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ON A COMMON FIXED POINT THEOREM IN INTUITIONISTIC MENGER SPACE VIA C CLASS AND INVERSE C CLASS FUNCTIONS WITH CLR PROPERTY

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Abstract. The objective of this paper is to ascertain the existence and uniqueness of common fixed point for four self mappings in intuitionistic Menger metric spaces under some conditions extending to (CLR) property and C-class functions. Some illustrative examples are furnished, which demonstrate the validity of the hypotheses. As an application to our main result, we derive a common fixed point theorem for four self-mappings in metric space. Our results generalize several works, including [4], [20].

1. Introduction

Fixed point theory is a fundamental tool in nonlinear analysis, especially in the context of solving differential equations, by providing powerful tools for establishing existence, uniqueness, and stability of solutions, as well as

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for developing numerical methods for estimate the solution. Anakira's algorithm [2] for non-linear Volterra integro-differential equations and Qawaqneh's application of fixed point results to fractional problems exemplify its importance. Anakira's algorithm offers a systematic approach to solving complex equations, incorporating integral and differential terms, while Qawaqneh [18] demonstrates how fixed point theory can be applied practically to address real-world problems involving fractional derivatives. In the same context, Farraj[9] presented an algorithm for solving fractional differential equations based on fixed point theorems. These contributions underscore the versatility and significance of fixed point methods in advancing our understanding and solving challenges in nonlinear analysis and differential equations.

The metric space is a classical space where many theorems of fixed points are based on it like: Banach's Fixed Point Theorem, Brouwer's fixed point theorem, Schauder fixed point theorem, There have been a number of generalizations of metric space. In 1942, Menger [16] introduced a new generalization of metric space, called statistical metric space, where the values of distance between two points was replaced by probabilistic distributions functions.

Many years later, this new space play an important role in filed of fixed point theorems. Because of the richest of the probabilistic space with new notions, mathematicians in filed of fixed point theorems have changed the direction from metric space and b-metric space to probabilistic space and Menger space. As consequence, many new theorems of fixed point was created under light conditions, including: See. [1, 5, 7, 8, 15] and references therein.

In 1998, Jungck and Rhoades [13] introduced the notion of weakly compatible and showed that compatible maps are weakly compatible but the inverse need not be true. Sintunavarat and Kuman [23] introduce the notion of CLR_g property, the importance of this notion is that it ensures that one does not require the closedness of range of subspaces. Thus two concepts was used by Singh and Chauhan [22] who proved a common fixed point theorem for a pair of weakly compatible self mappings in non-Archimedean Menger probabilistic metric space employing common limit range property.

Recently, Imdad and Pant [12] extended the notion of common limit range property to two pairs of self mappings which further relaxes the requirement on closedness of the subspaces. Since then, a number of fixed point theorems has been established by several researchers in different settings under common limit range property. We refer the reader to [11, 6] and references therein.

In this paper, we introduce a new fixed point theorems in intuitionistic Menger space with application in metric space. Our paper is organized as following: In section 2, we introduce certain notions which we use in following sections, In section 3, we prove our main results, then, we give some examples to support our theorem, also, some corollaries have been presented.

2. Preliminaries

We give some definitions and their properties for our main results.

Definition 2.1. ([10]) A triangular norm $*$ (t-norm for short) is a binary operation on the unit interval $[0,1]$, which is commutative, associative, non-decreasing in its second component and for all $x \in [0,1]$, $x * 1 = x$.

A triangular conorm (t-conorm for short) is also a binary operation on the unit interval $[0,1]$, which is commutative, associative, non-decreasing in its second component and for all $x \in [0,1]$, $x \diamond 0 = x$.

Remark 2.2. The monotonicity of a t-norm $*$ (respectively t-conorm) in its second component is, together with the commutativity, equivalent to the (joint) monotonicity in both components, that is, to

$$x_1 * y_1 \leq x_2 * y_2 \quad \text{whenever} \quad x_1 \leq x_2 \text{ and } y_1 \leq y_2.$$

Definition 2.3. ([10]) A distribution function on $[-\infty, +\infty]$ is a function $F : [-\infty, +\infty] \to [0,1]$ which is left-continuous on $\mathbb{R}$, non-decreasing and $F(-\infty) = 0$, $F(+\infty) = 1$. We denote by Δ the family of all distribution functions on $[-\infty, +\infty]$.

Definition 2.4. ([10]) A distance distribution function $F : [-\infty, +\infty] \to [0,1]$ is a distribution function with support contained in $[0, +\infty]$.

The family of all distance distribution functions will be denoted by Δ^+. We denote $\mathcal{D}^+ = \{F | F \in \Delta^+,\ \lim_{t\to+\infty} F(t) = 1\}$.

