}^{r,\delta;l}(\lambda z^r;p,\alpha). \tag{27}$$

Proof. Replace z by λz^r in (13) and multiply with z^s, then taking its derivative w.r.t. 'z', we get

$$\begin{aligned}\frac{d}{dz}[z^s J_{s,q}^{r,\delta;l}(\lambda z^r;p,\alpha)] &= \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-\lambda)^n (rn+s) z^{rn+s-1}}{B(\delta, l-\delta)\Gamma(rn+s+1)n!} \\ &= \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-\lambda)^n z^{rn+s-1}}{B(\delta, l-\delta)\Gamma(rn+s)n!} \\ &= z^{s-1} \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-\lambda z^r)^n}{B(\delta, l-\delta)\Gamma(rn+s)n!} \\ &= z^{s-1} J_{s-1,q}^{r,\delta;l}(\lambda z^r;p,\alpha).\end{aligned}$$

Differentiating n times with respect to z completes the proof of the theorem. □

4. Integral transforms of EGBMF

In this section we find some well known integral transforms such as Mellin, Euler-Beta, Laplace, K-transform and Whittaker transform.

Theorem 4.1. *(Mellin transform) Let,* $a, r, s, \delta, l, \eta \in \mathbb{C}; \Re(r) \geq 0, \Re(s) \geq -1, \Re(l) > \Re(\delta) > 0, \Re(\eta) > 0, \Re(a) > 0$ *and* $q \in \mathbb{N}$. *The Mellin transform of EGBMF holds true*

$$\begin{aligned}&\mathfrak{M}\left\{e^{-az} J_{s,q}^{r,\delta;l}(z;p,\alpha);\eta\right\} \\ &= \frac{1}{a^{\eta} B(\delta, l-\delta)} \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-1)^n}{\Gamma(rn+s+1)n!} \frac{\Gamma(\eta+n)}{a^n}.\end{aligned} \tag{28}$$

Proof. Using (16) in EGBMF (13), we get

$$\mathfrak{M}\left(e^{-az} J_{s,q}^{r,\delta;l}(z;p,\alpha);\eta\right) = \int_0^{\infty} t^{\eta-1} e^{-at} J_{s,q}^{r,\delta;l}(t;p,\alpha)dt.$$

Now using (17), we have

$$\mathfrak{M}\left(e^{-az}J_{s,q}^{r,\delta;l}(z;p,\alpha);\eta\right)=\int_0^\infty t^{\eta-1}e^{-at}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-t)^n}{B(\delta,l-\delta)\Gamma(rn+s+1)n!}dt$$

$$=\frac{1}{B(\delta,l-\delta)}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\int_0^\infty e^{-at}t^{\eta+n-1}dt$$

$$=\frac{1}{B(\delta,l-\delta)}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\frac{\Gamma(\eta+n)}{a^{\eta+n}}$$

$$=\frac{1}{a^{\eta}B(\delta,l-\delta)}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\frac{\Gamma(\eta+n)}{a^{n}}.$$

□

Theorem 4.2. *(Laplace transform) Let, $r,s,\delta,l,\eta\in\mathbb{C};\Re(r)\geq 0,\Re(s)\geq -1,\Re(l)>\Re(\delta)>0,\Re(\eta)>0$ and $q\in\mathbb{N}$. The Laplace transform of EGBMF is specified through*

$$\mathfrak{L}\left\{J_{s,q}^{r,\delta;l}(z;p,\alpha);\eta\right\}=\frac{1}{\eta B(\delta,l-\delta)}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-1)^n}{\Gamma(rn+s+1)\eta^n}. \tag{29}$$

Proof. Using Laplace transform (16) on extended generalized Bessel-Maitland function (13), we get

$$\mathfrak{L}\left(J_{s,q}^{r,\delta;l}(z;p,\alpha);\eta\right)=\int_0^\infty e^{-\eta t}J_{s,q}^{r,\delta;l}(t;p,\alpha)dt.$$

Now using (17), we have

$$\mathfrak{L}\left(J_{s,q}^{r,\delta;l}(z;p,\alpha);\eta\right)=\int_0^\infty e^{-\eta t}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-t)^n}{B(\delta,l-\delta)\Gamma(rn+s+1)n!}dt$$

$$=\frac{1}{B(\delta,l-\delta)}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\int_0^\infty e^{-\eta t}t^{n}dt$$

$$=\frac{1}{B(\delta,l-\delta)}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\frac{\Gamma(n+1)}{\eta^{n+1}}$$

$$=\frac{1}{\eta B(\delta,l-\delta)}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\frac{n!}{\eta^{n}}$$

$$=\frac{1}{\eta B(\delta,l-\delta)}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-1)^n}{\Gamma(rn+s+1)\eta^{n}}.$$

