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GAP FUNCTIONS AND ERROR BOUNDS FOR GENERAL SET-VALUED NONLINEAR VARIATIONAL-HEMIVARIATIONAL INEQUALITIES

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Abstract. The objective of this article is to study the general set-valued nonlinear variational-hemivariational inequalities and investigate the gap function, regularized gap function and Moreau-Yosida type regularized gap functions for the general set-valued nonlinear variational-hemivariational inequalities, and also discuss the error bounds for such inequalities using the characteristic of the Clarke generalized gradient, locally Lipschitz continuity, inverse strong monotonicity and Hausdorff Lipschitz continuous mappings.

1. Introduction

Stampacchia [38] first introduced the principle of variational inequality problem for modeling problems arising from mechanics to investigate the regularity problem for partial differential equations. The topic of variational

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inequalities can also be used as a core problem in optimization and nonlinear analysis to analyze various problems of complementarity and equilibrium in operational science, we often naturally meet the variational inequality problem for finding $x \in D$ such that

$$\langle A(x), y - x\rangle_X \geq 0, \ \forall y \in D, \tag{1.1}$$

where D is a nonempty closed convex subset of a normed space X representing constraints, $A : X \to X^*$ is a given operator, and $\langle \cdot, \cdot\rangle_X$ denotes the duality pairing between X and its dual X^*.

It is well known that the variational inequality (1.1) can be solved by transforming it into an equivalent optimization problem for the so-called merit function $\mu(\cdot;\alpha) : X \to R \cup \{+\infty\}$ dened by

$$\mu(x;\alpha) = \sup\{\langle A(x), x - z\rangle_X - \alpha\|x - z\|_X^2 \mid z \in D\} \text{ for } x \in D, \tag{1.2}$$

where α is a nonnegative parameter. Here, we note that

(i) If $\alpha = 0$ and X is finite dimensional, then (1.2) was first studied by Auslender in [6].
(ii) If $\alpha > 0$ and X is finite dimensional, then (1.2) was studied by Fukushima in [17].

The function $\mu(\cdot, 0)$ is usually known as the gap function, and the function $\mu(\cdot, \alpha)$ for $\alpha > 0$ is a regularized gap function.

Also, we notes that for all $\alpha > 0$, the function $\mu(\cdot, \alpha)$ is nonnegative on D and $\mu(x^*; \alpha) = 0$ whenever x^* satises the variational inequality (1.1), *see* [19].

The concept of gap function plays an vital role in the development of iterative algorithms, an evaluation of their convergence properties and usefull stopping methods for iterative algorithms. Error bounds are very important and used because they provide a measure of the distance between a solution set and a feasible arbitrary point. Solodov [37] developed some merit function associated with a generalized mixed variational inequality, and used those functions to achieve mixed variational error limits. Aussel *et al.* [8] introduced a new inverse quasi variational inequality, obtained local (global) error bounds for inverse quasi variational inequality in terms of certain gap functions to demonstrate the applicability of inverse quasi variational inequality. Focused on the Fukushima [18] concept, the regularized function of the Moreau-Yosida type has been introduced by Yamashita and Fukushima in [39]. They also suggested the so-called error bounds for variational inequalities *via* the regularized gap functions. Recently, there have been many studies on gap functions for different models on different topics such as iterative algorithms [21], the Painlev-Kuratowski convergence [2] and error bounds [1, 3, 4, 7, 9, 16, 20, 22].

In 2020, Chang *et al.* [11] introduce the mixed set-valued vector inverse quasi-variational inequality problems and to obtain error bounds for this kind of mixed set-valued vector inverse quasi-variational inequality problems in terms of the residual gap function, the regularized gap function, and the D-gap function. These bounds provide effective estimated distances between an arbitrary feasible point and the solution set of mixed set-valued vector inverse quasi-variational inequality problem. Recently, Chang *et al.* [10] studied the three types of gap functions, *i.e.,* the residual gap function, the regularized gap function and the global gap function by using the relaxed monotonicity and Hausdorff Lipschitz continuity and obtained the error bounds for generalized vector inverse variational inequality problems.

Hemivariational variational inequalities, which were first introduced by Panagiotopoulos [32, 33], deal with certain mechanical problems involving nonconvex and nonsmooth energy functions. If the energy function is convex, then hemivariational inequalities reduce to variational inequalities earlier studied by many authors, [5, 12, 15, 23, 24, 26, 28, 31]. On the other hand, the theory of elliptic type variational-hemivariational inequalities is known as a generalization of variational inequalities and hemivariational inequalities to the case involving both the convex and the nonconvex potentials, and based on the notion of the Clarke generalized gradient for locally Lipschitz functions, *see,* [13]. Interest in the study of variational-hemivariational inequality was originally motivated by various problems in mechanics, *see*, [25, 27, 29, 30, 34, 35, 36, 40, 41].

Our main purpose of this paper is to introduce the gap functions and regularized gap functions for a class of general set-valued nonlinear variational-hemivariational inequality problems and discuss the gap functions for the Minty version of these inequalities by utilizing the locally Lipschitz continuity, inverse strong monotonicity and Hausdorff Lipschitz continuous mapping and also provides two new error bounds for the general set-valued nonlinear variationalhemivariational inequality problems.

2. Mathematical prerequisites

In this section, we present some basic notations and concepts. Let $(X, \|\cdot\|_X)$ be a real Banach space with the dual X^*, and $\langle \cdot, \cdot \rangle_X$ be the duality pairing between X and X^*. Let $CB(X)$ be the family of all nonempty, closed and bounded subsets in X.

Definition 2.1. ([14]) A function $p : X \to R \cup \{+\infty\}$ is said to be

(a) proper, if $p \neq +\infty$.
(b) convex, if $p(tx + (1-t)y) \leq tp(x) + (1-t)p(y), \forall x, y \in X,\ t \in [0, 1]$.