Since any function from Δ^+ is equal 0 on $[-\infty, 0]$ we can consider the set Δ^+ consisting of non-decreasing functions F defined on $[0, +\infty]$ that satisfy $F(0) = 0$ and $F(+\infty) = 1$. Moreover, $\mathcal{D}^+$ then consists of non-decreasing functions F defined on $[0, +\infty)$ that satisfy $F(0) = 0$ and $\lim_{t\in+\infty} F(T) = 1$. The class $\mathcal{D}^+$ will play the important role in the probabilistic fixed point theorems.

H is a special element of $\mathcal{D}^+$ defined by

$$H(t) = \begin{cases} 0, & \text{if } t = 0, \\ 1, & \text{if } t > 0. \end{cases}$$

If X is a nonempty set, $F : X \times X \to \mathcal{D}^+$ is called a probabilistic distance on X and $F(x, y)$ is usually denoted by $F_{x,y}$.

Definition 2.5. ([14]) A non-distance distribution function is a function $L : [-\infty, +\infty] \to [0, 1]$ which is left continuous on $\mathbb{R}$, non-increasing and $L(-\infty) = 1$, $L(+\infty) = 0$. We will denote by $\bigtriangledown$ the family of all non-distance distribution functions on $[-\infty, +\infty]$ and denote by E the subsets of $\bigtriangledown$ containing functions L such that $\lim_{t\to+\infty} L(t) = 0$.

G is a special element of E defined by

$$G(t) = \begin{cases} 1, & \text{if } x \leq 0, \\ 0, & \text{if } x > 0. \end{cases}$$

If X is a non-empty set, $L : X \times X \to \bigtriangledown$ is called a probabilistic non-distance on X and $L(x, y)$ is usually denoted by $L_{x,y}$.

Definition 2.6. ([14]) A triple (X, F, L) is said to be an intuitionistic probabilistic metric space if X is a nonempty set, F is a probabilistic distance and L is a probabilistic non-distance on X satisfying the following conditions: for all $x,\ y,\ z \in X$, $t, s \geq 0$,

(1) $F_{x,y}(t) + L_{x,y}(t) \leq 1$;
(2) $F_{x,y}(0) = 0$;
(3) $F_{x,y}(t) = H(t)$ if and only if $x = y$;
(4) $F_{x,y}(t) = F_{y,x}(t)$;
(5) If $F_{x,z}(t) = 1$ and $F_{z,y}(s) = 1$, then $F_{x,y}(t + s) = 1$;
(6) $L_{x,y}(0) = 0$;
(7) $L_{x,y}(t) = G(t)$ if and only if $x = y$;
(8) $L_{x,y}(t) = L_{y,x}(t)$;
(9) If $L_{x,z}(t) = 0$ and $L_{z,y}(s) = 0$, then $L_{x,y}(t + s) = 0$.
If, in addition, the triangle inequalities
(10) $F_{x,y}(t + s) \geq F_{x,z}(t) * F_{z,y}(s)$;
(11) $L_{x,y}(t + s) \leq L_{x,z}(t) \diamond L_{z,y}(s)$.

Where $*$ is a t-norm and $\diamond$ is a t-conorm, then $(X, F, L, *, \diamond)$ is said to be an intuitionistic Menger space. The functions $F_{x,y}(t)$ and $L_{x,y}(t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Definition 2.7. ([21]) Suppose A and S be self-mappings of an intuitionistic Menger space $(X, F, L, *, \diamond)$. A point x in X is called a coincidence point of A and S if and only if $Ax = Sx$. In this case, $w = Ax = Sx$ is called a point of coincidence of A and S.

Definition 2.8. ([17]) Two self-maps A and B of an intuitionistic Menger space $(X, F, L, *, \diamond)$ are said to be weakly compatible if they commute at their coincidence points, i.e., if $Ax = Bx$ for some $x \in X$, then $ABx = BAx$.

Definition 2.9. ([23]) The pair (A, S) of self-mappings of an intuitionistic Menger space $(X, F, L, *, \diamond)$ is said to have the common limit range property with respect to the mapping S (denoted by (CLR_S)) if there exists a sequence $\{x_n\} \subset X$ such that, for $z \in S(X)$,

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = z.$$

Two pairs (A, S) and (B, T) of self-mappings of an intuitionistic Menger space $(X, F, L, *, \diamond)$ are said to have the common limit range property with respect to mappings S and T (denoted by (CLR_{ST})) if there exists two sequences $\{x_n\}$, $\{y_n\} \subset X$ such that for $z \in S(X) \cap T(X)$,

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = z.$$

In 2014 the concept of C-class functions was introduced by Saleem et al. [19], defined as (see [3]):

Definition 2.10. ([19]) We say that the continuous function $f : [0, +\infty)^2 \to \mathbb{R}$ is C-class function if the following conditions satisfies for all $s, t \in [0, +\infty)$,

(1) $f(s, t) \le s$;
(2) $f(s, t) = s$ implies that $s = 0$ or $t = 0$.