□

Theorem 4.3. *(Euler transform) Let,* $r, s, \delta, l, s \in \mathbb{C}; \Re(r) \geq 0, \Re(s) \geq -1, \Re(l) > \Re(\delta) > 0, \Re(s) > 0$ *and* $q \in \mathbb{N}$*. The Euler transform of EGBMF is*

$$B\left\{J_{s,q}^{r,\delta;l}(z^r;p) : s+1,1\right\} = J_{s+1,q}^{r,\delta;l}(1;p). \tag{30}$$

Proof. Using Euler transform (18) in (13), we get

$$B\left\{J_{s,q}^{r,\delta;l}(z^r;p,\alpha) : s+1,1\right\}$$
$$= \int_0^1 z^{s+1-1}(1-z)^{1-1} \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-z^r)^n}{B(\delta, l-\delta)\Gamma(rn+s+1)n!} dz.$$

Changing the order of summation and integral which is guaranteed under convergence condition, we get

$$= \sum_{n=0}^{\infty} \frac{(-1)^n B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}}{B(\delta, l-\delta)\Gamma(rn+s+1)n!} \int_0^1 z^{rn+s+1-1}(1-z)^{1-1} dz.$$

Applying (15) we acquire our result. □

Theorem 4.4. *(K-Transform) Let,* $r, s, \delta, l, \in \mathbb{C}; \Re(r) \geq 0, \Re(s) \geq -1, \Re(l) > \Re(\delta) > 0, \Re(\rho) > 0, \Re(\mu \pm \lambda) > 0$ *and* $q \in \mathbb{N}$*. The K- transform of EGBMF is*

$$\begin{aligned} &\int_0^{\infty} u^{\mu-1} K_{\lambda}(\rho u) J_{s,q}^{r,\delta;l}(u;p,\alpha) du \\ &= \frac{2^{\mu-2}}{\rho^{\mu} B(\delta, l-\delta)} \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-2\rho^{-1})^n}{\Gamma(rn+s+1)n!} \Gamma(\frac{\mu+n\pm\lambda}{2}). \end{aligned} \tag{31}$$

Proof. Use (13) in the definition of the K-transform, we obtain

$$\int_0^{\infty} u^{\mu-1} K_{\lambda}(\rho u) J_{s,q}^{r,\delta;l}(u;p,\alpha) du$$
$$= \int_0^{\infty} u^{\mu-1} K_{\lambda}(\rho u) \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-u)^n}{B(\delta, l-\delta)\Gamma(rn+s+1)n!} du$$
$$= \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-1)^n}{B(\delta, l-\delta)\Gamma(rn+s+1)n!} \int_0^{\infty} u^{\mu+n-1} K_{\lambda}(\rho u) du.$$

By using $\rho u = t$, we get

$$= \frac{1}{\rho^{\mu}} \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-\rho)^{-n}}{B(\delta, l-\delta)\Gamma(rn+s+1)n!} \int_0^{\infty} t^{\mu+n-1} K_{\lambda}(t) dt.$$

Now we use the integral formula [18],

$$\int_0^{\infty} y^{\mu-1} K_x(y) dy = 2^{\mu-2} \Gamma(\frac{\mu \pm x}{2})$$

in the above expression to complete the proof of the theorem,

$$= \frac{2^{\mu-2}}{\rho^{\mu} B(\delta, l-\delta)} \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-2\rho^{-1})^n}{\Gamma(rn+s+1)n!} \Gamma(\frac{\mu+n\pm\lambda}{2}).$$

□

Theorem 4.5. *(Whittaker-Transform) Let, $r, s, \delta, l, \rho \in \mathbb{C}; \Re(r) \geq 0, \Re(s) \geq -1, \Re(l) > \Re(\delta) > 0, \Re(\rho) > 0, \Re(\mu \pm \lambda) > 0$ and $q \in \mathbb{N}$. The Whittaker-transform of EGBMF is*

$$\int_0^\infty u^{\mu-1} e^{\frac{-\rho u}{2}} W_{\lambda,\nu}(\rho u) J_{s,q}^{r,\delta;l}(u;p,\alpha)du$$

$$= \frac{1}{\rho^\mu B(\delta, l-\delta)} \sum_{n=0}^\infty \frac{B_{p,q}^\alpha(\delta+qn, l-\delta)(l)_{qn}(-\rho)^{-n}}{\Gamma(rn+s+1)n!} \frac{\Gamma(\frac{1}{2} \pm \nu + \mu + n)}{\Gamma(1-\lambda+\mu+n)}. \tag{32}$$