(c) lower semicontinuous (l.s.c.) at $x \in X$, if for any sequence $\{x_n\} \subset X$ with $x_n \to x$,

$$p(x) \leq \liminf p(x_n).$$

(d) upper semicontinuous (u.s.c.) at $x \in X$, if for any sequence $\{x_n\} \subset X$ with $x_n \to x$,

$$\limsup p(x_n) \leq p(x).$$

(e) l.s.c (resp. u.s.c.) on X, if p is l.s.c (resp. u.s.c.) at every $x \in X$.

Definition 2.2. ([13]) Let $g : X \to R \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function. Then the convex subdifferential $\partial_c g : X \to X^*$ of g is defined by

$$\partial_c g(x) = \left\{x^* \in X^* \middle| \langle x^*, y - x\rangle_X \leq g(y) - g(x),\ \forall\ y \in X\right\},\ \forall\ x \in X.$$

An element $x^* \in \partial_c g(x)$ is called a subgradient of g at $x \in X$.

Definition 2.3. ([13]) A function $p : X \to R$ is said to be locally Lipschitz, if for every $x \in X$, there exist a neighbourhood U of x and a constant $\chi_x > 0$ such that

$$|p(z_1) - p(z_2)| \leq \chi_x \|z_1 - z_2\|_X,\ \forall\ z_1, z_2 \in U.$$

Let $p : X \to R$ be a locally Lipschitz function. Then the Clarke generalized directional derivative of p at the point $x \in X$ in the direction $y \in X$ is defined by

$$p^\circ(x; y) = \limsup_{z \to x, t \to 0^+} \frac{p(z + ty) - p(z)}{t}.$$

The generalized gradient of p at $x \in X$ is a subset of X^* defined by

$$\partial p(x) = \{x^* \in X^* | p^\circ(x; y) \geq \langle x^*, y\rangle_X,\ \forall\ y \in X\}.$$

Lemma 2.4. ([31, 35]) *Let X be a real Banach space and $p : X \to R$ be a locally Lipschitz function. Then the following statements are satisfied.*

(a) *For each $x \in X$, the function $X \ni y \rightarrowtail p^\circ(x; y) \in R$ is finite, positively homogeneous and subadditive, and*

$$|p^\circ(x; y)| \leq \chi_x \|y\|_X,\ \forall\ y \in X,$$

where $\chi_x > 0$ is a Lipschitz constant of p near x.

(b) *The function $X \times X \ni (x, y) \rightarrowtail p^\circ(x; y) \in R$ is upper semicontinuous.*

(c) *For every $x, y \in X$,*

$$p^\circ(x; y) = \max\{\langle \zeta, y\rangle_X \mid \zeta \in \partial p(x)\}.$$

Definition 2.5. ([1, 24]) An operator $B : X \to CB(X^*)$ is said to be pseudomonotone, if B is a bounded operator and for every sequence $\{u_n\} \subseteq X$ converging weakly to $u \in X$ with

$$\limsup \langle u_n, x_n - x\rangle \leq 0,\ \forall u_n \in B(x_n),$$

then we have

$$\langle u, x - y\rangle \leq \liminf \langle u_n, x_n - y\rangle,\ \forall\ y \in X,\ u \in B(x).$$

Let X be a reflexive Banach space and D be a nonempty, closed and convex subset of X. Let $B : D \to CB(X^*)$ be a set-valued mapping, $A : CB(X^*) \to CB(X^*)$ be a single-valued operator and $\varphi : D \times D \to R$ and $J : X \to R$ be the functionals, and $f \in X$.

Our purpose of this paper is to study following constrained general set-valued nonlinear variational-hemivatiational inequalities for finding $x \in D$ such that

$$\langle A(u) - f, y - x\rangle_X + \varphi(x, y) - \varphi(x, x) + J^\circ(x; y - x) \geq 0,\ \forall\ y \in D,\ u \in B(x). \tag{2.1}$$

Now, we have the following assumptions:

(1) the operator $A : CB(X^*) \to CB(X^*)$ is satisfying
  (a) A is continuous mapping.
  (b) A is locally Lipschitz continuous, that is, there exists $\alpha_A > 0$ such that

$$\|A(y_1) - A(y_2)\|_{X^*} \leq \alpha_A \|y_1 - y_2\|_X,\ \forall\ y_1, y_2 \in X. \tag{2.2}$$

(2) the operator $B : X \to CB(X^*)$ is satisfying
  (a) B is pseudomonotone.
  (b) B is inverse strongly monotone, that is, there exists $\alpha_B > 0$ such that

$$\begin{aligned}&\langle v_1 - v_2, y_1 - y_2\rangle_X \geq \alpha_B \|v_1 - v_2\|^2_{X^*},\\ &\forall\ y_1, y_2 \in X, v_1 \in B(y_1), v_2 \in B(y_2).\end{aligned} \tag{2.3}$$

  (c) B is Hausdorff Lipschitz continuous, that is, there exists $\beta_B > 0$ such that

$$\|v_1 - v_2\|_{X^*} \leq H(B(y_1), B(y_2)) \leq \beta_B \|y_1 - y_2\|_X, \tag{2.4}$$

  for all $y_1, y_2 \in X$. where $H(\cdot, \cdot)$ is the Hausdorff metric on $CB(X)$.

(3) $\varphi : D \times D \to R$ is such that
  (a) for each $x \in D$, $\varphi(x, \cdot) : \Omega \to R$ is convex and lower semicontinuous.

(b) there exists $\alpha_\varphi > 0$ such that

$$\varphi(x_1, y_2) - \varphi(x_1, y_1) + \varphi(x_2, y_1) - \varphi(x_2, y_2) \le \alpha_\varphi \|x_1 - x_2\|_X \|y_1 - y_2\|_X, \quad \forall\, x_1, x_2, y_1, y_2 \in D. \tag{2.5}$$

(4) $J: X \to R$ is a locally Lipschitz function such that
(a) $\|\partial J(y)\|_{X^*} \le \Upsilon_0 + \Upsilon_1 \|y\|_X, \ \forall\, y \in X$ with some $\Upsilon_0, \Upsilon_1 \ge 0$.
(b) there exists $\alpha_J \ge 0$ such that

$$J^\circ(y_1; y_2 - y_1) + J^\circ(y_2; y_1 - y_2) \le \alpha_J \|y_1 - y_2\|_X^2, \ \forall\, y_1, y_2 \in D. \tag{2.6}$$

For (2.1), we have the following existence and uniqueness result.