We will denote the set of all C-class functions by $\mathcal{C}$.

Definition 2.11. ([19]) We say that the continuous function $g : [0, +\infty)^2 \to \mathbb{R}$ is inverse C-class function if the following conditions satisfies for all $s, t \in [0, +\infty)$,

(1) $g(s, t) \ge s$;
(2) $g(s, t) = s$ implies that $s = 0$ or $t = 0$.

We will denote the set of all inverse C-class functions by $\mathcal{C}_{inv}$.

Definition 2.12. ([19]) We say that the continuous function $\psi : [0, +\infty) \to [0, +\infty)$ is an altering distance function, if the following conditions satisfies:

(1) ψ is non-decreasing on $[0, +\infty)$;
(2) $\psi(t) = 0$ if and only if $t = 0$.

We shall denote the class of altering distance functions by Ψ.

Alternatively, the continuous function $\varphi : [0, 1] \to [0, 1]$ is also called an altering distance function, if the following assumptions satisfies:

(3) φ is decreasing on $[0, 1]$;
(4) $\varphi(t) = 0$ if and only if $t = 1$.

We shall denote the set of such functions by Φ.

3. Main results

Lemma 3.1. *Let A, B, S and T be self-mappings of an intuitionistic Menger metric space $(X, F, L, *, \diamond)$ satisfying the following conditions:*

(1) *The pair (A, S) satisfies the (CLR_S) property (or the pair (B, T) satisfies the (CLR_T) property);*
(2) *$A(X) \subseteq T(X)$ (or $B(X) \subseteq S(X)$);*
(3) *$T(X)$ (or $S(X)$) is a closed subset of X;*
(4) *$B(y_n)$ converges for every sequence $\{y_n\}$ in X whenever $T(y_n)$ converges (or $A(x_n)$ converges for every sequence $\{x_n\}$ in X whenever $S(x_n)$ converges);*
(5)
$$\psi\big(F_{Ax,By}(t)\big) \geq g\big(\psi(M(x,y,t)); \varphi(M(x,y,t))\big) \tag{3.1}$$
and
$$\psi\big(L_{Ax,By}(t)\big) \leq f\big(\psi(N(x,y,t)); \varphi(N(x,y,t))\big), \tag{3.2}$$
where $\psi \in \Psi$, $\varphi \in \Phi$ and $f \in C$, $g \in C_{inv}$.
$$M(x,y,t) = \min\{F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t), F_{Sx,By}(t), F_{Ty,Ax}(t)\},$$
$$N(x,y,t) = \max\{L_{Sx,Ty}(t), L_{Ax,Sx}(t), L_{By,Ty}(t), L_{Sx,By}(t), L_{Ty,Ax}(t)\}$$
for all $x, y \in X$, $t > 0$.

Then the pairs (A, S) and (B, T) satisfy the (CLR_{ST}) property.

Proof. Suppose that the pair (A, S) satisfies the (CLR_S) property and $T(X)$ is a closed subset of X. Then, there exists a sequence $\{x_n\}$ in X such that
$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = z, \text{ where } z \in S(X).$$
Since $A(X) \subseteq T(X)$, there exists a sequence $\{y_n\}$ in X such that $Ax_n = Ty_n$. So
$$\lim_{n\to+\infty} Ty_n = \lim_{n\to+\infty} Ax_n = z,$$
where $z \in S(X) \cap T(X)$. Thus, $\lim_{n\to+\infty} Ax_n = z$, $\lim_{n\to+\infty} Sx_n = z$ and $\lim_{n\to+\infty} Ty_n = z$.

Now, we show that $\lim_{n\to+\infty} By_n = z$.

Let l is the limit, so, $\lim_{n\to+\infty} F_{By_n,l}(t) = 1$ and $\lim_{n\to+\infty} L_{By_n,l}(t) = 0$. We assert that $l = z$. By putting $x = x_n$ and $y = y_n$ in inequalities (3.1) and (3.2), we have
$$\begin{aligned}&\psi\big(F_{Ax_n,By_n}(t)\big)\\ &\geq g\big(\psi(\min\{F_{Sx_n,Ty_n}(t), F_{Ax_n,Sx_n}(t), F_{By_n,Ty_n}(t), F_{Sx_n,By_n}(t), F_{Ty_n,Ax_n}(t)\}),\\ &\quad \varphi(\min\{F_{Sx_n,Ty_n}(t), F_{Ax_n,Sx_n}(t), F_{By_n,Ty_n}(t), F_{Sx_n,By_n}(t), F_{Ty_n,Ax_n}(t)\})\big),\end{aligned}$$

and

$$
\begin{aligned}
&\psi\big(L_{Ax_n,By_n}(t)\big)\\
&\le f\big(\psi(\max\{L_{Sx_n,Ty_n}(t), L_{Ax_n,Sx_n}(t), L_{By_n,Ty_n}(t), L_{Sx_n,By_n}(t), L_{Ty_n,Ax_n}(t)\}),\\
&\quad \varphi(\max\{L_{Sx_n,Ty_n}(t), L_{Ax_n,Sx_n}(t), L_{By_n,Ty_n}(t), L_{Sx_n,By_n}(t), L_{Ty_n,Ax_n}(t)\})\big).
\end{aligned}
$$

