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GENERALIZED BI-QUASI-VARIATIONAL INEQUALITIES FOR QUASI-PSEUDO-MONOTONE TYPE III OPERATORS ON COMPACT SETS

Mohammad S. R. Chowdhury[1] **and Liliana Guran**[2]

[1]Department of Mathematics and Statistics, Faculty of Science,
The University of Lahore, Lahore, Pakistan
e-mails: `msrchowdhury@hotmail.com; showkat.rahim@math.uol.edu.pk`

[2]Faculty of Business, Babes-Bolyai University,
Cluj-Napoca, Romania
e-mails: `liliana.guran@ubbcluj.ro`

Abstract. A new type of more general form of variational inequalities for quasi-pseudo-monotone type III and strong quasi-pseudo-monotone type III operators has been obtained on compact domains in locally convex Hausdorff topological vector spaces. These more general forms of variational inequalities for the above types of operators used the more general form of minimax inequality by Chowdhury and Tan in [3] as the main tool to derive them. Our new results established in this paper should have potential applications in nonlinear analysis and related applications, e.g., see Aubin [1], Yuan [11] and references wherein.

1. Introduction

In 1989, Shih and Tan derived advanced form of variational inequalities in [9]. Now, for these advanced form of variational inequalities, we have derived some general theorems for quasi-pseudo-monotone type III operators and strong quasi-pseudo-monotone type III operators on comact domains in topological vector spaces(TVS). We now state quasi-pseudo-monotone type III operators and strong quasi-pseudo-mnotone type III operators below:

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[0]Corresponding author: Mohammad S. R. Chowdhury(`msrchowdhury@hotmail.com`).

Definition 1.1. Let G be a TVS, A be a nonempty subset of G and H be a TVS over Ψ. Let $\langle\cdot\,,\,\cdot\rangle : H \times G \to \Psi$ be a bilinear functional. Let us define three mappings as follows:
(1) $g : A \to \mathbb{R}$,
(2) $P : A \to 2^H$ and
(3) $J : A \to 2^H$.
Then J is said to be an g-quasi-pseudo-monotone (respectively, Sg-quasi-pseudo-monotone) type III operator if for each $b \in A$ and every net $\{b_\beta\}_{\beta\in\Gamma}$ in A converging to b (respectively, weakly to b) with

$$\limsup_{\beta}\left[\inf_{f\in P(b)}\inf_{u\in J(b_\beta)} Re\langle f-u, b_\beta - a\rangle + g(b_\beta) - g(a)\right] \leq 0,$$

we have

$$\begin{aligned}&\limsup\nolimits_{\beta}\left[\inf\nolimits_{f\in P(b)}\inf\nolimits_{u\in J(b_\beta)} Re\langle f-u, b_\beta - b\rangle + g(y_\beta) - g(b)\right]\\ &\geq \inf\nolimits_{f\in P(a)}\inf\nolimits_{w\in J(b)} Re\langle f-w, b-a\rangle + g(b) - g(a) \quad \text{for all } \ a\in A.\end{aligned}$$

J is said to be a quasi-pseudo-monotone (respectively, strong quasi-pseudo-monotone) type III operator if J is an g-quasi-pseudo-monotone (respectively, Sg-quasi-pseudo-monotone) type III operator with $g \equiv 0$.

We state and prove below a proposition which will provide examples of set-valued quasi-pseudo-monotone type III (respectively, a strong quasi-pseudo-monotone type III) operators:

Proposition 1.2. *Suppose that A is a subset of a topological vector space (TVS) G which is both nonempty and compact. Let $J : A \to 2^{G^*}$ be an upper semi-continuous mapping where we assume the relative weak topology on A and we assume the strong topology on G^* such that each $J(a)$ is a subset of G^* which is compact in the original topology of G^*. Let $P : A \to G^*$ be any single-valued map. Then J is both quasi-pseudo-monotone type III and strong quasi-pseudo-monotone type III operator.*

Proof. Let $a, b \in A$ and let $\{b_\beta\}_{\alpha\in\Gamma}$ be a net in A which converges to b (respec., $b_\beta \to b$ in the weak topology) with

$$\limsup_{\beta}\left[\inf_{u\in J(b_\beta)} Re\langle P(b) - u, b_\beta - a\rangle\right] \leq 0. \tag{1.1}$$