Theorem 2.6. *Let D be a nonempty, closed and convex subset of a reflexive Banach space X. Let $B: D \to CB(X^*)$ be a set-valued mapping, $A: CB(X^*) \to CB(X^*)$ be a single-valued operator and $\varphi: D \times D \to R$ and $J: X \to R$ be the functionals, and $f \in X$. Assume that* (1)-(4) *hold. If, in addition, the following condition is satisfied*

$$\frac{\alpha_j + \alpha_\varphi}{\alpha_B \alpha_A^2 \beta_B^2} < 1. \tag{2.7}$$

Then (2.1) *has a unique solution. Moreover, x solves* (2.1) *if and only if it solves the following general set-valued nonlinear Minty variational hemivariational inequalities for nding $x \in D$ such that*

$$\langle A(v) - f, y - x\rangle_X + \varphi(x, y) - \varphi(x, x) + J^\circ(y; y - x) \ge 0, \quad \forall\, y \in D, \ v \in B(y). \tag{2.8}$$

Proof. Let $x \in D$ be the unique solution of (2.1). First, we note that the assumption 4-(b) is equivalent to the following relaxed monotonicity condition of the generalized gradient

$$\langle \partial J(y) - \partial J(x), y - x\rangle_X \ge -\alpha_j \|y - x\|_X^2, \ \forall\, y, x \in D. \tag{2.9}$$

Next from the condition (2.7) together with (2.9), and the locally Lipscitz continuity of A, inverse strong monotonicity of B together with A, and Hausdorff Lipschitz continuity of B, we have, for all $\zeta_y \in \partial J(y), \zeta_x \in \partial J(x),\ x, y \in D, u \in B(x), v \in B(y)$,

$$\begin{aligned}
\langle A(v)-A(u), y-x\rangle_X + &\langle \zeta_y-\zeta_x, y-x\rangle_X\\
&\geq \alpha_B\|A(v)-A(u)\|^2_{X^*} - \alpha_j\|y-x\|^2_X,\\
&\geq \alpha_B\alpha_A^2\|v-u\|^2_{X^*} - \alpha_j\|y-x\|^2_X,\\
&\geq \alpha_B\alpha_A^2(H(B(y),B(x))_{X^*})^2 - \alpha_j\|y-x\|^2_X,\\
&\geq \alpha_B\alpha_A^2\beta_B^2\|y-x\|^2_X - \alpha_j\|y-x\|^2_X,\\
&\geq (\alpha_B\alpha_A^2\beta_B^2-\alpha_j)\|y-x\|^2_X. \qquad (2.10)
\end{aligned}$$

Let $y \in D$ be arbitrary. From (2.10), Lemma 2.4-(c) and the definition of generalized gradient, we have

$$\begin{aligned}
\langle A(v)-f, &y-x\rangle_X + \varphi(x,y) - \varphi(x,x) + J^\circ(y; y-x)\\
&\geq \langle A(v)-f+\zeta_y, y-x\rangle_X + \varphi(x,y)-\varphi(x,x)\\
&\geq \langle A(u)-f+\zeta_x, y-x\rangle_X + \varphi(x,y)-\varphi(x,x) + (\alpha_B\alpha_A^2\beta_B^2-\alpha_j)\|y-x\|^2_X\\
&\geq \langle A(u)-f+\zeta_x, y-x\rangle_X + \varphi(x,y)-\varphi(x,x)\\
&= \langle A(u)-f, y-x\rangle_X + \varphi(x,y)-\varphi(x,x) + J^\circ(x; y-x)\\
&\geq 0, \forall \zeta_y \in \partial J(y),
\end{aligned}$$

where $\zeta_x \in \partial J(x)$ is satisfied

$$J^\circ(x; y-x) = \langle \zeta_x, y-x\rangle_X.$$

Since $y \in D$ is arbitrary, hence $x \in D$ is a solution of (2.8).

Conversely, let $x \in D$ be a solution to the problem (2.8). For any $y \in D$ and $t \in (0,1)$, we denote $y_t = ty + (1-t)x \in D$. Inserting y_t into (2.8) and using the convexity of $y \rightarrowtail \varphi(x,y)$ and the positive homogeneity of $y \rightarrowtail J^\circ(x;y)$, we have

$$\begin{aligned}
0 &\leq t\langle A(v_t)-f, y-x\rangle_X + \varphi(x,y_t) - \varphi(x,x) + J^\circ(y_t; y_t-x)\\
&\leq t\langle A(v_t)-f, y-x\rangle_X + t\varphi(x,y) - t\varphi(x,x) + tJ^\circ(y_t; y-x),\ \forall v_t \in B(y_t).
\end{aligned}$$

Hence,

$$\langle A(v_t)-f, y-x\rangle_X + \varphi(x,y) - \varphi(x,x) + J^\circ(y_t; y-x) \geq 0,\ \forall v_t \in B(y_t). \quad (2.11)$$

Since B is pseudomonotone, therefore it is demicontinuous, *see* [28]. Passing to the upper limit as $t \to 0^+$ in (2.11), it follows from Lemma 2.6-(b) that

$$\begin{aligned}
\langle A(u)-f, &y-x\rangle_X + \varphi(x,y) - \varphi(x,x) + J^\circ(x; y-x)\\
&\geq \limsup_{t\to 0^+}\langle A(v_t)-f, y-x\rangle_X + \varphi(x,y)-\varphi(x,x) + \limsup_{t\to 0^+} J^\circ(y_t; y-x)\\
&\geq \limsup_{t\to 0^+}\{\langle A(v_t)-f, y-x\rangle_X + \varphi(x,y)-\varphi(x,x) + J^\circ(y_t; y-x)\}\\
&\geq 0,\ \forall v_t \in B(y_t).
\end{aligned}$$

Since $y \in D$ is an arbitrary, hence, we conclude that $x \in D$ is a solution of (2.1) and proof is completed. □

3. Main results

The purpose of this section is to discuss the gap function, regularized gap function and the Moreau-Yosida type regularized gap function utilizing local Lipschitz continuity, inverse strongly monotone and Hausdorff Lipschitz continuity associates with (2.1).