Taking the limit as $n \to +\infty$, we get

$$
\begin{aligned}
\psi\big(F_{z,l}(t)\big) \ge g\big(&\psi(\min\{F_{z,z}(t), F_{z,z}(t), F_{l,z}(t), F_{z,l}(t), F_{z,z}(t)\})),\\
&\varphi(\min\{F_{z,z}(t), F_{z,z}(t), F_{l,z}(t), F_{z,l}(t), F_{z,z}(t)\})\big)
\end{aligned}
$$

and

$$
\begin{aligned}
\psi\big(L_{z,l}(t)\big) \le f\big(&\psi(\max\{L_{z,z}(t), L_{z,z}(t), L_{z,l}(t), L_{z,l}(t), L_{z,z}(t)\}),\\
&\varphi(\max\{L_{z,z}(t), L_{z,z}(t), L_{l,z}(t), L_{z,l}(t), L_{z,z}(t)\})\big).
\end{aligned}
$$

So, we have

$$\psi\big(F_{z,l}(t)\big) \ge g\big(\psi(F_{z,l}(t)); \varphi(F_{z,l}(t))\big) \ge \psi(F_{z,l}(t))$$

and

$$\psi\big(L_{z,l}(t)\big) \le f\big(\psi(L_{z,l}(t)); \varphi(L_{z,l}(t))\big) \le \psi(L_{z,l}(t)).$$

Hence

$$g\big(\psi(F_{z,l}(t)); \varphi(F_{z,l}(t))\big) = \psi(F_{z,l}(t)).$$

This implies that either $\psi(F_{z,l}(t)) = 0$ or $\varphi(F_{z,l}(t)) = 0$.
That is,

$$F_{z,l}(t) = 1. \tag{3.3}$$

And

$$f\big(\psi(L_{z,l}(t)); \varphi(L_{z,l}(t))\big) = \psi(L_{z,l}(t)).$$

This implies that either $\psi(L_{z,l}(t)) = 0$ or $\varphi(L_{z,l}(t)) = 0$.
That is

$$L_{z,l}(t) = 0. \tag{3.4}$$

From (3.3) and (3.4), we have $l = z$. Thus, the pairs (A, S) and (B, T) satisfy the CLR_{ST} property. □

Theorem 3.2. *Let A, B, S and T be four self-mappings of an intuitionistic Menger space $(X, F, L, *, \diamond)$ satisfying the inequalities* (3.1) *and* (3.2) *in Lemma* 3.1. *If the pair (A, S) and (B, T) satisfy the CLR_{ST} property, then (A, S) and (B, T) have a coincidence points. Moreover, if (A, S) and (B, T) are weakly compatible, then A, B, S and T have a unique common fixed point in X.*

Proof. Since the pair (A,S) and (B,T) satisfy the CLR_{ST} property, there exist two sequences $\{x_n\}_{n\in\mathbb{N}}$, $\{y_n\}_{n\in\mathbb{N}}$ in X such that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = z,$$

where $z \in S(X) \cap T(X)$. Hence, there exist $u, v \in X$ such that $Su = Tv = z$.

Now, we show that $Au = Su = z$. As in the proof of Lemma 3.1, we can prove that $Au = Su = z$ by putting $x = u$ and $y = y_n$ in the inequalities (3.1) and (3.2). Therefore, u is a coincidence point of the pair (A,S).

Now, we assert that v is a coincidence points of the pair (B,T), that is, we show that $Bv = Tv = z$. Just show that $Bv = z$.
By putting $x = u$ and $y = v$ in the inequalities (3.1) and (3.2) we find

$$\begin{aligned}&\psi(F_{Au,Bv}(t))\\ &\geq g\big(\psi(\min\{F_{Su,Tv}(t), F_{Au,Su}(t), F_{Bv,Tv}(t), F_{Su,Bv}(t), F_{Tv,Au}(t)\}),\\ &\qquad \varphi(\min\{F_{Su,Tv}(t), F_{Au,Su}(t), F_{Bv,Tv}(t), F_{Su,Bv}(t), F_{Tv,Au}(t)\})\big)\end{aligned}$$

and

$$\begin{aligned}&\psi(L_{Au,Bv}(t))\\ &\leq f\big(\psi(\max\{L_{Su,Tv}(t), L_{Au,Su}(t), L_{Bv,Tv}(t), L_{Su,Bv}(t), L_{Tv,Au}(t)\}),\\ &\qquad \varphi(\max\{L_{Su,Tv}(t), L_{Au,Su}(t), L_{Bv,Tv}(t), L_{Su,Bv}(t), L_{Tv,Au}(t)\})\big).\end{aligned}$$