Since A is compact and J is upper semi-continuous on A with relative weak topology on A and we consider strong topology on G^*, then for all $\beta \in \Gamma$ there exists $u_\beta \in J(b_\beta)$ such that

$$\lim_\beta \left[\inf_{w\in J(b_\beta)} Re\langle P(b) - w, b_\beta - a\rangle \right] = \lim_\beta Re\langle P(b) - u_\beta, b_\beta - a\rangle$$
$$= Re\langle P(b) - u, b - a\rangle,$$

where $u = \lim_\beta u_\beta$ and $u \in J(b)$. Then

$$\begin{aligned}
&\limsup_\beta \left[\inf_{u'\in J(b_\beta)} Re\langle P(b) - u', b_\beta - a\rangle \right] \\
&= \lim_\beta \left[\inf_{w\in J(b_\beta)} Re\langle P(b) - w, b_\beta - a\rangle \right] \\
&= Re\langle P(b) - u, b - a\rangle \\
&\geq \inf_{w\in J(b)} Re\langle P(b) - w, b - a\rangle.
\end{aligned}$$

Hence , we have

$$\inf_{w\in J(b)} Re\langle P(b) - w, b - a\rangle \leq 0. \tag{1.2}$$

by (1.1). Again,

$$\begin{aligned}
&\limsup_\beta \left[\inf_{u\in J(b_\beta)} Re\langle P(b) - u, b_\beta - b\rangle \right] \\
&= \lim_\beta Re\langle P(b) - w_\beta, b_\beta - b\rangle] \\
&= Re\langle P(b) - w, b - b\rangle \\
&= 0, \qquad (1.3)
\end{aligned}$$

where $w = \lim_\beta w_\beta$ and $w \in J(b)$. Therefore, by (1.2) and (1.3) we have

$$\begin{aligned}
&\limsup_\beta \left[\inf_{u\in J(b_\beta)} Re\langle P(b) - u, b_\beta - b\rangle \right] \\
&= 0 \\
&\geq \inf_{w\in J(b)} Re\langle P(b) - w, b - a\rangle.
\end{aligned}$$

Consequently, J is both quasi-pseudo-monotone type III and strong quasi-pseudo-monotone type III operators. □

Chowdhury and Tan derived in [3] an advanced form of minimax inequality of Ky Fan [6] which will be used mainly to obtain our main results on quasi-pseudo-monotone type III operators.

2. Preliminaries

The following is a definition of quasi-pseudo-monotone type II and strongly quasi-pseudo-monotone type II operators in [4]:

Definition 2.1. Let E be a topological vector space, X be a non-empty subset of E and F be a topological vector space over Φ. Let $\langle \cdot \,, \cdot \rangle : F \times E \to \Phi$ be a bilinear functional. Suppose we have the following three maps:
(i) $h : X \to \mathbb{R}$,
(ii) $M : X \to 2^F$ and
(iii) $T : X \to 2^F$.

Then T is said to be an h-quasi-pseudo-monotone (respectively, strongly h-quasi-pseudo-monotone) type II operator if for each $y \in X$ and every net $\{y_\alpha\}_{\alpha \in \Gamma}$ in X converging to y (respectively, weakly to y) with

$$\limsup_{\alpha} \left[\inf_{f \in M(y_\alpha)} \inf_{u \in T(y_\alpha)} \Re\langle f - u, y_\alpha - y \rangle + h(y_\alpha) - h(y) \right] \leq 0,$$

we have

$$\begin{aligned} &\limsup_\alpha \left[\inf_{f \in M(y_\alpha)} \inf_{u \in T(y_\alpha)} \Re\langle f - u, y_\alpha - x\rangle + h(y_\alpha) - h(x)\right] \\ &\geq \inf_{f \in M(y)} \inf_{w \in T(y)} \Re\langle f - w, y - x\rangle + h(y) - h(x) \quad \text{for all} \;\; x \in X. \end{aligned}$$

T is said to be a quasi-pseudo-monotone (respectively, strongly quasi-pseudo-monotone) type II operator if T is an h-quasi-pseudo-monotone (respectively, strongly h-quasi-pseudo-monotone) type II operator with $h \equiv 0$.

The following is an example of quasi-pseudo-monotone type II operators in [4]:

Example 2.2. Consider $X = [-1, 1]$ and $E = \mathbb{R}$. Then $E^* = \mathbb{R}$. Let $M : X \to 2^{\mathbb{R}}$ be defined by

$$M(x) = \begin{cases} [0, 2x], & \text{if } x \geq 0; \\ [2x, 0], & \text{if } x < 0. \end{cases}$$

Again, let $T : X \to 2^{\mathbb{R}}$ be defined by

$$T(x) = \begin{cases} \{1, 3\}, & \text{if } x < 1, \\ \{1, 2, 3\}, & \text{if } x = 1. \end{cases}$$

Then M is lower semi-continuous and T is upper semi-continuous. It can be shown that T becomes a quasi-pseudo-monotone type II operator on $X = [-1, 1]$.