Definition 3.1. ([6, 32]) A real-valued function $\mu : D \to R$ is said to be a gap function for (2.1), if it satises the following assertions:

(a) $\mu(x) \geq 0, \ \forall x \in D$.
(b) For $x^* \in D$, $\mu(x^*) = 0$ if and only if x^* is a solution of (2.1).

Consider the functions $\Upsilon^f, \ \Upsilon^f_* : D \to R$ defined by

$$\Upsilon^f(x) = \sup_{y \in D} \left\{ \langle A(u) - f, x - y \rangle_X + \varphi(x,x) - \varphi(x,y) - J^\circ(x; y - x) \right\}, \quad \forall x \in D, u \in B(x), \tag{3.1}$$

$$\Upsilon^f_*(x) = \sup_{y \in D} \left\{ \langle A(v) - f, x - y \rangle_{\mathbb{X}} + \varphi(x,x) - \varphi(x,y) - J^\circ(y; y - x) \right\}, \quad \forall x \in D, v \in B(y). \tag{3.2}$$

The following lemma shows that functions Υ^f and Υ^f_* are gap functions for (2.1).

Lemma 3.2. *Assume that the assumptions of Theorem* 2.6 *hold. Then, the functions* Υ^f *and* Υ^f_* *defined by* (3.1) *and* (3.2) *are two gap functions for* (2.1).

Proof. First of all, we prove that Υ^f is a gap function for (2.1). It is not difficult to demonstrate in an analogous way that the function Υ^f_* is also a gap function for (2.1). We will review two conditions of Denition 3.1.

(a) From the definition of Υ^f, for all $x \in D$, we have

$$\begin{aligned} \Upsilon^f(x) &\geq \langle A(u) - f, x - x \rangle_X + \varphi(x,x) - \varphi(x,x) - J^\circ(x; x - x) \\ &= -J^\circ(x; 0) \\ &= 0, \ \forall u \in B(x). \end{aligned} \tag{3.3}$$

It implies that $\Upsilon^f(x) \geq 0$ for all $x \in D$.

(b) Suppose that for $x^* \in D$, $\Upsilon^f(x^*) = 0$, that is, for all $u^* \in B(x^*)$,

$$\sup_{y \in D} \{\langle A(u^*) - f, x^* - y\rangle_X + \varphi(x^*, x^*) - \varphi(x^*, y) - J^\circ(x^*; y - x^*)\} = 0. \tag{3.4}$$

This together with the fact

$$\langle A(u^*) - f, x^* - x^*\rangle_X + \varphi(x^*, x^*) - \varphi(x^*, x^*) - J^\circ(x^*; x^* - x^*) = 0, \forall u^* \in B(x^*)$$

implies that (3.4) is equivalent to

$$\langle A(u^*) - f, y - x^*\rangle_X + \varphi(x^*, y) - \varphi(x^*, x^*) - J^\circ(x^*; y - x^*) \geq 0$$

for all $y \in D, u^* \in B(x^*)$. Therefore, we infer that x^* is a solution of (2.1) if and only if

$$\Upsilon^f(x^*) = 0.$$

This means that Υ^f is a gap function for (2.1). This completes the proof. □

Let $\lambda > 0$ be a fixed parameter. We consider the following functions $\Upsilon^{f,\lambda}, \Upsilon_*^{f,\lambda} : D \to R$ defined by

$$\Upsilon^{f,\lambda}(x) = \sup_{y \in D} \left\{ \langle A(u) - f, x - y\rangle_X + \varphi(x, x) - \varphi(x, y) - J^\circ(x; y - x) - \frac{1}{2\lambda}\|x - y\|_X^2 \right\} \tag{3.5}$$

for all $x \in D, u \in B(x)$,

$$\Upsilon_*^{f,\lambda}(x) = \sup_{y \in D} \left\{ \langle A(v) - f, x - y\rangle_X + \varphi(x, x) - \varphi(x, y) - J^\circ(y; y - x) - \frac{1}{2\lambda}\|x - y\|_X^2 \right\} \tag{3.6}$$

for all $x \in D, v \in B(y)$.

In what follows, the functions $\Upsilon^{f,\lambda}$ and $\Upsilon_*^{f,\lambda}$ are called the regularized gap functions for (2.1).

Theorem 3.3. *Suppose the assertions of Theorem* 2.6 *hold. Then, for any* $\lambda > 0$*, the functions* $\Upsilon^{f,\lambda}$ *and* $\Upsilon_*^{f,\lambda}$ *are two gap functions for* (2.1).

Proof. Now, we prove that $\Upsilon^{f,\lambda}$ is a gap function for (2.1). Applying the analogous techniques, we can easily show that $\Upsilon_*^{f,\lambda}$ is also a gap function for (2.1).

(a) For any $\lambda > 0$ fixed, it is trivial that for each $x \in \mathcal{D}$, $\Upsilon^{f,\lambda}(x) \geq 0$. Since for $x \in D$,

$$\begin{aligned}\Upsilon^{f,\lambda}(x) &= \langle A(u) - f, x - x\rangle_X + \varphi(x,x) - \varphi(x,x) - J^\circ(x; x - x) - \frac{1}{2\lambda}\|x - x\|_X^2 \\ &= -J^\circ(x;0) \\ &= 0, \quad \forall u \in B(x).\end{aligned}$$

(b) Assume that for $x^* \in D$, $\Upsilon^{f,\lambda}(x^*) = 0$. Then for all $u^* \in B(x^*)$,

$$\sup_{y\in D}\left\{\langle A(u^*) - f, x^* - y\rangle_X + \varphi(x^*,x^*) - \varphi(x^*,y) - J^\circ(x^*; y- x^*) - \frac{1}{2\lambda}\|x^* - y\|_X^2\right\} \\ = 0.$$

