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$\mathscr{I}$-CONVERGENCE OF DOUBLE SEQUENCES IN NEUTROSOPHIC 2-NORMED SPACES

Nesar Hossain

Abstract. In this paper, we study the notion of $\mathscr{I}$-convergence of double sequences in neutrosophic 2-normed spaces which is more generalized version of statistical convergence of double sequences. Also we define $\mathscr{I}_2$-Cauchy sequence and discuss on $\mathscr{I}_2$-completeness with regards to neutrosophic 2-norm.

1. Introduction

As a generalization of ordinary convergence of sequences of real numbers, the idea of statistical convergence was first introduced independently by Fast [18], Steinhaus [58] and Schoenberg [55]. Some recent relevant studies on statistical convergence and applications may be referred to attract a wider audience as [8, 30, 33, 42, 43, 44]. One of its interesting generalization is $\mathscr{I}$-convergence introduced by Kostyrko et al. [39] where $\mathscr{I}$ is an ideal of subsets of the set of natural numbers. Since then this concept is being nurtured as several applications in different settings by various researchers like [3, 4, 16, 14, 17, 24, 29, 32, 34, 48, 50, 54].

After the introduction of fuzzy set theory by Zadeh [61], there has been extensive effort to find applications and fuzzy analogues of the classical theories and it is being applied in various branches of engineering and science, namely [5, 19, 22, 31, 40]. The reader can refer to the recent monographs [6] and [7] on certain developments of the spaces of double sequences and usage of four dimensional triangle matrices, and classical sets of fuzzy valued sequences, and related topics. Later on, the notion of fuzzy set theory has been developed effectively and generalized into new notion as its extension like intuitionistic fuzzy set [1], interval valued fuzzy set [60], interval valued intuitionistic fuzzy set [2], vague fuzzy set [11]. As a generalization of crisp set, fuzzy set, intuitionistic fuzzy set, Pythagorean fuzzy set, Smarandache [57] studied the concept of neutrosophic set. Later on, Bera and Mahapatra introduced the notion of

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neutrosophic soft linear space [9] and neutrosophic soft normed linear space [10].

In 2020, Kirişci and Şimşek [35] defined neutrosophic normed space and studied the notion of statistical convergence. Granados and Dhital [23] introduced the idea of statistical convergence of double sequences and Kişi [36] discussed on ideal convergence for single sequences in the same space. Recently, Murtaza et al. [49] defined neutrosophic 2-normed space and studied the notion of statistical convergence. In this paper, we study the concept of $\mathscr{I}$-convergence of double sequences and $\mathscr{I}_2$-completeness with respect to neutrosophic 2-norm.

2. Preliminaries

Throughout the paper $\mathbb{N}$ and $\mathbb{R}$ indicate the set of natural numbers and the set of reals respectively. $|A|$ denotes the number of elements of the set A. First we recall some basic definitions and notations which will be useful in the sequal.

Definition 2.1. [46] *Let $\mathscr{K} \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $\mathscr{K}(m,n)$ be the numbers of (j,k) in $\mathscr{K}$ such that $j \leq m$ and $k \leq n$. Then the two-dimensional analogue of natural density can be defined as follows.*
The lower asymptotic density of the set $\mathscr{K} \subseteq \mathbb{N} \times \mathbb{N}$ is defined as

$$\underline{\delta_2}(\mathscr{K}) = \liminf_{m,n} \frac{\mathscr{K}(m,n)}{mn}$$

. In case the sequence $(\frac{\mathscr{K}(m,n)}{mn})$ has a limit in Pringsheim's sense then we say that $\mathscr{K}$ has a double natural density and is defined as $\lim_{m,n} \frac{\mathscr{K}(m,n)}{mn} = \delta_2(\mathscr{K})$.

Example 2.2. [46] *Let $\mathscr{K} = \{(i^2, j^2) : i,j \in \mathbb{N}\}$. Then $\delta_2(\mathscr{K}) = \lim_{m,n} \frac{\mathscr{K}(m,n)}{mn} \leq \lim_{m,n} \frac{\sqrt{m}\sqrt{n}}{mn} = 0$ i.e. the set $\mathscr{K}$ has double natural density zero, while the set $\{(i,2j) : i,j \in \mathbb{N}\}$ has double natural density $\frac{1}{2}$.*

Note that, if we set $m = n$, we have a two-dimensional natural density due to Christopher [12].

Definition 2.3. [46] *A real double sequence $\{l_{mn}\}$ is said to be statistically convergent to the number ξ if for each $\varepsilon > 0$, the set $\{(m,n), m \leq i, n \leq j : |l_{mn} - \xi| \geq \varepsilon\}$ has double natural density zero.*

Definition 2.4. [39] *A family $\mathscr{I}$ of subsets of a non empty set $\mathscr{X}$ is said to be an ideal in $\mathscr{X}$ if the following conditions hold:*

1. $\emptyset \in \mathscr{I}$;
2. $\mathscr{A}, \mathscr{B} \in \mathscr{I}$ implies $\mathscr{A} \cup \mathscr{B} \in \mathscr{I}$;
3. $\mathscr{A} \in \mathscr{I}$ and $\mathscr{B} \subset \mathscr{A}$ implies $\mathscr{B} \in \mathscr{I}$.

An ideal $\mathscr{I}$ is called non trivial if $\mathscr{X} \notin \mathscr{I}$ and $\mathscr{I} \neq \emptyset$.

Definition 2.5. [39] *A non trivial ideal* $\mathscr{I} \subset 2^{\mathscr{X}}$ *is called admissible if* $\{\{x\} : x \in \mathscr{X}\} \subset \mathscr{I}$.

Definition 2.6. [39] *A non empty family* $\mathscr{F}$ *of subsets of a non empty set* $\mathscr{X}$ *is called a filter in* $\mathscr{X}$ *if the following properties hold:*

1. $\emptyset \notin \mathscr{F}$;
2. $\mathscr{A}, \mathscr{B} \in \mathscr{F}$ *implies* $\mathscr{A} \cap \mathscr{B} \in \mathscr{F}$;
3. $\mathscr{A} \in \mathscr{F}$ and $\mathscr{A} \subset \mathscr{B}$ *implies* $\mathscr{B} \in \mathscr{F}$.

If $\mathscr{I} \subset 2^{\mathscr{X}}$ is a non trivial ideal then the class $\mathscr{F}(\mathscr{I}) = \{\mathscr{X} \setminus \mathscr{A} : \mathscr{A} \in \mathscr{I}\}$ is a filter on $\mathscr{X}$ which is called filter associated with the ideal $\mathscr{I}$ [39].

Definition 2.7. [39] *An admissible ideal* $\mathscr{I} \subset 2^{\mathbb{N}}$ *is said to satisfy the condition* (AP) *if for every countable family of mutually disjoint sets* $\{\mathscr{A}_1, \mathscr{A}_2, \ldots\}$ *belonging to* $\mathscr{I}$ *there exists a countable family of sets* $\{\mathscr{B}_1, \mathscr{B}_2, \ldots\}$ *such that the symmetric difference* $\mathscr{A}_i \triangle \mathscr{B}_i$ *is finite for each* $i \in \mathbb{N}$ *and* $\bigcup_{i=1}^{\infty} \mathscr{B}_i \in \mathscr{I}$.

Definition 2.8. [13] *A non trivial ideal* $\mathscr{I}_2$ *of* $\mathbb{N} \times \mathbb{N}$ *is said to be strongly admissible if* $\{i\} \times \mathbb{N}$ *and* $\mathbb{N} \times \{i\}$ *belong to* $\mathscr{I}_2$ *for each* $i \in \mathbb{N}$.

It is clear that a strongly admissible ideal is also admissible. Throughout the discussion $\mathscr{I}_2$ stands for an admissible ideal of $\mathbb{N} \times \mathbb{N}$ unless otherwise stated.