So

$$\psi(F_{Bv,z}(t)) \geq g\big(\psi(F_{Bv,z}(t)), \varphi(F_{Bv,z}(t))\big)$$

and

$$\psi(L_{Bv,z}(t)) \leq f\big(\psi(L_{Bv,z}(t)), \varphi(L_{Bv,z}(t))\big).$$

Because of $g \in \mathcal{C}_{inv}$ and $f \in \mathcal{C}$, we find

$$\psi(F_{Bv,z}(t)) \geq g\big(\psi(F_{Bv,z}(t)), \varphi(F_{Bv,z}(t))\big) \geq \psi(F_{Bv,z}(t))$$

and

$$\psi(L_{Bv,z}(t)) \leq f\big(\psi(L_{Bv,z}(t)), \varphi(L_{Bv,z}(t))\big) \leq \psi(L_{Bv,z}(t)).$$

So

$$g\big(\psi(F_{Bv,z}(t)), \varphi(F_{Bv,z}(t))\big) = \psi(F_{Bv,z}(t)).$$

This implies that either $\psi(F_{Bv,z}(t)) = 0$ or $\varphi(F_{Bv,z}(t)) = 0$.
That is

$$F_{Bv,z}(t) = 1. \tag{3.5}$$

And

$$f\big(\psi(L_{Bv,z}(t)), \varphi(L_{Bv,z}(t))\big) = \psi(L_{Bv,z}(t)).$$

This implies that either $\psi(L_{Bv,z}(t)) = 0$ or $\varphi(L_{Bv,z}(t)) = 0$.
That is

$$L_{Bv,z}(t) = 0. \tag{3.6}$$

From (3.5) and (3.6) we have $Bv = z$, so $Bv = Tv = z$. Therefore, v is a coincidence point of the pair (B, T). Since the pair (A, S) is weakly compatible and $Au = Su$, we obtain $Az = Sz$.

Now we prove that z is a common fixed point of A and S. Applying the inequalities (3.1) and (3.2) with $x = z$ and $y = v$, we get

$$\begin{aligned}\psi(F_{Az,z}(t)) \geq g\big(&\psi(\min\{F_{Az,z}(t),\, F_{Az,Az}(t),\, F_{z,z}(t),\, F_{Az,z}(t),\, F_{Az,z}(t)\}),\\ &\varphi(\min\{F_{Az,z}(t),\, F_{Az,Az}(t),\, F_{z,z}(t),\, F_{Az,z}(t),\, F_{Az,z}(t)\})\big)\end{aligned}$$

and

$$\begin{aligned}\psi(L_{Az,z}(t)) \leq f\big(&\psi(\max\{L_{Az,z}(t),\, L_{Az,Az}(t),\, L_{z,z}(t),\, L_{Az,z}(t),\, L_{Az,z}(t)\}),\\ &\varphi(\max\{L_{Az,z}(t),\, L_{Az,Az}(t),\, L_{z,z}(t),\, L_{Az,z}(t),\, L_{Az,z}(t)\})\big).\end{aligned}$$

Because of $g \in \mathcal{C}_{inv}$ and $f \in \mathcal{C}$, we find

$$\psi(F_{Az,z}(t)) \geq g\big(\psi(F_{Az,z}(t)), \varphi(F_{Az,z}(t))\big) \geq \psi(F_{Az,z}(t))$$

and

$$\psi(L_{Az,z}(t)) \leq f\big(\psi(L_{Az,z}(t)), \varphi(L_{Az,z}(t))\big) \leq \psi(L_{Az,z}(t)).$$

So,

$$g\big(\psi(F_{Az,z}(t)), \varphi(F_{Az,z}(t))\big) = \psi(F_{Az,z}(t)).$$

This implies that either $\psi(F_{Az,z}(t)) = 0$ or $\varphi(F_{Az,z}(t)) = 0$, that is

$$F_{Az,z}(t) = 1. \tag{3.7}$$

And

$$f\big(\psi(L_{Az,z}(t)), \varphi(L_{Az,z}(t))\big) = \psi(L_{Az,z}(t)).$$

This implies that either $\psi(L_{Az,z}(t)) = 0$ or $\varphi(L_{Az,z}(t)) = 0$, that is

$$L_{Az,z}(t) = 0. \tag{3.8}$$

From (3.7) and (3.8), we have $Az = z = Sz$, which shows that z is a common fixed point of A and S. Since the pair (B, T) is weakly compatible, we get $Bz = Tz$.