This implies that

$$\begin{aligned}&\langle A(u^*) - f, y - x^*\rangle_X - \varphi(x^*,x^*) + \varphi(x^*,y) + J^\circ(x^*; y - x^*) \\ &\qquad \geq -\frac{1}{2\lambda}\|x^* - y\|_X^2, \ \forall y \in D, u^* \in B(x^*). \qquad (3.7)\end{aligned}$$

For any $z \in D$ and $t \in (0,1)$, we put $y = y_t = (1-t)x^* + tz \in D$ in (3.7), and using the convexity of $y \rightarrowtail \varphi(x,y)$ and positive homogeneity of $y \rightarrowtail J^\circ(x;y)$, then we have

$$\begin{aligned}&t\langle A(u^*) - f, z - x^*\rangle_X - t\varphi(x^*,x^*) + t\varphi(x^*,z) + tJ^\circ(x^*; z - x^*) \\ &\quad \geq \langle A(u^*) - f, y_t - x^*\rangle_X - \varphi(x^*,x^*) + \varphi(x^*,y_t) + J^\circ(x^*; y_t - x^*) \\ &\quad \geq -\frac{1}{2\lambda}\|x^* - y_t\|_X^2 \\ &\quad = -\frac{t^2}{2\lambda}\|x^* - z\|_X^2, \quad \forall u^* \in B(x^*).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\langle A(u^*) - f, z - x^*\rangle_X - \varphi(x^*,x^*) + \varphi(x^*,z) - J^\circ(x^*; z - x^*) \\ &\qquad\qquad \geq -\frac{t}{2\lambda}\|x^* - z\|_X^2, \ \forall z \in D, u^* \in B(x^*).\end{aligned}$$

Letting $t \to 0^+$ for the above inequality, we get

$$\langle A(u^*) - f, z - x^*\rangle_X - \varphi(x^*,x^*) + \varphi(x^*,z) + J^\circ(x^*; z - x^*) \geq 0, \ \forall z \in D, u^* \in B(x^*).$$

Hence, x^* is a solution of (2.1).

Conversely, suppose that $x^* \in D$ is a solution of (2.1), that is,

$$\langle A(u^*) - f, y - x^*\rangle_X - \varphi(x^*,x^*) + \varphi(x^*,y) + J^\circ(x^*; y - x^*) \geq 0, \ \forall y \in D, u^* \in B(x^*).$$

This ensures that

$$\sup_{y\in D}\left\{\langle A(u^*)-f,x^*-y\rangle_X+\varphi(x^*,x^*)-\varphi(x^*,y)-J^\circ(x^*;y-x^*)-\frac{1}{2\lambda}\|x^*-y\|_X^2\right\}$$
$$\le 0$$

for all $u^* \in B(x^*)$. The latter combined with the fact

$$\Upsilon^{f,\lambda}(x) \ge 0,\ \forall x \in D$$

and imply that

$$\Upsilon^{f,\lambda}(x^*) = 0.$$

This completes the proof. □

Lemma 3.4. *Assume that the assumptions of Theorem* 2.6 *are satisfied. If, in addition, $\varphi : D \times D \to R$ is continuous, then, for each $\lambda > 0$, the functions $\Upsilon^{f,\lambda}$ and $\Upsilon_*^{f,\lambda}$ are both lower semicontinuous.*

Proof. We can prove that $\Upsilon^{f,\lambda}$ is a lower semicontinuous for each $\lambda > 0$. It is not difficult to use a similar argument to verify that $\Upsilon_*^{f,\lambda}$ has the same property.

Consider the function $\hat{\Upsilon}^{f,\lambda} : D \times D \to R$ defined by

$$\hat{\Upsilon}^{f,\lambda}(x,y) = \langle A(u) - f, x - y\rangle_X + \varphi(x,x) - \varphi(x,y)$$
$$- J^\circ(x; y - x) - \frac{1}{2\lambda}\|x - y\|_X^2, \forall u \in B(x).$$

Since the operators $A : CB(X^*) \to CB(X^*)$ and $B : X \to CB(X^*)$ are demicontinuous being pseudomonotone, this means that the function $x \rightarrowtail \langle A(u), x\rangle_X$ is continuous. The latter together with the lower semicontinuity of $(x,y) \rightarrowtail -J^\circ(x;y)$, and the continuity of $(x,y) \rightarrowtail \varphi(x,y)$ and $x \rightarrowtail \|x\|_{\mathbb{X}}$ guarantees that $x \rightarrowtail \hat{\Upsilon}^{f,\lambda}(x,y)$ is lower semicontinuous for all $y \in D$.

Next, we see that

$$\Upsilon^{f,\lambda}(x) = \sup_{y\in D} \hat{\Upsilon}^{f,\lambda}(x,y),\ \forall\ x \in D.$$

Let $\{x_n\} \subset D$ and $x_n \to x$ as $n \to \infty$. Then, we have

$$\begin{aligned}\liminf_{n\to\infty} \Upsilon^{f,\lambda}(x_n) &= \liminf_{n\to\infty} \sup_{y\in D} \hat{\Upsilon}^{f,\lambda}(x_n,y)\\ &\ge \liminf_{n\to\infty} \hat{\Upsilon}^{f,\lambda}(x_n,z)\\ &\ge \hat{\Upsilon}^{f,\lambda}(x,z),\quad \forall z \in D.\end{aligned}$$

Passing to supremum with $z \in D$ for the above inequality, it gives

$$\begin{aligned}\liminf_{n\to\infty} \Upsilon^{f,\lambda}(x_n) &\geq \sup_{z\in D} \hat{\Upsilon}^{f,\lambda}(x,z)\\ &= \Upsilon^{f,\lambda}(x).\end{aligned}$$

Therefore, the function $\Upsilon^{f,\lambda}$ is lower semicontinuous and proof is completed. $\square$

Let $\lambda, \tau > 0$ be two parameters. Moreover, let us consider the following functions

$$\urcorner_{\Upsilon^{f,\lambda,\tau}},\ \urcorner_{\Upsilon_*^{f,\lambda,\tau}} : D \to R$$

are defined by

$$\urcorner_{\Upsilon^{f,\lambda,\tau}}(x) = \inf_{z\in D}\left\{\Upsilon^{f,\lambda}(z) + \tau\|x-z\|_X^2\right\},\ \ \forall x \in D, \tag{3.8}$$

$$\urcorner_{\Upsilon_*^{f,\lambda,\tau}}(x) = \inf_{z\in D}\left\{\Upsilon_*^{f,\lambda}(z) + \tau\|x-z\|_X^2\right\},\ \ \forall x \in D. \tag{3.9}$$

In the sequel, we invoke the functions $\urcorner_{\Upsilon^{f,\lambda,\tau}}$ and $\urcorner_{\Upsilon_*^{f,\lambda,\tau}}$ to be the Moreau-Yosida regularized gap functions for (2.1). Subsequently, we will verify that these functions are two gap functions for (2.1).