The following result is Lemma 3 of Takahashi in [10]:

Lemma 2.3. *Let X and Y be topological spaces, $f : X \to \mathbb{R}$ be non-negative and continuous and $g : Y \to \mathbb{R}$ be lower semi-continuous. Then the map $F : X \times Y \to \mathbb{R}$, defined by $F(x, y) = f(x)g(y)$ for all $(x, y) \in X \times Y$, is lower semi-continuous.*

We shall need the following Kneser's minimax theorem given in [7, pp.2418-2420]:

Theorem 2.4. *Let X be a non-empty convex subset of a vector space and Y be a nonempty compact convex subset of a Hausdorff topological vector space. Suppose that f is a real-valued function on $X \times Y$ such that for each fixed $x \in X$, the map $y \mapsto f(x, y)$, i.e., $f(x, \cdot)$ is lower semi-continuous and convex on Y and for each fixed $y \in Y$, the map $x \mapsto f(x, y)$, i.e., $f(\cdot, y)$ is concave on X. Then*

$$\min_{y \in Y} \sup_{x \in X} f(x,\, y) = \sup_{x \in X} \min_{y \in Y} f(x,\, y).$$

3. Existence theorems for more advanced form of BQVI of Quasi-Pseudo-Monotone Type III and Strong Quasi-Pseudo-Monotone Type operators

Quasi-pseudo-monotone type III (respectively, strong quasi-pseudo-monotone type III) operators with compact domain will be used in deriving some new results for generalized bi-quasi-variational inequalities (GBQVI) in locally convex topological vector spaces (LCTVS). Certainly these extends and generalizes similar results in [9].

Use of the Lemma 3.1 in [5] is necessary for our research finding below:

Theorem 3.1. *Suppose G is a locally convex Hausdorff topological vector spaces over Ψ, A is a subset of G which is convex, nonempty and compact and H a topological vector space (TVS) with whose field is Ψ and which is a T_2 topology. We assume that the bilinear functional $\langle \cdot\,, \cdot \rangle : H \times G \to \Psi$ is continuous. Assume also the following:*

(1) *$L : A \to 2^A$ is upper semi-continuous with every $L(a)$ a subset of A which is a convex and closed subset of A;*
(2) *$g : A \to \mathbb{R}$ is a single-valued continuous function which is also a convex function;*
(3) *$J : A \to 2^H$ is a g-quasi-pseudo-monotone type III (respectively, sg-quasi-pseudo-monotone type III) operator and is upper semi-continuous such that each $J(a)$ is a compact set (respectively, a compact set in weak topology) and a convex set and $J(A)$ is also bounded subset of H in the original topology of H;*

(4) *$P : A \to 2^H$ is a single valued mapping for all $a \in A$ which is both a continuous and a linear mapping also;*

(5) *$\Sigma = \{b \in A : \sup_{a\in L(b)} \inf_{u\in J(b)} Re\langle P(b) - u, b - a\rangle + g(b) - g(a)) > 0\}$ is an open subset of A.*

In addition, we assume the following conditions are satisfied:

(i) *for each $B \in \mathcal{F}(A)$ (where $\mathcal{F}(A)$ is the set of all non-empty subsets of A) and each $a, b \in co(B)$ and any net $\{b_\beta\}_{\alpha\in\Gamma}$ in A converging to y we have $\limsup_\beta[\inf_{u\in J(b_\beta)} Re\langle P(b) - u, b_\beta - b\rangle + g(b_\beta) - g(b)] \le 0$ whenever $\limsup_\beta[\inf_{u\in J(b_\beta)} Re\langle P(b) - u, b_\beta - a\rangle + g(b_\beta) - g(a)] \le 0$, and*

(ii) *$\limsup_\beta[\inf_{w\in J(b_\beta)} Re\langle P(b) - w, b_\beta - a\rangle + g(b_\beta) - g(a)]$ $\ge \inf_{w\in J(b)} Re\langle P(b) - w, b - a\rangle + g(b) - g(a)$ whenever $\limsup_\beta[\inf_{w\in J(b_\beta)} Re\langle P(b) - w, b_\beta - b\rangle + g(b_\beta) - g(b)]$ $\ge \inf_{w\in J(b)} Re\langle P(b) - w, b - a\rangle + g(b) - g(a)$ for all $a \in A$.*

Then there exists a point $\hat{b} \in A$ such that

(i)' *$\hat{b} \in S(\hat{b})$ and*

(ii)' *there exist a point $\hat{w} \in J(\hat{b})$ with $Re\langle P(\hat{b}) - \hat{w}, \hat{b} - a\rangle \le g(a) - g(\hat{b})$ for all $a \in S(\hat{b})$.*

Moreover, it is not necessary that G will be a locally convex (LC) space if $L(a) = A$ for all $a \in A$, and if $J \equiv 0$, the continuity assumption on $\langle\cdot\,,\cdot\rangle$ can be weakened to the assumption that for each $f \in H$, the map $x \mapsto \langle f, x\rangle$ is continuous (respectively, weakly continuous) on A.