Definition 2.9. *(see* [13]*) A double sequence* $\{x_{mn}\}$ *of real numbers is said to be* $\mathscr{I}_2$*-convergent to* $\xi \in \mathbb{R}$ *if for every* $\varepsilon > 0$, *the set* $\{(m, n) \in \mathbb{N} \times \mathbb{N} : |x_{mn} - \xi| \geq \varepsilon\} \in \mathscr{I}_2$.

Remark 2.10. *(see* [13]*)* (a) *If we take* $\mathscr{I}_2 = \mathscr{I}_2^0$, *where* $\mathscr{I}_2^0 = \{A \subset \mathbb{N} \times \mathbb{N} : \exists\, m(A) \in \mathbb{N} : i, j \geq m(A) \implies (i, j) \notin A\}$, *then* $\mathscr{I}_2^0$ *will be a non trivial strongly admissible ideal and clearly an ideal* $\mathscr{I}_2$ *is strongly admissible if and only if* $\mathscr{I}_2^0 \subset \mathscr{I}_2$. *In this case* $\mathscr{I}_2$*-convergence coincides with ordinary convergence of double sequences of real numbers.*
(b) *If we take* $\mathscr{I}_2 = \mathscr{I}_2^\delta$, *where* $\mathscr{I}_2^\delta = \{A \subset \mathbb{N} \times \mathbb{N} : \delta_2(A) = 0\}$, *then* $\mathscr{I}_2^\delta$*-convergence becomes statistical convergence of double sequences of real numbers.*

Definition 2.11. *(see* [13]*) An admissible ideal* $\mathscr{I}_2 \subset 2^{\mathbb{N} \times \mathbb{N}}$ *is said to satisfy the condition* $(AP2)$ *if for every countable family of mutually disjoint sets* $\{\mathscr{A}_1, \mathscr{A}_2, \ldots\}$ *belonging to* $\mathscr{I}$, *there exists a countable family of sets* $\{\mathscr{B}_1, \mathscr{B}_2, \ldots\}$ *such that the symmetric difference* $\mathscr{A}_i \triangle \mathscr{B}_i \in \mathscr{I}_2^0$ *i.e.* $\mathscr{A}_i \triangle \mathscr{B}_i$ *is included in the finite union of rows and columns in* $\mathbb{N} \times \mathbb{N}$ *for each* $i \in \mathbb{N}$ *and* $\bigcup_{i=1}^{\infty} \mathscr{B}_i \in \mathscr{I}_2$ *(hence* $\mathscr{B}_i \in \mathscr{I}_2$ *for each* $i \in \mathbb{N}$*).*

The notion of linear 2-normed space was introduced by Gähler in 1960 and later on, this idea has been developed in different manners [20, 25, 26, 28].

Definition 2.12. [21] *Let* $\mathscr{Z}$ *be a real vector space of dimension* d, *where* $2 \leq d < \infty$. *A 2-norm on* $\mathscr{Z}$ *is a function* $\|.,.\| : \mathscr{Z} \times \mathscr{Z} \to \mathbb{R}$ *which satisfies the following conditions:*

1. $\|x, y\| = 0$ *if and only if* x *and* y *are linearly dependent in* $\mathscr{X}$*;*
2. $\|x, y\| = \|y, x\|$ *for all* x, y *in* $\mathscr{X}$*;*
3. $\|\alpha x, y\| = |\alpha|\,\|x, y\|$ *for all* α *in* $\mathbb{R}$ *and for all* x, y *in* $\mathscr{X}$*;*
4. $\|x + y, z\| \leq \|x, z\| + \|y, z\|$ *for all* x, y, z *in* $\mathscr{X}$.

Example 2.13. [52] *Let* $\mathscr{X} = \mathbb{R}^2$*. Define* $\|\cdot,\cdot\|$ *on* $\mathbb{R}^2$ *by* $\|x, y\| = |x_1y_2 - x_2y_1|$*, where* $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$*. Then* $(\mathscr{X}, \|\cdot,\cdot\|)$ *is a 2-normed space.*

Definition 2.14. [53] *A double sequence* $\{u_{mn}\}$ *in a 2-normed space* $\mathscr{X}$ *is said to be* $\mathscr{I}_2$*-convergent to* $\xi \in \mathscr{X}$ *if for every* $\varepsilon > 0$ *and nonzero* $z \in \mathscr{X}$*, the set* $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|u_{mn} - \xi, z\| \geq \varepsilon\} \in \mathscr{I}_2$.

Definition 2.15. [15] *Let* $\{u_{mn}\}$ *be a double sequence in a 2-normed space* $\mathscr{X}$ *and* $\mathscr{I}_2$ *be a strongly admissible ideal of* $\mathbb{N}\times\mathbb{N}$*. Then* $\{u_{mn}\}$ *is said to be* $\mathscr{I}_2$*-Cauchy if for every* $\varepsilon > 0$ *and nonzero* $z \in \mathscr{X}$ *there exist* $m_0 = m_0(\varepsilon, z), n_0 = n_0(\varepsilon, z) \in \mathbb{N}$ *such that* $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \|u_{mn} - u_{m_0n_0} - \xi\| \geq \varepsilon\} \in \mathscr{I}_2$.

Definition 2.16. [56] *A binary operation* $\boxslash : [0, 1] \times [0, 1] \to [0, 1]$ *is named to be a continuous t-norm if the following conditions hold.*

1. $\boxslash$ *is associative and commutative;*
2. $\boxslash$ *is continuous;*
3. $x \boxslash 1 = x$ *for all* $x \in [0, 1]$*;*
4. $x \boxslash y \leq z \boxslash w$ *whenever* $x \leq z$ *and* $y \leq w$ *for each* $x, y, z, w \in [0, 1]$.

Definition 2.17. [56] *A binary operation* $\odot : [0, 1] \times [0, 1] \to [0, 1]$ *is named to be a continuous t-conorm if the following conditions are satisfied.*

1. $\odot$ *is associative and commutative;*
2. $\odot$ *is continuous;*
3. $x \odot 0 = x$ *for all* $x \in [0, 1]$*;*
4. $x \odot y \leq z \odot w$ *whenever* $x \leq z$ *and* $y \leq w$ *for each* $x, y, z, w \in [0, 1]$.

Example 2.18. [38] *The following are the examples of t-norms:*

1. $x \boxslash y = min\{x, y\}$*;*
2. $x \boxslash y = x.y$*;*
3. $x \boxslash y = max\{x + y - 1, 0\}$*. This t-norm is known as Lukasiewicz t-norm.*

Example 2.19. [38] *The following are the examples of t-conorms:*

1. $x \odot y = max\{x, y\}$*;*
2. $x \odot y = x + y - x.y$*;*
3. $x \odot y = min\{x + y, 1\}$*. This is known as Lukasiewicz t-conorm.*

Lemma 2.20. [51] *If* $\boxslash$ *is a continuous t-norm,* $\odot$ *is a continuous t-conorm,* $r_i \in (0, 1)$ *and* $1 \leq i \leq 7$*, then the following statements hold:*

1. *If* $r_1 > r_2$*, there are* $r_3, r_4 \in (0, 1)$ *such that* $r_1 \boxslash r_3 \geq r_2$ *and* $r_1 \geq r_2 \odot r_4$
2. *If* $r_5 \in (0, 1)$*, there are* $r_6, r_7 \in (0, 1)$ *such that* $r_6 \boxslash r_6 \geq r_5$ *and* $r_5 \geq r_7 \odot r_7$.

Now we recall the notion of neutrosophic 2-normed space introduced by Murtaza et al. [49].