Theorem 3.5. *Assume that the assumptions of Lemma* 3.4 *are satisfied. Then, for all* $\lambda, \tau > 0$*, the functions* $\urcorner_{\Upsilon^{f,\lambda,\tau}}$ *and* $\urcorner_{\Upsilon_*^{f,\lambda,\tau}}$ *are two gap functions for* (2.1)*.*

Proof. We can prove that $\urcorner_{\Upsilon^{f,\lambda,\tau}}$ is a gap function for (2.1). It is possible to prove, in an analogous way, that $\urcorner_{\Upsilon_*^{f,\lambda,\tau}}$ is also a gap function for (2.1).

(a) For any $\lambda, \tau > 0$ fixed, recall that $\Upsilon^{f,\lambda,\tau}$ is a gap function for (2.1), hence

$$\Upsilon^{f,\lambda,\tau}(x) \geq 0,\ \forall\ x \in D.$$

In consequence,

$$\urcorner_{\Upsilon^{f,\lambda,\tau}}(x) \geq 0,\ \forall\ x \in D.$$

(b) Suppose that $x \in D$ is a solution of (2.1). Theorem 3.3 show that

$$\Upsilon^{f,\lambda,\tau}(x^*) = 0.$$

Moreover, the inequality

$$\begin{aligned}\urcorner_{\Upsilon^{f,\lambda,\tau}}(x^*) &= \inf_{z\in D}\left\{\Upsilon^{f,\lambda}(z) + \tau\|x^*-z\|_X^2\right\}\\ &\leq \Upsilon^{f,\lambda}(x^*) + \tau\|x^*-x^*\|_X^2\\ &= 0\end{aligned}$$

and the fact $\urcorner_{\Upsilon^{f,\lambda,\tau}}(x^*) \geq 0$ implies that $\urcorner_{\Upsilon^{f,\lambda,\tau}}(x^*) = 0$.

Conversely, let for $x^* \in D$, $\daleth_{\Upsilon^{f,\lambda,\tau}}(x^*) = 0$, that is,

$$\inf_{z \in D} \left\{ \Upsilon^{f,\lambda}(z) + \tau \|x^* - z\|_X^2 \right\} = 0.$$

Then, there exists a minimizing sequence $\{z_n\}$ in D such that

$$0 \leq \Upsilon^{f,\lambda}(z_n) + \tau \|x^* - z_n\|_X^2 < \frac{1}{n}. \tag{3.10}$$

It is obvious that

$$\Upsilon^{f,\lambda}(z_n) \to 0$$

and

$$\|x^* - z_n\|_{\mathbb{X}} \to 0 \quad \text{as} \quad n \to \infty,$$

that is, $z_n \to x^*$ as $n \to +\infty$.

From Lemma 3.4 and nonnegativity of $\Upsilon^{f,\lambda}$, we have

$$\begin{aligned} 0 &\leq \Upsilon^{f,\lambda}(x^*) \\ &\leq \liminf_{n \to +\infty} \Upsilon^{f,\lambda}(z_n) \\ &= 0. \end{aligned}$$

Thus $\Upsilon^{f,\lambda}(x^*) = 0$. Since $\Upsilon^{f,\lambda}$ is a gap function, x^* is a solution of (2.1), and proof is completed. □

4. The error bounds

In this section, we discuss two error bounds for (2.1) associated with the regularized gap function $\Upsilon^{f,\lambda,\tau}$ and the Moreau-Yosida regularized gap function $\daleth_{\Upsilon^{f,\lambda,\tau}}$, respectively. These error estimates measure the distance between any admissible point and the unique solution of (2.1).

Theorem 4.1. *Let $x^* \in D$ be the unique solution of* (2.1) *and $\lambda > 0$ be such that*

$$\alpha_B \beta_B^2 \alpha_A^2 - \alpha_\varphi - \alpha_J > \frac{1}{2\lambda}. \tag{4.1}$$

Assume that the assertions of Theorem 2.6 *hold. Then, for each $x \in D$, we have*

$$\|x - x^*\|_X \leq \sqrt{\frac{\Upsilon^{f,\lambda}(x)}{\alpha_B \beta_B^2 \alpha_A^2 - \alpha_\varphi - \alpha_J - \dfrac{1}{2\lambda}}}. \tag{4.2}$$

Proof. Let $x^* \in D$ be the unique solution of (2.1), that is, for $y \in D, u^* \in B(x^*)$.

$$\langle A(u^*) - f, y - x^* \rangle_X + \varphi(x^*, y) - \varphi(x^*, x^*) + J^\circ(x^*; y - x^*) \geq 0. \tag{4.3}$$

Then, for any $x \in D$ fixed, we put $y = x$ in (4.3), we obtain, for $u^* \in B(x^*)$,

$$\langle A(u^*) - f, x - x^* \rangle_X + \varphi(x^*, x) - \varphi(x^*, x^*) + J^\circ(x^*; x - x^*) \geq 0. \tag{4.4}$$

By virtue of the definition of $\Upsilon^{f,\lambda}$, we have

$$\begin{aligned}\Upsilon^{f,\lambda}(x) \geq \langle A(u) - f, x - x^* \rangle_X + \varphi(x, x) - \varphi(x, x^*) \\ - J^\circ(x; x^* - x) - \frac{1}{2\lambda}\|x - x^*\|_X^2.\end{aligned} \tag{4.5}$$