Proof. Use (13) in the definition of the Whittaker transform, we have
$\int_0^\infty u^{\mu-1} e^{\frac{-\rho u}{2}} W_{\lambda,\nu}(\rho u) J_{s,q}^{r,\delta;l}(u;p,\alpha)du$

$$= \int_0^\infty u^{\mu-1} e^{\frac{-\rho u}{2}} W_{\lambda,\nu}(\rho u) \sum_{n=0}^\infty \frac{B_{p,q}^\alpha(\delta+qn, l-\delta)(l)_{qn}(-u)^n}{B(\delta, l-\delta)\Gamma(rn+s+1)n!} du$$

$$= \sum_{n=0}^\infty \frac{B_{p,q}^\alpha(\delta+qn, l-\delta)(l)_{qn}(-1)^n}{B(\delta, l-\delta)\Gamma(rn+s+1)n!} \int_0^\infty u^{\mu+n-1} e^{\frac{-\rho u}{2}} W_{\lambda,\nu}(\rho u)du.$$

By using $\rho u = t$, we get

$$= \frac{1}{\rho^\mu B(\delta, l-\delta)} \sum_{n=0}^\infty \frac{B_{p,q}^\alpha(\delta+qn, l-\delta)(l)_{qn}(-\rho)^{-n}}{\Gamma(rn+s+1)n!} \int_0^\infty t^{\mu+n-1} e^{\frac{-t}{2}} W_{\lambda,\nu}(t)dt.$$

Now we use the integral formula [18],

$$\int_0^\infty y^{\mu-1} e^{\frac{-y}{2}} W_{\lambda,\nu}(y)dy = \frac{\Gamma(\frac{1}{2} \pm \nu + \mu)}{\Gamma(1-\lambda+\mu)}$$

in the above expression to complete the proof of the theorem,

$$= \frac{1}{\rho^\mu B(\delta, l-\delta)} \sum_{n=0}^\infty \frac{B_{p,q}^\alpha(\delta+qn, l-\delta)(l)_{qn}(-\rho)^{-n}}{\Gamma(rn+s+1)n!} \frac{\Gamma(\frac{1}{2} \pm \nu + \mu + n)}{\Gamma(1-\lambda+\mu+n)}.$$

□

5. Riemann-Liouville fractional integrals

Theorem 5.1. *Let, $r, s, a, \delta, l \in \mathbb{C}; \Re(r) \geq 0, \Re(s) \geq -1, \Re(a) \geq -1, \Re(l) > \Re(\delta) > 0, \Re(\mu)$ and $q \in \mathbb{N}$. The fractional integral of EGBMF is*

$$(I_{0+}^\mu [z^a J_{s,q}^{r,\delta;l}(z;p,\alpha)])(x)$$

$$= \frac{x^{\mu+a}}{\Gamma(\mu)B(\delta, l-\delta)} \sum_{n=0}^\infty \frac{B_{p,q}^\alpha(\delta+qn, l-\delta)(l)_{qn}(-x)^n}{\Gamma(rn+s+1)n!} B(a+n+1, \mu). \tag{33}$$

Proof. Applying the (19) and using the convergence condition to reverse the order of the integration and the summation, we can prove the theorem as follows.

$$
\begin{aligned}
&(I_{0+}^{\mu}[z^a J_{s,q}^{r,\delta;l}(z;p,\alpha)])(x) = \frac{1}{\Gamma(\mu)}\int_0^x (x-u)^{\mu-1}u^a J_{s,q}^{r,\delta;l}(u;p,\alpha)du \\
&= \frac{1}{\Gamma(\mu)}\int_0^x (x-u)^{\mu-1}u^a \sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-u)^n}{B(\delta,l-\delta)\Gamma(rn+s+1)n!}du \\
&= \frac{1}{\Gamma(\mu)B(\delta,l-\delta)}\int_0^x (x-u)^{\mu-1}u^a \sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-u)^n}{\Gamma(rn+s+1)n!}du \\
&= \frac{1}{\Gamma(\mu)B(\delta,l-\delta)}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\int_0^x (x-u)^{\mu-1}u^{a+n}du \\
&= \frac{x^{\mu+a}}{\Gamma(\mu)B(\delta,l-\delta)}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-x)^n}{\Gamma(rn+s+1)n!}\int_0^1 (1-t)^{\mu-1}t^{a+n}du \\
&= \frac{x^{\mu+a}}{\Gamma(\mu)B(\delta,l-\delta)}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-x)^n}{\Gamma(rn+s+1)n!}B(a+n+1,\mu).
\end{aligned}
$$