Definition 2.21. [49] *Let $\mathscr{W}$ be a vector space, $\mathscr{N}_2 = \{<(l,z), \psi(l,z), \varphi(l,z), \vartheta(l,z) >: (l,z) \in \mathscr{W} \times \mathscr{W}\}$ be a normed space such that $\mathscr{N}_2 : \mathscr{W} \times \mathscr{W} \times \mathbb{R}^+ \to [0,1]$. Let $\boxtimes$ and $\odot$ be the continuous t-norm and continuous t-conorm respectively. Then the four tuple $\mathscr{X} = (\mathscr{W}, \mathscr{N}_2, \boxtimes, \odot)$ is named to be neutrosophic 2normed space (in short N2-NS) if for all $l, s, z \in \mathscr{W}$ and $\eta, \zeta > 0$ and for each $\beta \neq 0$, the following conditions hold:*

1. $0 \leq \psi(l,z;\eta) \leq 1$, $0 \leq \varphi(l,z;\eta) \leq 1$, $0 \leq \vartheta(l,z;\eta) \leq 1$, *for all* $\eta \in \mathbb{R}^+$;
2. $\psi(l,z;\eta) + \varphi(l,z;\eta) + \vartheta(l,z;\eta) \leq 3$, *for* $\eta \in \mathbb{R}^+$;
3. $\psi(l,z;\eta) = 1$ *(for* $\eta > 0$*) iff* $l = \theta$, *zero element in* $\mathscr{W}$;
4. $\psi(\beta l,z;\eta) = \psi(l,z;\frac{\eta}{|\beta|})$;
5. $\psi(l,z;\eta) \boxtimes \psi(s,z;\zeta) \leq \psi(l+s,z;\eta+\zeta)$;
6. $\psi(l,z;\cdot)$ *is a continuous non-decreasing function;*
7. $\lim_{\eta\to\infty} \psi(l,z;\eta) = 1$;
8. $\varphi(l,z;\eta) = 0$ *(for* $\eta > 0$*) iff* $l = \theta$;
9. $\varphi(\beta l,z;\eta) = \varphi(l,z;\frac{\eta}{|\beta|})$;
10. $\varphi(l,z;\eta) \odot \varphi(s,z;\zeta) \geq \varphi(l+s,z;\eta+\zeta)$;
11. $\varphi(l,z;\cdot)$ *is a continuous non-increasing function;*
12. $\lim_{\eta\to\infty} \varphi(l,z;\eta) = 0$;
13. $\vartheta(l,z;\eta) = 0$ *(for* $\eta > 0$*) iff* $l = \theta$;
14. $\vartheta(\beta l,z;\eta) = \vartheta(l,z;\frac{\eta}{|\beta|})$;
15. $\vartheta(l,z;\eta) \odot \vartheta(s,z;\zeta) \geq \vartheta(l+s,z;\eta+\zeta)$;
16. $\vartheta(l,z;\cdot)$ *is a continuous non-increasing function;*
17. $\lim_{\eta\to\infty} \vartheta(l,z;\eta) = 0$;
18. *If* $\eta \leq 0$, *then* $\psi(l,z;\eta) = 0$, $\varphi(l,z;\eta) = 1$, $\vartheta(l,z;\eta) = 1$.

In this case $\mathscr{N}_2 = (\psi, \varphi, \vartheta)$ *is called neutrosophic 2-norm.*

Example 2.22. [49] *Let $(\mathscr{W}, \|,.,\|)$ be a 2-normed space. Consider continuous t-norm and continuous t-conorm as $a \boxtimes b = ab$ and $a \odot b = a + b - ab$ for all $a, b \in [0,1]$ respectively. Now, for $x, y \in \mathscr{W}$ and $\eta > 0$ with $\eta > \|x,y\|$ consider*

$$\psi(x,y;\eta) = \frac{\eta}{\eta + \|x,y\|},\ \varphi(x,y;\eta) = \frac{\|x,y\|}{\eta + \|x,y\|},\ \vartheta(x,y;\eta) = \frac{\|x,y\|}{\eta}.$$

If we take $\eta \leq \|x,y\|$ then

$$\psi(x,y;\eta) = 0,\ \varphi(x,y;\eta) = 1 \text{ and } \vartheta(x,y;\eta) = 1.$$

Then $(\mathscr{W}, \mathscr{N}_2, \boxtimes, \odot)$ is a neutrosophic 2-normed space where $\mathscr{N}_2 : \mathscr{W} \times \mathscr{W} \times \mathbb{R}^+ \to [0,1]$.

Definition 2.23. [45] *A double sequence $\{l_{mn}\}$ in a N2-NS $\mathscr{X}$ is said to be convergent to $\xi \in \mathscr{X}$ with respect to $\mathscr{N}_2$ if for every $\sigma \in (0,1), \eta > 0$ there exists $n_0 \in \mathbb{N}$ such that $\psi(l_{mn} - \xi, z; \eta) > 1 - \sigma$, $\varphi(l_{mn} - \xi, z; \eta) < \sigma$ and*

$\vartheta(l_{mn}-\xi, z; \eta) < \sigma$ *for all* $m, n \geq n_0$ *and nonzero* $z \in \mathscr{X}$, *i.e.* $\lim_{m,n\to\infty} \psi(l_{mn}-\xi, z; \eta) = 1$, $\lim_{m,n\to\infty} \varphi(l_{mn}-\xi, z; \eta) = 0$ *and* $\lim_{m,n\to\infty} \vartheta(l_{mn}-\xi, z; \eta) = 0$. *In this case, we write* $\mathscr{N}_2 - \lim l_{mn} = \xi$ *or* $l_{mn} \xrightarrow{\mathscr{N}_2} \xi$.

Definition 2.24. [45] *A double sequence* $\{l_{mn}\}$ *in a N2-NS* $\mathscr{X}$ *is said to be statistically convergent to* $\xi \in \mathscr{X}$ *with respect to* $\mathscr{N}_2$ *if for every* $\sigma \in (0,1), \eta > 0$ *and nonzero* $z \in \mathscr{X}$, $\delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(l_{mn}-\xi, z; \eta) \leq 1-\sigma$ *or* $\varphi(l_{mn}-\xi, z; \eta) \geq \sigma$ *and* $\vartheta(l_{mn}-\xi, z; \eta) \geq \sigma\}) = 0$ *or equivalently* $\lim_{i,j} \frac{1}{ij}|\{m \leq i, n \leq j : \psi(l_{mn}-\xi, z; \eta) \leq 1-\sigma$ *or* $\varphi(l_{mn}-\xi, z; \eta) \geq \sigma$ *and* $\vartheta(l_{mn}-\xi, z; \eta) \geq \sigma\}| = 0$. *In this case we write* $st_2(\mathscr{N}_2) - \lim l_{mn} = \xi$ *or* $l_{mn} \xrightarrow{st_2(\mathscr{N}_2)} \xi$ *and* ξ *is called* $st_2(\mathscr{N}_2)$*-limit of* $\{l_{mn}\}$.

Definition 2.25. [45] *Let* $\{l_{mn}\}$ *be a double sequence in a N2-NS* $\mathscr{X}$, $\sigma \in (0,1)$ *and* $\eta > 0$. $\{l_{mn}\}$ *is named to be statistically Cauchy with respect to* $\mathscr{N}_2$ *if there exist* $m_0 = m_0(\sigma), n_0 = n_0(\sigma) \in \mathbb{N}$ *such that* $\delta_2(\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(l_{mn}-l_{m_0n_0}, z; \eta) \leq 1-\sigma$ *or* $\varphi(l_{mn}-l_{m_0n_0}, z; \eta) \geq \sigma$ *and* $\vartheta(l_{mn}-l_{m_0n_0}, z; \eta) \geq \sigma\}) = 0$ *for nonzero* $z \in \mathscr{X}$.