It follows from the locally Lipschitz continuity of A with respect to constant α_A, inverse strongly monotone of B with respect to the constant $\alpha_B > 0$ and Hausdorff Lipschitz continuity with constant β_B, assumptions (3)-(b) and (4)-(b), and inequality (4.4), we have

$$\begin{aligned}&\langle A(u) - f, x - x^* \rangle_X + \varphi(x, x) - \varphi(x, x^*) - J^\circ(x; x^* - x) - \frac{1}{2\lambda}\|x - x^*\|_X^2 \\ &\quad \geq \langle A(u^*) - f, x - x^* \rangle_X + \varphi(x^*, x) - \varphi(x^*, x^*) + J^\circ(x^*; x - x^*) \\ &\qquad + (\beta_B^2 \alpha_B \alpha_A^2 - \alpha_j - \alpha_\varphi - \frac{1}{2\lambda})\|x - x^*\|_X^2 \\ &\quad \geq \left(\beta_B^2 \alpha_B \alpha_A^2 - \alpha_j - \alpha_\varphi - \frac{1}{2\lambda}\right)\|x - x^*\|_X^2, \ \forall u \in B(x), u^* \in B(x^*).\end{aligned} \tag{4.6}$$

Combining (4.5) and (4.6), we have

$$\Upsilon^{f,\lambda}(x) \geq \left(\beta_B^2 \alpha_B \alpha_A^2 - \alpha_j - \alpha_\varphi - \frac{1}{2\lambda}\right)\|x - x^*\|_X^2. \tag{4.7}$$

Hence, the desired inequality (4.2) is valid. □

Theorem 4.2. *Let $x^* \in D$ be the unique solution of* (2.1) *and $\lambda > 0$ be such that*

$$\beta_B^2 \alpha_B \alpha_A^2 - \alpha_J - \alpha_\varphi \geq \frac{1}{2\lambda}. \tag{4.8}$$

Assume that the assumptions of Theorem 2.6 *hold. Then, for each $x \in D$ and all $\tau > 0$, we have*

$$\|x - x^*\|_X \leq \sqrt{\frac{2\daleth_{\Upsilon^{f,\lambda,\tau}}(x)}{\min\left\{\beta_B^2 \alpha_B \alpha_A^2 - \alpha_\varphi - \alpha_J - \frac{1}{2\lambda}, \tau\right\}}}. \tag{4.9}$$

Proof. Let $x^* \in D$ be the unique solution of (2.1). By the definition of the function $\daleth_{\Upsilon^{f,\lambda,\tau}}$,

$$\begin{aligned}
\daleth_{\Upsilon^{f,\lambda,\tau}}(x) &= \inf_{z\in D}\left\{\Upsilon^{f,\lambda}(z)+\tau\|x-z\|_X^2\right\}\\
&\geq \inf_{z\in D}\left\{\left(\beta_B^2\alpha_B\alpha_A^2-\alpha_J-\alpha_\varphi-\frac{1}{2\lambda}\right)\|x^*-z\|_X^2+\tau\|x-z\|_X^2\right\}\\
&\geq \min\left\{\beta_B^2\alpha_B\alpha_A^2-\alpha_J-\alpha_\varphi-\frac{1}{2\lambda},\tau\right\}\inf_{z\in D}\left\{\|x^*-z\|_X^2+\|x-z\|_X^2\right\}\\
&\geq \frac{1}{2}\min\left\{\beta_B^2\alpha_B\alpha_A^2-\alpha_J-\alpha_\varphi-\frac{1}{2\lambda},\tau\right\}\|x-x^*\|_X^2,\ \forall x\in D.
\end{aligned}$$

Hence

$$\|x-x^*\|_X \leq \sqrt{\frac{2\daleth_{\Upsilon^{f,\lambda,\tau}}(x)}{\min\left\{\beta_B^2\alpha_B\alpha_A^2-\alpha_\varphi-\alpha_J-\frac{1}{2\lambda},\tau\right\}}},\ \forall x\in D,$$

which completes the proof of the theorem. □

References

[1] A.A.H. Ahmadini, Salahuddin and J.K. Kim, *Error bounds for nonlinear mixed variational-hemivariational inequality problems,* Nonlinear Funct. Anal. and Appl., **29**(1) (2024), 15-33.

[2] L.Q. Anh, T. Bantaojai, N.V. Hung, V.M. Tam and R. Wangkeeree, *PainleveKuratowski convergences of the solution sets for generalized vector quasi-equilibrium problems.* Comput. Appl. Math., **37** (2018), 38323845.

[3] L.Q. Anh and N.V. Hung, *Gap functions and Hausdorff continuity of solution mappings to parametric strong vector quasiequilibrium problems.* J. Ind. Manag. Optim. **14** (2018), 6579.

[4] L.Q. Anh, N.V. Hung and V.M. Tam, *Regularized gap functions and error bounds for generalized mixed strong vector quasi equilibrium problems,* Comput. Appl. Math., **37** (2018), 59355950.

[5] P.N. Anh, H.T.C. Thach and J.K. Kim, *Proximal-like subgradient methods for solving multi-valued variational inequalities,* Nonlinear Funct. Anal. and Appl., **25**(3) (2020), 437-451, doi.org/10.22771/nfaa.2020.25.03.03.

[6] A. Auslender, *Optimisation: Mthodes Numriques.* Masson, Paris, 1976.

[7] D. Aussel and J. Dutta, *On gap functions for multivalued Stampacchia variational inequalities,* J. Optim. Theory Appl., **149** (2011), 513527.

[8] D. Aussel, R. Gupta and A. Mehrab, *Gap functions and error bounds for inverse quasi-variational inequality problems,* J. Math. Anal. Appl., **407** (2013), 270280.

[9] G. Bigi and M. Passacantando, *D-gap functions and descent techniques for solving equilibrium problems,* J. Glob. Optim., **62** (2015), 183203.