□

Theorem 5.2. *Let, $r, s, a, \delta, l \in \mathbb{C}; \Re(r) \geq 0, \Re(s) \geq -1, \Re(a) \geq -1, \Re(l) > \Re(\delta) > 0, \Re(\mu)$ and $q \in \mathbb{N}$. The fractional integral of EGBMF is*

$$
\begin{aligned}
&(I_{-}^{\mu}[z^a J_{s,q}^{r,\delta;l}(z;p,\alpha)])(x) \\
&= \frac{x^{\mu+a}}{\Gamma(\mu)B(\delta,l-\delta)}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-x^{-1})^n}{\Gamma(rn+s+1)n!}B(n-a-\mu,\mu).
\end{aligned} \tag{34}
$$

Proof. Applying (20) and using the convergence condition to reverse the order of the integration and the summation,we can prove the theorem as follows.

$$
\begin{aligned}
&(I_{-}^{\mu}[z^a J_{s,q}^{r,\delta;l}(z;p,\alpha)])(x) = \frac{1}{\Gamma(\mu)}\int_x^{\infty} (u-x)^{\mu-1}u^a J_{s,q}^{r,\delta;l}(u;p,\alpha)du \\
&= \frac{1}{\Gamma(\mu)}\int_x^{\infty} (u-x)^{\mu-1}u^a \sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-u)^n}{B(\delta,l-\delta)\Gamma(rn+s+1)n!}du \\
&= \frac{1}{\Gamma(\mu)B(\delta,l-\delta)}\int_x^{\infty} (u-x)^{\mu-1}u^a \sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-u)^n}{\Gamma(rn+s+1)n!}du \\
&= \frac{1}{\Gamma(\mu)B(\delta,l-\delta)}\sum_{n=0}^{\infty}\frac{B_{p,q}^{\alpha}(\delta+qn,l-\delta)(l)_{qn}(-1)^n}{\Gamma(rn+s+1)n!}\int_x^{\infty} (u-x)^{\mu-1}u^{a+n}du
\end{aligned}
$$

$$= \frac{x^{\mu+a}}{\Gamma(\mu)B(\delta, l-\delta)} \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(\frac{-1}{x})^n}{\Gamma(rn+s+1)n!} \int_0^1 (1-t)^{\mu-1} t^{n-a-\mu-1} du$$

$$= \frac{x^{\mu+a}}{\Gamma(\mu)B(\delta, l-\delta)} \sum_{n=0}^{\infty} \frac{B_{p,q}^{\alpha}(\delta+qn, l-\delta)(l)_{qn}(-x^{-1})^n}{\Gamma(rn+s+1)n!} B(n-a-\mu, \mu).$$

□

6. Conclusion

It is worth stressing that the generalized Bessel-Maitland function expressed in terms of beta function and the results computed are amenable for further generalization and future investigation. Motivated by numerous applications in various field of sciences such as physics, chemistry, biology, engineering, and applied science, in this paper we used beta function (14) to define a new extension of generalized Bessel-Maitland function and discussed its properties such as integral representation, recurrence relation, integral transform and fractional integral. We have attempted to exploited the closed connections of generalized Bessel-Maitland function with several important special functions such as Struve's function, Mittag-Leffler function, Fox-H function, confluent hypergeometric function and hypergeometric function etc.

Further, we mentioned here another generalization of beta function as

$$\begin{aligned} &B_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau,\alpha,\beta,p,q}(x,y) \\ &= \int_0^1 t^{x-1}(1-t)^{y-1}\ E_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau}\left(\frac{-p}{t^{\alpha}}\right)\ E_{\sigma_1,\lambda_1,\sigma_2,\lambda_2}^{u,v,\tau}\left(\frac{-q}{(1-t)^{\beta}}\right)\ dt. \end{aligned} \tag{35}$$

With the help of (35), researchers can easily obtain various new and interesting results involving Struve's function, Mittag-Leffler function, Fox-H function, confluent hypergeometric function and hypergeometric function etc. with different arguments after some suitable parametric replacements.

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