3. Main Results

Das et al. [13] first introduced the idea of $\mathscr{I}$-convergence of double sequences in metric spaces. Later on, this idea has been studied in different settings by many authors [27, 47, 59]. Mohiuddine et al. [41] studied the notion of $\mathscr{I}$-convergence of double sequences in random 2-normed spaces. Kişi [37] defined and studied $\mathscr{I}_2$-convergence in neutrosophic normed spaces. In this section, we define and study $\mathscr{I}_2$-convergence and $\mathscr{I}_2$-Cauchy sequence and prove some associated results in the line of investigations of them with respect to neutrosophic 2-norm. Throughout this section $\mathscr{X}$ stands for neutrosophic 2-normed space unless otherwise stated. First we define the following:

Definition 3.1. *Let* $\{u_{mn}\}$ *be a double sequence in a N2-NS* $\mathscr{X}$. *Then* $\{u_{mn}\}$ *is named to be* $\mathscr{I}_2$*-convergent to* $\xi \in \mathscr{X}$ *if for each* $\sigma \in (0,1)$, $\eta > 0$ *and nonzero* $z \in \mathscr{X}$, *the set* $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(u_{mn}-\xi, z; \eta) \leq 1-\sigma$ *or* $\varphi(u_{mn}-\xi, z; \eta) \geq \sigma$, $\vartheta(u_{mn}-\xi, z; \eta) \geq \sigma\} \in \mathscr{I}_2$. *In this case we write* $\mathscr{I}_2(\mathscr{N}_2) - \lim u_{mn} = \xi$ *or* $u_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} \xi$ *and* ξ *is called* $\mathscr{I}_2(\mathscr{N}_2)$*-limit of* $\{u_{mn}\}$.

Lemma 3.2. *Let* $\{u_{mn}\}$ *be a double sequence in a N2-NS* $\mathscr{X}$. *Then for every* $\sigma \in (0,1)$, $\eta > 0$ *and for nonzero* $z \in \mathscr{X}$ *the following statements are equivalent:*

1. $\mathscr{I}_2(\mathscr{N}_2) - \lim u_{mn} = \xi$;
2. $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(u_{mn}-\xi, z; \eta) \leq 1-\sigma\} \in \mathscr{I}_2$ *and* $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \varphi(u_{mn}-\xi, z; \eta) \geq \sigma\} \in \mathscr{I}_2$, $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(u_{mn}-\xi, z; \eta) \geq \sigma\} \in \mathscr{I}_2$;
3. $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(u_{mn}-\xi, z; \eta) > 1-\sigma$ *and* $\varphi(u_{mn}-\xi, z; \eta) < \sigma$, $\vartheta(u_{mn}-\xi, z; \eta) < \sigma\} \in \mathscr{F}(\mathscr{I}_2)$;

4. $\{(m,n)\in\mathbb{N}\times\mathbb{N}:\psi(u_{mn}-\xi,z;\eta)>1-\sigma\}\in\mathscr{F}(\mathscr{I}_2)$ *and* $\{(m,n)\in\mathbb{N}\times\mathbb{N}:\varphi(u_{mn}-\xi,z;\eta)<\sigma\}\in\mathscr{F}(\mathscr{I}_2)$, $\{(m,n)\in\mathbb{N}\times\mathbb{N}:\vartheta(u_{mn}-\xi,z;\eta)<\sigma\}\in\mathscr{F}(\mathscr{I}_2)$;
5. $\mathscr{I}_2(\mathscr{N}_2)-\lim\psi(u_{mn}-\xi,z;\eta)=1$ *and* $\mathscr{I}_2(\mathscr{N}_2)-\lim\varphi(u_{mn}-\xi,z;\eta)=0$, $\mathscr{I}_2(\mathscr{N}_2)-\lim\vartheta(u_{mn}-\xi,z;\eta)=0$.

Theorem 3.3. *Let $\{u_{mn}\}$ be a double sequence in a N2-NS $\mathscr{X}$. If $\mathscr{N}_2-\lim u_{mn}=\xi$ then $\mathscr{I}_2(\mathscr{N}_2)-\lim u_{mn}=\xi$.*

Proof. Suppose that $\mathscr{N}_2-\lim u_{mn}=\xi$. Then for every $\sigma\in(0,1)$ and $\eta>0$ there exists $n_0\in\mathbb{N}$ such that $\psi(u_{mn}-\xi,z;\eta)>1-\sigma$, $\varphi(u_{mn}-\xi,z;\eta)<\sigma$ and $\vartheta(u_{mn}-\xi,z;\eta)<\sigma$ for all $m,n>n_0$ and nonzero $z\in\mathscr{X}$. So, $\{(m,n)\in\mathbb{N}\times\mathbb{N}:\psi(u_{mn}-\xi,z;\eta)\leq 1-\sigma$ or $\varphi(u_{mn}-\xi,z;\eta)\geq\sigma,\ \vartheta(u_{mn}-\xi,z;\eta)\geq\sigma\}\subset\{1,2,\ldots,n_0\}\times\{1,2,\ldots,n_0\}$. Since $\mathscr{I}_2$ is an admissible ideal, $\{(m,n)\in\mathbb{N}\times\mathbb{N}:\psi(u_{mn}-\xi,z;\eta)\leq 1-\sigma$ or $\varphi(u_{mn}-\xi,z;\eta)\geq\sigma,\ \vartheta(u_{mn}-\xi,z;\eta)\geq\sigma\}\in\mathscr{I}_2$ i.e., $\mathscr{I}_2(\mathscr{N}_2)-\lim u_{mn}=\xi$. This completes the proof. □

Theorem 3.4. *Let $\{u_{mn}\}$ be a double sequence in a N2-NS $\mathscr{X}$. If $u_{mn}\xrightarrow{\mathscr{I}_2(\mathscr{N}_2)}\xi$, $\mathscr{I}_2(\mathscr{N}_2)$-limit of $\{u_{mn}\}$ is unique.*