[10] S.S. Chang, Salahuddin, M. Liu, X.R. Wang and J.F. Tang, *Error bounds for generalized vector inverse quasi-variational inequality problems with point to set mappings,* AIMS. Mathematics. **6**(2) (2020), 1800-1815.

[11] S.S. Chang, Salahuddin, L. Wang and Z.L. Ma, *Error bounds for mixed set valued vector inverse quasi-variational inequalities,* J. Inequal. Appl., **2020** (2020):160.doi.org/10.1186/s13660.020.02424-7.

[12] S.S. Chang, Salahuddin, X.R. Wang, L.C. Zhao and J.F. Tang, *Generalized* (η, g, φ)*-mixed vector equilibrium problems in fuzzy events,* Commun. Appl. Nonlinear Anal., **27**(4) (2020), 49-64.

[13] F.H. Clarke, *Optimization and Nonsmooth Analysis,* Wiley, New York , 1983.

[14] Z. Denkowski, S. Migórski and N.S. Papageorgiou, *An Introduction to Nonlinear Analysis: Theory,* Kluwer Academic, Plenum, Dordrecht, New York, 2003.

[15] Z. Denkowski, S. Migórski and N.S. Papageorgiou, *An Introduction to Nonlinear Analysis: Applications,* Kluwer Academic, Plenum, Dordrecht, New York, 2003.

[16] J.H. Fan and X.G. Wang, *Gap functions and global error bounds for set-valued variational inequalities,* J. Comput. Appl. Math., **233** (2010), 29562965.

[17] M. Fukushima, *Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems,* Math. Program., **53** (1992), 99110.

[18] M. Fukushima, *A class of gap functions for quasi-variational inequality problems.* J. Ind. Manag. Optim., **3** (2007), 165171.

[19] D.W. Hearn, *The gap function of a convex program,* Oper. Res. Lett., **1** (1982), 6771.

[20] N.V. Hung, S. Migórski, V.M. Tam and S. Zeng, *Gap functions and error bounds for variational-hemivariational inequalities,* Acta. Appl. Math., **169** (2020), 691-709, doi.org 10.1007/s10440-020-00319-9.

[21] N.V. Hung, V.M. Tam, K. Elisabeth and J.C. Yao, *Existence of solutions and algorithm for generalized vector quasi-complementarity problems with application to traffic network problems,* J. Nonlinear Convex Anal., **20** (2019), 17511775.

[22] N.V. Hung, V.M. Tam and A. Pitea, *Global error bounds for mixed quasi-hemivariational inequality problems on Hadamard manifolds,* Optimization **(2020)**, doi.org.10.1080.02331934.2020.1718126.

[23] J.K. Kim, M. Alesemi and Salahuddin, *Convergence theorem of relaxed quasimonotone variational inequality problems,* Nonlinear and Convex Anal., **22**(12) (2021), 2672-2678.

[24] J.K. Kim, Salahuddin and A.H. Dar, *Existence solution for the generalized relaxed pseudomonotone variational inequalitie,* Nonlinear Funct. Anal. Appl., **25**(1) (2020), 25-34.

[25] B.S. Lee and Salahuddin, *Solution for general class of hemivariational like inequality systems,* J. Nonlinear Convex Anal., **16**(1) (2015), 141-150.

[26] B.S. Lee and Salahuddin, *Minty lemma for inverted vector variational inequalities,* Optimization. **66**(3) (2017), 351-359.

[27] S. Migórski, A.A. Khan and S.D. Zeng, *Inverse problems for nonlinear quasi-hemivariational inequalities with application to mixed boundary value problems.* Inverse Problems, **36** (2020), ID: 024008.

[28] S. Migórski, A, Ochal and M. Sofonea, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems,* Adv. Mechanics Math., **26**, Springer, New York, 2013.

[29] S. Migórski, A. Ochal and M. Sofonea, *A class of variational hemivariational inequalities in reexive Banach spaces,* J. Elast., **127** (2017), 151178.

[30] S. Migórski and S.D. Zeng, *Mixed variational inequalities driven by fractional evolution equations,* Acta Math. Sci., **39** (2019), 461468.

[31] Z. Naniewicz and P.D. Panagiotopoulos, *Mathematical Theory of Hemivariational Inequalities and Applications,* Marcel Dekker, New York, 1995.

[32] P.D. Panagiotopoulos, *Hemivariational Inequalities,* Appl. Mechanics and Eng., Springer, Berlin, 1993.

[33] P.D. Panagiotopoulos, *Nonconvex problems of semipermeable media and related topics,* Z. Angew. Math. Mech., **65** (1985), 2936.

[34] Salahuddin, *Regularized equilibrium problems in Banach spaces,* Korean J. Math., **24**(1) (2016), 51-63.

[35] M. Sofonea and S. Migórski, *Variational Hemivariational Inequalities with Applications,* Pure and Appl. Math. Chapman and Hall/CRC Press, Boca Raton, London, 2018.

[36] M. Sofonea and S. Migórski, *A class of history-dependent variational hemivariational inequalities,* Nonlinear Differ. Equ. Appl., **23**:38 (2016), doi.org 10.1007.s00030-016-0391-0.

[37] M.V. Solodov, *Merit functions and error bounds for generalized variational inequalities,* J. Math. Anal. Appl., **287** (2003), 405414.

[38] G. Stampacchia, *Variational Inequalities, in Theory and Applications of Monotone Operators,* Proc. the NATO Advanced Study Institute, Venice, Italy, (EdizioniOdersi, Gubbio, Italy) (1968). 102-192.

[39] N. Yamashita and M. Fukushima, *Equivalent unconstrained minimization and global error bounds for vari- ational inequality problems,* SIAM J. Control Optim., **35** (1997), 273284.

[40] B. Zeng, Z. Liu and S. Migórski, *On convergence of solutions to variational hemivariational inequalities,* Z. Angew. Math. Phys., **69** (2018): 87, doi.org 10.1007.s00033-018-0980-3.

[41] B. Zeng and S. Migórski, *Variationalhemivariational inverse problems for unilateral frictional contact,* Appl. Anal., **99**(2) (2020), 293-312.