Similarly, we can prove that z is a common fixed point of B and T. Hence, z is a common fixed point of A, B, S and T. The uniqueness of z follows easily by the inequalities (3.1) and (3.2). □

Now, we give an example to support our theorem.

Example 3.3. Let $X = [0, 1]$ with the metric d define by $d(x, y) = |x - y|$ for all $x, y \in X$. We define the distance and non distance distribution (F, L) induced by the metric distance d, which given by the following expressions: for each $t \in [0, +\infty)$ and $x, y \in X$.

$$F_{x,y}(t) = H(t - d(x, y))$$

and

$$L_{x,y}(t) = G(t - d(x, y)).$$

Let define four self-maps $A,\ B,\ S$ and T as follows, for all $x \in [0, 1]$,

$$Ax = E(x) \ \text{(integer part function,)} \quad Bx = x^2,$$
$$Sx = x, \quad T(x) = \sqrt{x}.$$

We can choose the sequences $\{x_n = y_n = \dfrac{1}{n}\}_{n\in\mathbb{N}^*}$ to prove that the pairs (A, S) and (B, T) satisfies the CLR_{ST} property. Moreover, it is clear that the both pairs (A, S) and (B, T) are weakly compatible.

It remains to verify that the inequalities (3.1) and (3.2) are holds, we choose the C-class and inverse C-class functions as $f(s, t) = s(1-t)$ and $g(s, t) = s(1+t)$ for all $(s, t) \in [0, +\infty)^2$, respectively. For any altering distance functions $\psi \in \Psi$ and $\varphi \in \Phi$. The unique case were the inequalities are not satisfies is when, $F_{Ax,By}(t) = 0$ and $M(x, y, t) = 1$ or $L_{Ax,By}(t) = 1$ and $N(x, y, t) = 0$.

This case is impossible in this example because the mappings A, B, S and T verified the following inequality:

$$\max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\} - d(Ax, By) \geq 0 \tag{3.9}$$

The following curve represent the function

$$\begin{aligned} h(x, y) = \max\{&d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\} \\ &- d(Ax, By) \end{aligned} \tag{3.10}$$

on $[0; 1] \times [0; 1[$, which confirm the validity of the inequality (3.9).(If $y = 1$, $f(x, 1) = 0$).

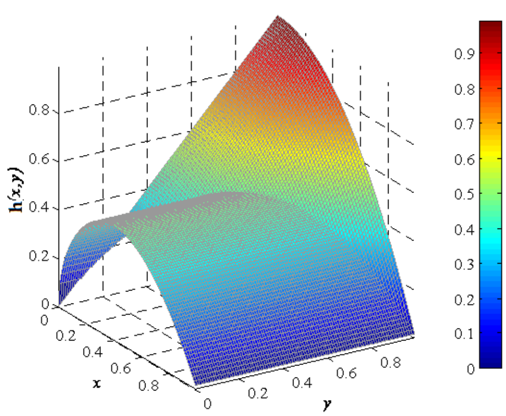

FIGURE 1. Curve of the function h on $[0; 1] \times [0; 1[$.

All conditions of Theorem 3.2 are satisfies, then A, B, S and T have a unique common fixed point $x = 0$ in X.

By according the Lemma 3.1 with the Theorem 3.2, we find the following theorem.

Theorem 3.4. *Let A, B, S and T be self-mappings of an intuitionistic Menger metric space $(X, F, L, *, \diamond)$ satisfying the following conditions:*

(1) *The pair (A, S) satisfies the (CLR_S) property (or the pair (B, T) satisfies the (CLR_T) property);*
(2) *$A(X) \subseteq T(X)$ (or $B(X) \subseteq S(X)$);*
(3) *$T(X)$ (or $S(X)$) is a closed subset of X;*
(4) *$B(y_n)$ converges for every sequence $\{y_n\}$ in X whenever $T(y_n)$ converges (or $A(x_n)$ converges for every sequence $\{x_n\}$ in X whenever $S(x_n)$ converges);*
(5)
$$\psi\big(F_{Ax,By}(t)\big) \geq g\big(\psi(M(x, y, t)); \varphi(M(x, y, t))\big)$$
and
$$\psi\big(L_{Ax,By}(t)\big) \leq f\big(\psi(N(x, y, t)); \varphi(N(x, y, t))\big),$$
where $\psi \in \Psi$, $\varphi \in \Phi$, and $f \in C$, $g \in C_{inv}$,
$$M(x, y, t) = \min\{F_{Sx,Ty}(t), F_{Ax,Sx}(t), F_{By,Ty}(t), F_{Sx,By}(t), F_{Ty,Ax}(t)\},$$
$$N(x, y, t) = \max\{L_{Sx,Ty}(t), L_{Ax,Sx}(t), L_{By,Ty}(t), L_{Sx,By}(t), L_{Ty,Ax}(t)\}$$
for all $x, y \in X$, $t > 0$;
(6) *The both pairs (A, S) and (B, T) are weakly compatible.*

Then A, B, S and T have a unique common fixed point in X.