Proof. Suppose that $u_{mn}\xrightarrow{\mathscr{I}_2(\mathscr{N}_2)}\xi$ and $u_{mn}\xrightarrow{\mathscr{I}_2(\mathscr{N}_2)}\gamma$ where $\xi\neq\gamma$. For a given $\sigma\in(0,1)$ choose $\lambda\in(0,1)$ such that $(1-\lambda)\boxtimes(1-\lambda)>1-\sigma$ and $\lambda\odot\lambda<\sigma$. Then for every $\eta>0$ and nonzero $z\in\mathscr{X}$, the sets $\{(m,n)\in\mathbb{N}\times\mathbb{N}:\psi(u_{mn}-\xi,z;\frac{\eta}{2})\leq 1-\lambda$ or $\varphi(u_{mn}-\xi,z;\frac{\eta}{2})\geq\lambda,\ \vartheta(u_{mn}-\xi,z;\frac{\eta}{2})\geq\lambda\}\in\mathscr{I}_2$ and $\{(m,n)\in\mathbb{N}\times\mathbb{N}:\psi(u_{mn}-\gamma,z;\frac{\eta}{2})\leq 1-\lambda$ or $\varphi(u_{mn}-\gamma,z;\frac{\eta}{2})\geq\lambda,\ \vartheta(u_{mn}-\gamma,z;\frac{\eta}{2})\geq\lambda\}\in\mathscr{I}_2$. Consider $\mathscr{A}_{\psi1}=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\psi(u_{mn}-\xi,z;\frac{\eta}{2})\leq 1-\lambda\}$; $\mathscr{A}_{\varphi1}=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\varphi(u_{mn}-\xi,z;\frac{\eta}{2})\geq\lambda\}$; $\mathscr{A}_{\vartheta1}=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\vartheta(u_{mn}-\xi,z;\frac{\eta}{2})\geq\lambda\}$ and $\mathscr{A}_{\psi2}=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\psi(u_{mn}-\gamma,z;\frac{\eta}{2})\leq 1-\lambda\}$; $\mathscr{A}_{\varphi2}=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\varphi(u_{mn}-\gamma,z;\frac{\eta}{2})\geq\lambda\}$; $\mathscr{A}_{\vartheta2}=\{(m,n)\in\mathbb{N}\times\mathbb{N}:\vartheta(u_{mn}-\gamma,z;\frac{\eta}{2})\geq\lambda\}$. Let $\mathscr{A}_{\psi,\varphi,\vartheta}=[\mathscr{A}_{\psi1}\cup\mathscr{A}_{\psi2}]\cap[\mathscr{A}_{\varphi1}\cup\mathscr{A}_{\varphi2}]\cap[\mathscr{A}_{\vartheta1}\cup\mathscr{A}_{\vartheta2}]$. Using Lemma3.2, we get $\mathscr{A}_{\psi,\varphi,\vartheta}\in\mathscr{I}_2$. So, let $(m,n)\in\mathscr{A}^c_{\psi,\varphi,\vartheta}$. There arise three possible cases.
<u>$Case-i$</u> : If $(m,n)\in\mathscr{A}^c_{\psi1}\cap\mathscr{A}^c_{\psi2}$ then for nonzero $z\in\mathscr{X}$ we have $\psi(\xi-\gamma,z;\eta)\geq\psi(u_{mn}-\xi,z;\frac{\eta}{2})\boxtimes\psi(u_{mn}-\gamma,z;\frac{\eta}{2})>(1-\lambda)\boxtimes(1-\lambda)>1-\sigma$. Since $\sigma>0$ is arbitrary, $\psi(\xi-\gamma,z;\eta)=1$ for every $\eta>0$. Hence $\xi=\gamma$.
<u>$Case-ii$</u> : If $(m,n)\in\mathscr{A}^c_{\varphi1}\cap\mathscr{A}^c_{\varphi2}$ then for nonzero $z\in\mathscr{X}$, we have $\varphi(\xi-\gamma,z;\eta)\leq\varphi(u_{mn}-\xi,z;\frac{\eta}{2})\odot\varphi(u_{mn}-\gamma,z;\frac{\eta}{2})<\lambda\odot\lambda<\sigma$. Since $\sigma>0$ is arbitrary, $\varphi(\xi-\gamma,z;\eta)=0$. This implies $\xi=\gamma$.
<u>$Case-iii$</u> : If $(m,n)\in\mathscr{A}^c_{\vartheta1}\cap\mathscr{A}^c_{\vartheta2}$ then for nonzero $z\in\mathscr{X}$ we have $\vartheta(\xi-\gamma,z;\eta)\leq\vartheta(u_{mn}-\xi,z;\frac{\eta}{2})\odot\vartheta(u_{mn}-\gamma,z;\frac{\eta}{2})<\lambda\odot\lambda<\sigma$. Since $\sigma>0$ is arbitrary, $\vartheta(\xi-\gamma,z;\eta)=0$. This implies $\xi=\gamma$.
Therefore, we conclude that $\mathscr{I}_2(\mathscr{N}_2)$-limit of $\{u_{mn}\}$ is unique. This completes the proof. □

Theorem 3.5. *Let $\{u_{mn}\}$ be a double sequence in a N2-NS $\mathscr{X}$. Then we have*

1. *If* $u_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} \xi$ and $w_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} \gamma$, $u_{mn} + w_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} \xi + \gamma$;
2. *If* $u_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} \xi$, $cu_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} c\xi$.

Proof. 1. For a given $\sigma \in (0,1)$ choose $\lambda \in (0,1)$ such that $(1-\lambda) \boxtimes (1-\lambda) > 1-\sigma$ and $\lambda \odot \lambda < \sigma$. Since $u_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} \xi$, for every $\eta > 0$ and nonzero $z \in \mathscr{X}$ we get

$$\mathscr{A}_{\psi 1} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(u_{mn} - \xi, z; \frac{\eta}{2}) \leq 1-\lambda\} \in \mathscr{I}_2$$
$$\mathscr{A}_{\varphi 1} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \varphi(u_{mn} - \xi, z; \frac{\eta}{2}) \geq \lambda\} \in \mathscr{I}_2$$
$$\mathscr{A}_{\vartheta 1} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(u_{mn} - \xi, z; \frac{\eta}{2}) \geq \lambda\} \in \mathscr{I}_2.$$

Again, since $w_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} \gamma$,

$$\mathscr{A}_{\psi 2} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(w_{mn} - \gamma, z; \frac{\eta}{2}) \leq 1-\lambda\} \in \mathscr{I}_2$$
$$\mathscr{A}_{\varphi 2} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \varphi(w_{mn} - \gamma, z; \frac{\eta}{2}) \geq \lambda\} \in \mathscr{I}_2$$
$$\mathscr{A}_{\vartheta 2} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \vartheta(w_{mn} - \gamma, z; \frac{\eta}{2}) \geq \lambda\} \in \mathscr{I}_2.$$

Now, let $\mathscr{A}_{\psi,\varphi,\vartheta} = [\mathscr{A}_{\psi 1} \cup \mathscr{A}_{\psi 2}] \cap [\mathscr{A}_{\varphi 1} \cup \mathscr{A}_{\varphi 2}] \cap [\mathscr{A}_{\vartheta 1} \cup \mathscr{A}_{\vartheta 2}]$. Using Lemma 3.2, we get $\mathscr{A}_{\psi,\varphi,\vartheta} \in \mathscr{I}_2$. So, let $(i,j) \in \mathscr{A}^c_{\psi,\varphi,\vartheta}$. Now, for $(i,j) \in \mathscr{A}^c_{\psi 1} \cap \mathscr{A}^c_{\psi 2}$ we have

$$\begin{aligned}\psi(u_{ij} + w_{ij} - (\xi+\gamma), z; \eta) &\geq \psi(u_{ij} - \xi, z; \frac{\eta}{2}) \boxtimes \psi(w_{ij} - \gamma, z; \frac{\eta}{2})\\ &> (1-\lambda) \boxtimes (1-\lambda)\\ &> 1-\sigma.\end{aligned}$$

Again for $(i,j) \in \mathscr{A}^c_{\varphi 1} \cap \mathscr{A}^c_{\varphi 2}$,

$$\begin{aligned}\varphi(u_{ij} + w_{ij} - (\xi+\gamma), z; \eta) &\leq \varphi(u_{ij} - \xi, z; \frac{\eta}{2}) \odot \varphi(w_{ij} - \gamma, z; \frac{\eta}{2})\\ &< \lambda \odot \lambda\\ &< \sigma.\end{aligned}$$

Similarly, for $(i,j) \in \mathscr{A}^c_{\vartheta 1} \cap \mathscr{A}^c_{\vartheta 2}$, $\vartheta(u_{ij} + w_{ij} - (\xi+\gamma), z; \eta) < \sigma$. Therefore $\{(i,j) \in \mathbb{N}\times\mathbb{N} : \psi(u_{ij} + w_{ij} - (\xi+\gamma), z; \eta) \leq 1-\sigma$ or $\varphi(u_{ij} + w_{ij} - (\xi+\gamma), z; \eta) \geq \sigma,\ \vartheta(u_{ij} + w_{ij} - (\xi+\gamma), z; \eta) \geq \sigma\} \subseteq \mathscr{A}_{\psi,\varphi,\vartheta} \in \mathscr{I}_2$ i.e., $u_{mn} + w_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} \xi + \gamma$.

2. It is obvious if $c = 0$. So, we consider $c \neq 0$. Since $u_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} \xi$, for every $\sigma \in (0,1)$, $\eta > 0$ and nonzero $z \in \mathscr{X}$, the set $\mathscr{A} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(u_{mn} - \xi, z; \eta) \leq 1-\sigma$ or $\varphi(u_{mn} - \xi, z; \eta) \geq \sigma,\ \vartheta(u_{mn} - \xi, z; \eta) \geq \sigma\} \in \mathscr{I}_2$. Now for $(m,n) \in \mathscr{A}^c$, we get $\psi(cu_{mn} - c\xi, z; \eta) = \psi(u_{mn} - \xi, z; \frac{\eta}{|c|}) \geq \psi(u_{mn} - \xi, z; \eta) \boxtimes \psi(\theta, z; \frac{\eta}{|c|} - \eta) = \psi(u_{mn} - \xi, z; \eta) \boxtimes 1 = \psi(u_{mn} -$

$\xi, z;\eta) > 1-\sigma$, $\varphi(cu_{mn}-c\xi, z;\eta) = \varphi(u_{mn}-\xi, z;\frac{\eta}{|c|}) \leq \varphi(u_{mn}-\xi, z;\eta) \odot \varphi(\theta, z;\frac{\eta}{|c|}-\eta) = \varphi(u_{mn}-\xi, z;\eta) \odot 0 = \varphi(u_{mn}-\xi, z;\eta) < \sigma$. Similarly $\vartheta(cu_{mn}-c\xi, z;\eta) < \sigma$. Therefore $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(cu_{mn}-c\xi, z;\eta) \leq 1-\sigma$ or $\varphi(cu_{mn}-c\xi, z;\eta) \geq \sigma,\ \vartheta(cu_{mn}-c\xi, z;\eta) \geq \sigma\} \subseteq \mathscr{A} \in \mathscr{I}_2$. Therefore $cu_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} c\xi$.

This completes the proof. □

Now we define $\mathscr{I}^*$-convergence of double sequences with respect to $\mathscr{N}_2$.

Definition 3.6. *Let $\{u_{mn}\}$ be a double sequence in a N2-NS $\mathscr{X}$. Then $\{u_{mn}\}$ is said to be $\mathscr{I}_2^*$-convergent to $\xi \in \mathscr{X}$ with respect to the neutrosophic 2-norm $\mathscr{N}_2$ if there exists a set $\mathscr{M} = \{m_1 < m_2 < \cdots < m_p < \cdots; n_1 < n_2 < \cdots < n_q < \cdots\} \subset \mathbb{N}\times\mathbb{N}$ such that $\mathscr{M} \in \mathscr{F}(\mathscr{I}_2)$ and $\mathscr{N}_2 - \lim u_{m_p n_q} = \xi$. In this case we write $\mathscr{I}_2^*(\mathscr{N}_2) - \lim u_{mn} = \xi$ or $u_{mn} \xrightarrow{\mathscr{I}_2^*(\mathscr{N}_2)} \xi$ and ξ is called $\mathscr{I}_2^*(\mathscr{N}_2)$-limit of $\{u_{mn}\}$.*

Theorem 3.7. *Let $\{u_{mn}\}$ be a double sequence in a N2-NS $\mathscr{X}$ and $\mathscr{I}_2$ be a strongly admissible ideal. If $u_{mn} \xrightarrow{\mathscr{I}_2^*(\mathscr{N}_2)} \xi$ then $u_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} \xi$.*

Proof. Suppose that $u_{mn} \xrightarrow{\mathscr{I}_2^*(\mathscr{N}_2)} \xi$. Then there exists a set $\mathscr{M} = \{m_1 < m_2 < \cdots < m_p < \cdots; n_1 < n_2 < \cdots < n_q < \cdots\} \subset \mathbb{N}\times\mathbb{N}$ such that $\mathscr{M} \in \mathscr{F}(\mathscr{I}_2)$ (i.e. $\mathbb{N}\times\mathbb{N} \setminus \mathscr{M} = \mathscr{A} \in \mathscr{I}_2$) and $\mathscr{N}_2 - \lim u_{m_p n_q} = \xi$. So for each $\sigma \in (0,1)$, $\eta > 0$ and nonzero $z \in \mathscr{X}$, there exists a $p_0 \in \mathbb{N}$ such that $\psi(u_{m_p n_q}-\xi, z;\eta) > 1-\sigma$, $\varphi(u_{m_p n_q}-\xi, z;\eta) < \sigma$ and $\vartheta(u_{m_p n_q}-\xi, z;\eta) < \sigma$ for all $p, q > p_0$. Hence $\{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(u_{mn}-\xi, z;\eta) \leq 1-\sigma$ or $\varphi(u_{mn}-\xi, z;\eta) \geq \sigma,\ \vartheta(u_{mn}-\xi, z;\eta) \geq \sigma\} \subset \mathscr{A} \cup (\mathscr{M} \cap (\{m_1, m_2, \ldots, m_{p_0}\} \times \mathbb{N} \cup \mathbb{N} \times \{n_1, n_2, \ldots, n_{p_0}\})) \in \mathscr{I}_2$. Therefore $u_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} \xi$. This completes the proof. □

In general, the converse of Theorem 3.7 is not true which can be shown by the following example.

Example 3.8. *Let $\mathscr{W} = \mathbb{R}^2$ and $(\mathscr{W}, \|.,.\|)$ be a 2-normed space with $\|x, y\| = |x_1y_2 - x_2y_1|$, where $x = (x_1, x_2), y = (y_1, y_2) \in \mathbb{R}^2$. Consider continuous t-norm and continuous t-conorm as $a \boxtimes b = ab$ and $a \odot b = \min\{a+b, 1\}$ for all $a, b \in [0,1]$ respectively. Now, for $x, y \in \mathscr{W}$ and $\eta > 0$ with $\eta > \|x, y\|$, consider*

$$\psi(x,y;\eta) = \frac{\eta}{\eta + \|x,y\|},\ \varphi(x,y;\eta) = \frac{\|x,y\|}{\eta + \|x,y\|},\ \vartheta(x,y;\eta) = \frac{\|x,y\|}{\eta}.$$

Then $(\mathscr{W}, \mathscr{N}_2, \boxtimes, \odot)$ is a neutrosophic 2-normed space with regards to neutrosophic 2-norm $\mathscr{N}_2 = (\psi, \varphi, \vartheta)$. Let $\mathbb{N}\times\mathbb{N} = \bigcup_{p,q} \mathscr{D}_{pq}$ be a decomposition of $\mathbb{N}\times\mathbb{N}$ such that for any $(m,n) \in \mathbb{N}\times\mathbb{N}$ each $\mathscr{D}_{pq}$ contains infinitely many $(p,q)'s$ where $p \geq m, q \geq n$ and $\mathscr{D}_{pq} \cap \mathscr{D}_{mn} = \emptyset$ where $(p,q) \neq (m,n)$. Now we *define the double sequence* $\{u_{mn}\}$ *by* $u_{mn} = \left(\frac{1}{pq}, 0\right)$ *if* $(m,n) \in \mathscr{D}_{pq}$. *Then, for* $\eta > 0$ *and nonzero* $z = (z_1, z_2) \in \mathscr{W}$ *we have*

$$\psi(u_{mn}, z; \eta) = \frac{\eta}{\eta + \|u_{mn}, z\|} \to 1$$
$$\varphi(u_{mn}, z; \eta) = \frac{\|u_{mn}, z\|}{\eta + \|u_{mn}, z\|} \to 0$$
$$\vartheta(u_{mn}, z; \eta) = \frac{\|u_{mn}, z\|}{\eta} \to 0$$

as $m, n \to \infty$. *Therefore* $\mathscr{N}_2 - \lim u_{mn} = 0$. *Since* $\mathscr{I}_2$ *is an admissible ideal,* $\mathscr{I}_2(\mathscr{N}_2) - \lim u_{mn} = 0$.