By taking $S = I$ and $T = I$ in previous theorem, we find the next corollary.

Corollary 3.5. *Let A, B be self-mappings of an intuitionistic Menger metric space $(X, F, L, *, \diamond)$ satisfying the following conditions:*

(1) *The pair (A, B) satisfies the (CLR_B) property;*
(2) *$A(X) \subseteq B(X)$;*
(3) *$B(X)$ is a closed subset of X;*
(4) *$B(y_n)$ converges for every sequence $\{y_n\}$ in X whenever $A(y_n)$ converges;*
(5)
$$\psi\big(F_{Ax,By}(t)\big) \geq g\big(\psi(M(x, y, t)); \varphi(M(x, y, t))\big)$$
and
$$\psi\big(L_{Ax,By}(t)\big) \leq f\big(\psi(N(x, y, t)); \varphi(N(x, y, t))\big),$$

where $\psi \in \Psi$, $\varphi \in \Phi$, *and* $f \in \mathcal{C}$, $g \in \mathcal{C}_{inv}$,

$M(x, y, t) = \min\{F_{x,y}(t), F_{Ax,x}(t), F_{By,y}(t), F_{x,By}(t), F_{y,Ax}(t)\}$,

$N(x, y, t) = \max\{L_{x,y}(t), L_{Ax,x}(t), L_{By,y}(t), L_{x,By}(t), L_{y,Ax}(t)\}$

for all $x, y \in X$,$t > 0$*;*

(6) (A, B) *are weakly compatible.*

Then, the mapping A, B *have a unique fixed point.*

We can also derive corollaries similar to [20].

4. Application to metric space

As application to our main results, we derive the corresponding common fixed point theorem in metric space.

Theorem 4.1. *Let* (X, d) *be a metric space and* A, B, S *and* T *four self-mappings on* X*. If the following conditions are satisfied,*

(1) *there exist sequences* $\{x_n\}$ *and* $\{y_n\}$ *in* X *and* $z \in S(X) \cap T(X)$ *such that*

$$\lim_{n\to+\infty} Ax_n = \lim_{n\to+\infty} Sx_n = \lim_{n\to+\infty} By_n = \lim_{n\to+\infty} Ty_n = z;$$

(2) $ASx = SAx$*, whenever* $Ax = Sx$ *for some* $x \in X$*;*

(3) $BTy = TBy$*, whenever* $By = Ty$ *for some* $y \in X$*;*

(4) *for all* $x, y \in X$*,*

$$d(Ax, By) \leq \max\{d(Sx, Ty), d(Ax, Sx), d(By, Ty), d(Sx, By), d(Ty, Ax)\}.$$

Then A, B, S *and* T *have a unique common fixed point in* X*.*

Proof. (Applying Theorem 3.2) Let (X, d) be a metric space. By choosing the distribution distance functions and non-distance as the following:

$$F_{x,y} = H(t - d(x, y))$$

and

$$L_{x,y} = G(t - d(x, y)).$$

With C-class function f and inverse C-class function g define by

$$f(s, t) = s(1 - t)$$

and

$$g(s, t) = s(1 + t).$$

For arbitrary $\varphi \in \Phi$ and $\psi \in \Psi$, the inequalities (3.1)-(3.2) was satisfying, then all conditions of Theorem 3.2 are hold. Then, A, B, S and T have a unique common fixed point in X. $\square$

5. Conclusion

In this paper, we presented the concept of intuitionistic Menger metric space and gave the results of common fixed point theory for four self mappings in intuitionistic Menger metric spaces under some conditions extending to (CLR) property and C-class functions., which is an important issue in applications. This study is the extended and generalization form of many theorems.

This study can be extended in different structures such as intuitionistic fuzzy b-metric space; intuitionistic fuzzy metric like space; intuitionistic fuzzy b-metric like space; etc

Also, among the open problems is the use and development of the notion of measure of non compactness in the probabilistic topology exactly in theory of fixed point in fuzzy metric space and Menger space, one can use the reference therein. For future applied works, these obtained results can provide a deeper understanding of the structure of intuitionistic Menger metric spaces.

References

[1] A. Abdelhalim, A. Aliouche, L. Benaoua and T.E. Ousseif, *Common coupled fixed point theorems for two pairs of weakly compatible mappings in Menger metric spaces*, Moroccan J. Pure Appl. Anal., **5**(2) (2019), 197-221.