But $\{u_{mn}\}$ *is not* $\mathscr{I}_2^*$*-convergent to* $0 \in \mathscr{W}$. *If possible, let* $\mathscr{I}_2^*(\mathscr{N}_2) - \lim u_{mn} = 0$. *Then there exists a set* $\mathscr{M} = \{m_1 < m_2 < \cdots < m_s < \cdots ; n_1 < n_2 < \cdots < n_t < \cdots\} \subset \mathbb{N} \times \mathbb{N}$ *such that* $\mathscr{M} \in \mathscr{F}(\mathscr{I}_2)$ *and* $\mathscr{N}_2 - \lim u_{m_s n_t} = 0$. *Since* $\mathscr{M} \in \mathscr{F}(\mathscr{I}_2)$, *there is a set* $\mathscr{A} \in \mathscr{I}_2$ *such that* $\mathscr{M} = \mathbb{N} \times \mathbb{N} \setminus \mathscr{A}$. *Now from the definition of* $\mathscr{I}_2$, *there exist* $i, j \in \mathbb{N}$ *such that*

$$\mathscr{A} \subset (\cup_{m=1}^{i} (\cup_{n=1}^{\infty} \mathscr{D}_{mn})) \cup (\cup_{n=1}^{j} (\cup_{m=1}^{\infty} \mathscr{D}_{mn})).$$

But then $\mathscr{D}_{i+1,j+1} \subset \mathscr{M}$. *So,* $u_{m_s n_t} = \left(\frac{1}{(i+1)(j+1)}, 0\right)$ *for infinitely many* $(m_s, n_t) \in \mathscr{M}$ *which contradicts the fact* $\mathscr{N}_2 - \lim u_{m_s n_t} = 0$.

Now we see under what condition the converse of Theorem 3.7 is true.

Theorem 3.9. *Let* $\{u_{mn}\}$ *be a double sequence in a N2-NS* $\mathscr{X}$. *If* $\mathscr{I}_2$ *satisfies the condition* $(AP2)$ *then* $u_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} \xi \implies u_{mn} \xrightarrow{\mathscr{I}_2^*(\mathscr{N}_2)} \xi$.

Proof. Suppose that $\mathscr{I}_2$ satisfies the condition $(AP2)$ and $u_{mn} \xrightarrow{\mathscr{I}_2(\mathscr{N}_2)} \xi$. Then for each $\sigma \in (0,1)$,$\eta > 0$ and nonzero $z \in \mathscr{X}$ the set $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \psi(u_{mn} - \xi, z; \eta) \leq 1 - \sigma$ or $\varphi(u_{mn} - \xi, z; \eta) \geq \sigma,\ \vartheta(u_{mn} - \xi, z; \eta) \geq \sigma\} \in \mathscr{I}_2$. For $k \in \mathbb{N}$, $\eta > 0$ and nonzero $z \in \mathscr{X}$, consider $\mathscr{B}_k = \{(m,n) \in \mathbb{N} \times \mathbb{N} : 1 - \frac{1}{k} \leq \psi(u_{mn} - \xi, z; \eta) < 1 - \frac{1}{k+1}$ or $\frac{1}{k+1} < \varphi(u_{mn} - \xi, z; \eta) \leq \frac{1}{k},\ \frac{1}{k+1} < \vartheta(u_{mn} - \xi, z; \eta) \leq \frac{1}{k}\}$. Clearly $\{\mathscr{B}_1, \mathscr{B}_2, \ldots\}$ is countable and pairwise disjoint and each $\mathscr{B}_k \in \mathscr{I}_2$. Since $\mathscr{I}_2$ satisfies the condition $(AP2)$, there exists a countable family $\{\mathscr{C}_1, \mathscr{C}_2, \ldots\}$ such that the symmetric difference $\mathscr{B}_k \triangle \mathscr{C}_k$ is included in finite union of rows and columns in $\mathbb{N} \times \mathbb{N}$ for each k and $\mathscr{C} = \bigcup_{k=1}^{\infty} \mathscr{C}_k \in \mathscr{I}_2$. Now from associated filter of $\mathscr{I}_2$ there is $\mathscr{M} \in \mathscr{F}(\mathscr{I}_2)$ such that $\mathscr{M} = \mathbb{N} \times \mathbb{N} \setminus \mathscr{C}$. It is sufficient to prove that the subsequence $\{u_{mn}\}_{(m,n) \in \mathscr{M}}$ is convergent to ξ with regards to neutrosophic 2-norm $\mathscr{N}_2$. Let $\lambda \in (0,1)$, $\eta > 0$. Choose $s \in \mathbb{N}$ such that $\frac{1}{s} < \lambda$. Then $\{(m,n) \in \mathbb{N} \times \mathbb{N} : \psi(u_{mn} - \xi, z; \eta) \leq 1 - \lambda$ or $\varphi(u_{mn} - \xi, z; \eta) \geq \lambda,\ \vartheta(u_{mn} - \xi, z; \eta) \geq \lambda\} \subset \{(m,n) \in \mathbb{N} \times \mathbb{N} : \psi(u_{mn} - \xi, z; \eta) \leq 1 - \frac{1}{s}$ or $\varphi(u_{mn} - \xi, z; \eta) \geq \frac{1}{s},\ \vartheta(u_{mn} - \xi, z; \eta) \geq \frac{1}{s}\} \subset \bigcup_{k=1}^{s+1} \mathscr{B}_k$. Since $\mathscr{B}_k \triangle \mathscr{C}_k$ are included in finite union of rows and columns for $k = 1, 2, \ldots$, there exists

$(s_0, t_0) \in \mathbb{N} \times \mathbb{N}$ such that

$$\left(\bigcup_{k=1}^{s+1} \mathscr{C}_k\right) \bigcap \{m \geq s_0 \text{ and } n \geq t_0\} = \left(\bigcup_{k=1}^{s+1} \mathscr{B}_k\right) \bigcap \{m \geq s_0 \text{ and } n \geq t_0\}.$$

If $m \geq s_0,\ n \geq t_0$ and $(m, n) \notin \mathscr{C}$ then $(m, n) \notin \bigcup_{k=1}^{s+1} \mathscr{C}_k$. Hence $(m, n) \notin \bigcup_{k=1}^{s+1} \mathscr{B}_k$. Therefore for every $m \geq s_0,\ n \geq t_0$ and $(m.n) \in \mathscr{M}$ we get $\psi(u_{mn} - \xi, z; \eta) > 1 - \lambda$, $\varphi(u_{mn} - \xi, z; \eta) < \lambda$ and $\vartheta(u_{mn} - \xi, z; \eta) < \lambda$. So, $u_{mn} \xrightarrow{\mathscr{I}_2^*(\mathscr{N}_2)} \xi$. This completes the proof. □

Now we define the notion of $\mathscr{I}_2$-Cauchy sequence with regards to $\mathscr{N}_2$.

Definition 3.10. *Let $\{u_{mn}\}$ be a double sequence in a N2-NS $\mathscr{X}$ and $\sigma \in (0, 1)$ and $\eta > 0$. Then $\{u_{mn}\}$ is named to be $\mathscr{I}_2$-Cauchy with regards to neutrosophic 2-norm $\mathscr{N}_2$ if for nonzero $z \in \mathscr{X}$ there exist $n_0 = n_0(\sigma, z)$, $m_0 = m_0(\sigma, z) \in \mathbb{N}$ such that $\{(m, n) \in \mathbb{N} \times \mathbb{N} : \psi(u_{mn} - u_{m_0 n_0}, z; \eta) \leq 1 - \sigma$ or $\varphi(u_{mn} - u_{m_0 n_0}, z; \eta) \geq \sigma,\ \vartheta(u_{mn} - u_{m_0 n_0}, z; \eta) \geq \sigma\} \in \mathscr{I}_2$.*

Now we proceed with the investigations of relation between $\mathscr{I}_2$-Cauchy sequence and $\mathscr{I}_2$-convergence with respect to the neutrosophic 2-norm $\mathscr{N}_2$.

Theorem 3.11. *Let $\{u_{mn}\}$ be a double sequence in a N2-NS $\mathscr{X}$. If $\{u_{mn}\}$ is $\mathscr{I}_2$-convergent with regards to $\mathscr{N}_2$ then it is $\mathscr{I}_2$-Cauchy with regards to $\mathscr{N}_2$.*

Proof. Let $\{u_{mn}\}$ be $\mathscr{I}_2$-convergent to $\xi \in \mathscr{X}$ and $\sigma \in (0, 1)$ be given. Choose $\lambda \in (0, 1)$ such that $(1 - \lambda) \boxtimes (1 - \lambda) > 1 - \sigma$ and $\lambda \odot \lambda < \sigma$. Then for every $\eta > 0$ and nonzero $z \in \mathscr{X}$, the set $\mathscr{A} = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \psi(u_{mn} - \xi, z; \frac{\eta}{2}) \leq 1 - \lambda$ or $\varphi(u_{mn} - \xi, z; \frac{\eta}{2}) \geq \lambda,\ \vartheta(u_{mn} - \xi, z; \frac{\eta}{2}) \geq \lambda\} \in \mathscr{I}_2$. Then $\mathscr{A}^c \in \mathscr{F}(\mathscr{I}_2)$. So $\mathscr{A}^c \neq \emptyset$. Then there is $(m_0, n_0) \in \mathscr{A}^c$. Now, we define $\mathscr{B} = \{(m, n) \in \mathbb{N} \times \mathbb{N} : \psi(u_{mn} - u_{m_0 n_0}, z; \eta) \leq 1 - \sigma$ or $\varphi(u_{mn} - u_{m_0 n_0}, z; \eta) \geq \sigma,\ \vartheta(u_{mn} - u_{m_0 n_0}, z; \eta) \geq \sigma\}$. It is sufficient to prove the theorem that $\mathscr{B} \subset \mathscr{A}$. Let $(i, j) \in \mathscr{B}$. Then we get

$$\psi(u_{ij} - u_{m_0 n_0}, z; \eta) \leq 1 - \sigma \text{ or } \varphi(u_{ij} - u_{m_0 n_0}, z; \eta) \geq \sigma,\ \vartheta(u_{ij} - u_{m_0 n_0}, z; \eta) \geq \sigma.$$

$\underline{Case - i}$: We consider $\psi(u_{ij} - u_{m_0 n_0}, z; \eta) \leq 1 - \sigma$. We show $\psi(u_{ij} - \xi, z; \frac{\eta}{2}) \leq 1 - \lambda$. If possible, let $\psi(u_{ij} - \xi, z; \frac{\eta}{2}) > 1 - \lambda$. Then we have $1 - \sigma \geq \psi(u_{ij} - u_{m_0 n_0}, z; \eta) \geq \psi(u_{ij} - \xi, z; \frac{\eta}{2}) \boxtimes \psi(u_{m_0 n_0} - \xi, z; \frac{\eta}{2}) > (1 - \lambda) \boxtimes (1 - \lambda) > 1 - \sigma$, which is not possible. So, $\psi(u_{ij} - \xi, z; \frac{\eta}{2}) \leq 1 - \lambda$.
$\underline{Case - ii}$: We consider $\varphi(u_{ij} - u_{m_0 n_0}, z; \eta) \geq \sigma$. We show $\varphi(u_{ij} - \xi, z; \frac{\eta}{2}) \geq \lambda$. If possible, let $\varphi(u_{ij} - \xi, z; \frac{\eta}{2}) < \lambda$. Then we have $\sigma \leq \varphi(u_{ij} - u_{m_0 n_0}, z; \eta) \leq \varphi(u_{ij} - \xi, z; \frac{\eta}{2}) \odot \varphi(u_{m_0 n_0} - \xi, z; \frac{\eta}{2}) < \lambda \odot \lambda < \sigma$, which is not possible. So, $\varphi(u_{ij} - \xi, z; \frac{\eta}{2}) \geq \lambda$.
$\underline{Case - iii}$: If we consider $\vartheta(u_{ij} - u_{m_0 n_0}, z; \eta) \geq \sigma$ then in the line of Case-II we can show that $\vartheta(u_{ij} - \xi, z; \frac{\eta}{2}) \geq \lambda$.

Combining the above three cases we get $(i, j) \in \mathscr{A}$ i.e. $\mathscr{B} \subset \mathscr{A} \in \mathscr{I}_2$. Hence $\{u_{mn}\}$ is $\mathscr{I}_2$-Cauchy with regards to $\mathscr{N}_2$. This completes the proof. □

Theorem 3.12. *Let $\{u_{mn}\}$ be a double sequence in a N2-NS $\mathscr{X}$. If $\{u_{mn}\}$ is $\mathscr{I}_2$-Cauchy with regards to $\mathscr{N}_2$ then it is $\mathscr{I}_2$-convergent with regards to $\mathscr{N}_2$.*

Proof. Let $\{u_{mn}\}$ be $\mathscr{I}_2$-Cauchy with regards to $\mathscr{N}_2$ but it is not $\mathscr{I}_2$-convergent to $\xi \in \mathscr{X}$ with regards to $\mathscr{N}_2$. Then for $\sigma \in (0,1)$, $\eta > 0$ and nonzero $z \in \mathscr{X}$, there exist $n_0 = n_0(\sigma, z)$, $m_0 = m_0(\sigma, z) \in \mathbb{N}$ such that $\mathscr{A} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(u_{mn} - u_{m_0n_0}, z; \eta) \le 1-\sigma$ or $\varphi(u_{mn} - u_{m_0n_0}, z; \eta) \ge \sigma$, $\vartheta(u_{mn} - u_{m_0n_0}, z; \eta) \ge \sigma\} \in \mathscr{I}_2$ and $\mathscr{B} = \{(m,n) \in \mathbb{N}\times\mathbb{N} : \psi(u_{mn} - \xi, z; \frac{\eta}{2}) > 1-\sigma$ and $\varphi(u_{mn} - \xi, z; \frac{\eta}{2}) < \sigma$, $\vartheta(u_{mn} - \xi, z; \frac{\eta}{2}) < \sigma\} \in \mathscr{I}_2$. So, $\mathscr{B}^c \in \mathscr{F}(\mathscr{I}_2)$. Since $\psi(u_{mn} - u_{m_0n_0}, z; \eta) \ge 2\psi(u_{mn} - \xi, z; \frac{\eta}{2}) > 1-\sigma$ and $\varphi(u_{mn} - u_{m_0n_0}, z; \eta) \le \varphi(u_{mn} - \xi, z; \frac{\eta}{2}) < \sigma$, $\vartheta(u_{mn} - u_{m_0n_0}, z; \eta) \le \vartheta(u_{mn} - \xi, z; \frac{\eta}{2}) < \sigma$ if $\psi(u_{mn} - \xi, z; \frac{\eta}{2}) > \frac{1-\sigma}{2}$ and $\varphi(u_{mn} - \xi, z; \frac{\eta}{2}) < \frac{\eta}{2}$, $\vartheta(u_{mn} - \xi, z; \frac{\eta}{2}) < \frac{\eta}{2}$. This implies $\mathscr{A}^c \in \mathscr{I}_2$ which leads to a contradiction because $\{u_{mn}\}$ is $\mathscr{I}_2$-Cauchy with regards to $\mathscr{N}_2$. Hence $\{u_{mn}\}$ is $\mathscr{I}_2$-convergent with regards to $\mathscr{N}_2$. This completes the proof. □

Definition 3.13. *A N2-NS $\mathscr{X}$ is named to be $\mathscr{I}_2$-complete with regards to $\mathscr{N}_2$ if every $\mathscr{I}_2$-Cauchy sequence is $\mathscr{I}_2$-convergent with regards to $\mathscr{N}_2$.*

Remark 3.14. *In the light of Theorems 3.11 and 3.12, we see every neutrosophic 2-normed space is $\mathscr{I}_2$-complete.*

Conclusion and future developments

In this paper, we have dealt with $\mathscr{I}_2$ convergent sequences in N2-NS and have shown that every N2-NS is $\mathscr{I}_2$-complete. Later on, these results may be the opening of new tools to generalize this notion in various direction such as $\mathscr{I}_2$-statistical and $\mathscr{I}_2$-lacunary statistical convergence with respect to $\mathscr{N}_2$. Also, this idea can be used in the field of convergence related problems in many branches of science and engineering.

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NESAR HOSSAIN
Department of Mathematics, The University of Burdwan,
Golapbag, Burdwan-713104,
West Bengal, India.
E-mail: nesarhossain24@gmail.com