[2] N.R. Anakira, Adel Almalki, D. Katatbeh, G.B. Hani, A.F. Jameel, Khamis S. Al Kalbani and M. Abu-Dawas, *An Algorithm for Solving Linear and Non-Linear Volterra Integro-Differential Equations*, Int. J. Adv. Soft Comput. Appl., **15**(3) (2023), 69-83.

[3] A.H. Ansari and A. Mutlu, *C-class functions on coupled fixed point theorem for mixed monotone mappings on partial ordered dislocated quasi metric spaces*, Nonlinear Funct. Anal. Appl., **22** (1) (2017), 99-106.

[4] A.H. Ansari, V. Popa, Y.M. Singh and M.S. Khan, *Fixed point theorems of an implicit relation via C- class function in metric spaces*, J. Adv. Math. Stud., **13**(1) (2020), 1-10.

[5] L. Benaoua and A. Aliouche, *A common coupled fixed point theorem in intuitionistic Menger metric space*, Mathematica Moravica, **20**(2) (2016), 59-85.

[6] L. Benaoua and A. Aliouche, *Coupled fixed point theorems for weakly compatible mappings along with CLR property in Menger metric spaces*, Carpathian Math. Publ., **8**(2) (2016), 195-210.

[7] L. Benaoua and A. Aliouche, *Common fixed point theorems in intuitionistic Menger space using property EA and an application to Fredholm integral equations*, J. Interdisciplinary Math., **25**(1) (2022), 1-21.

[8] L. Benaoua, V. Parvaneh, T. Oussaeif, L. Guran, G.H. Laid and C. Park, *Common fixed point theorems in intuitionistic fuzzy metric spaces with an application for Volterra integral equations*, Comm. Nonlinear Sci. Numer. Simu., **127** (2023), 107524.

[9] G. Farraj, M. Banan, K. Rushdi and B. Wahiba, *An algorithm for solving fractional differential equations using conformable optimized decomposition method,* Int. J. Adv. Soft Comput. Appl., **15**(1) (2023), 187-196.

[10] O. Hadzic and E. Pap, *Fixed point theory in probabilistic metric spaces*, Springer Science and Business Media, 536, 2001.

[11] M. Imdad, S. Chauhan and Z. Kadelburg, *Fixed point theorems for mappings with common limit range property satisfying generalized* (ψ, φ)*-weak contractive conditions*, Math. Sci., Springer, **7** (2013), 1-8.

[12] M. Imdad, B.D. Pant and S. Chauhan, *Fixed point theorems in Menger spaces using the* (CLR_{ST}) *property and applications*, J. Nonlinear Anal. Optim., **3** (2012), 225-237.

[13] G. Jungck and B.E. Rhoades, *Fixed point for set valued functions without continuity*, Indian J. Pure Appl. Math., **3** (1998), 227-238.

[14] S. Kutukcu, *On intuitionistic Menger spaces*, Bull. Math. Statistics Research, **6** (2018), 28-32.

[15] B. Laouadi, T.E. Oussaeif, L. Benaoua, L. Guran and S. Radenovic, *Some new fixed point results in b-metric space with rational generalized contractive condition*, J. Sib. Fed. Univ. Math. Phys., **16**(4) (2023), 506518.

[16] K. Menger, *Statistical Metrics*, Proc. Nat. Acad. Sci. USA, **28**(12) (1942), 535-537.

[17] B.D. Pant, S. Chauhan and V. Pant, *Common fixed point theorems in intuitionistic Menger spaces*, J. Adv. Studies Top., **1** (2010), 54-57.

[18] H. Qawaqneh, *Fractional analytic solutions and fixed point results with some applications*, Adv. Fixed Point Theory, **14**(1) (2024), 1-14.

[19] N. Saleem, A.H. Ansari and M.K. Jain, *Some fixed point theorems of inverse* C*-class function under weak semi-compatibility*, J. Fixed Point Theory, **9** (2018).

[20] R. Sharma and A.H. Ansari, *Some fixed point theorems in an intuitionistic Menger space via* C*-class and inverse* C*-class functions,* Comput. Math. Meth., **2** (2020), https://doi.org/10.1002/cmm4.1090.

[21] A. Sharma, A. Jain and S. Choudhari, *Sub-compatibility and fixed point theorem in intuitionistic Menger space*, Int. J. Theory and Appl. Sci., **3** (2011), 9-12.

[22] S.L. Singh, B.D. Pant and S. Chauhan, *Fixed point theorems in non-Archimedean Menger PM spaces*, J. Nonlinear Anal. Optim., **3** (2012), 153-160.

[23] W. Sintunavarat and P. Kumam, *Common fixed point theorems for a pair of weakly compatible mappings in fuzzy metric spaces*, J. Appl. Math., (2011), https://doi.org/10.1155/2011/637958.