Proof. The main thing here is to show that there exists a point $\hat{b} \in A$ such that $\hat{b} \in S(\hat{b})$ and

$$\sup_{a\in S(\hat{b})} \left[\inf_{u\in J(\hat{b})} Re\langle P(\hat{b}) - u, \hat{b} - a\rangle + g(\hat{b}) - g(a)\right] \le 0.$$

If we do not agree with the above consequence, then for each $b \in A$, either $b \notin L(b)$ or there exists $a \in L(b)$ such that

$$\inf_{u\in J(b)} Re\langle P(b) - u, b - a\rangle + g(b) - g(a) > 0;$$

that is, for each $b \in A$, either $b \notin L(b)$ or $b \in \Sigma$. Suppose that $b \notin L(b)$. Then we can use an advanced form of Hahn Banach Theorem in functional analysis for locally convex Hausdorff topological vector spaces to derive that there exists $p \in G^*$ such that $Re\langle p, b\rangle - \sup_{a\in L(b)} Re\langle p, a\rangle > 0$.

Let $\gamma(b) = \sup_{a\in L(b)} \inf_{u\in J(b)} Re\langle P(b) - u, b - a\rangle + g(b) - g(a)$ and suppose that

$$\Sigma = W_0 := \{b \in A \mid \gamma(b) > 0\}$$

and for each $h \in G^*$, set

$$W_h := \left\{ b \in A : Re\langle h, b\rangle - \sup_{a \in L(b)} Re\langle h, a\rangle > 0 \right\}.$$

So, we obtain $A = W_0 \cup \bigcup_{h \in G^*} W_h$. Clearly, W_h is an open subset of A for each h, by Lemma 2.2 in [10] and by our assumption, W_0 is an open subset of A by hypothesis. So, we see that $\{W_0, W_h : h \in G^*\}$ is a covering for A which is also open. But A is compact, so, there exists $h_1, h_2, \cdots, h_n \in G^*$ such that $A = W_0 \cup \bigcup_{i=1}^{n} V_{h_i}$. To make it more convenient, we write $W_i = W_{h_i}$ for $i = 1, 2, \cdots, n$. Suppose that $\{\delta_0, \delta_1, \cdots, \delta_n\}$ is a partition of unity which is continuous on A and which also subordinates the covering $\{W_0, W_1, \cdots, W_n\}$. Then $\delta_0, \delta_1, \cdots, \delta_n$ are functions on A which are real valued and non negative and are also continuous such that δ_i becomes zero on $A \setminus W_i$, for all $i = 0, 1, \cdots, n$ and $\sum_{i=0}^{n} \delta_i(a) = 1$ for all $a \in A$.

Define $\psi : A \times A \to \mathbb{R}$ by

$$\psi(a, b) = \delta_0(b) \left[\inf_{u \in J(b)} Re\langle P(b) - u, b - a\rangle + g(b) - g(a) \right] + \sum_{i=1}^{n} \delta_i(b) Re\langle p_i, b - a\rangle$$

for each $x, b \in A$. Then following derivations are obtained:

(1) Because G has a topology which is T_2, for all $B \in \mathcal{F}(A)$ and for all $a \in co(B)$ arbitrarily chosen, the mapping

$$b \mapsto \inf_{u \in J(b)} Re\langle P(b) - u, b - a\rangle + g(b) - g(a)$$

is lower semi-continuous (lsc) (respectively, weakly lower semi-continuous (wlsc)) on $co(B)$ by the Lemma 3.1 in [5] and hence we conclude that

$$b \mapsto \delta_0(b) \left[\inf_{u \in J(b)} Re\langle P(b) - w, b - a\rangle + g(b) - g(a) \right]$$

is lower semi-continuous (respectively, weakly lower semi-continuous (wlsc)) on $co(B)$ by Lemma 2.2 in [10] and the Lemma 3 in [9]. Also for each fixed $a \in A$,

$$b \mapsto \Sigma_{i=1}^{n} \delta_i(b) Re\langle p_i, b - a\rangle$$

is a function on A which is also continuous. Consequently, for all $B \in \mathcal{F}(A)$ and $a \in co(B)$ arbitrarily chosen, the function $b \mapsto \psi(a, b)$ is lsc (respectively, wlsc) on $co(B)$.

(2) For each $B \in \mathcal{F}(A)$ and for each $b \in co(B)$, $\min_{a \in A} \psi(a, b) \leq 0$. Suppose this is not true. Then for some $B = \{a_1, a_2, \cdots, a_n\} \in \mathcal{F}(A)$ and some

$b \in co(B)$ (say $b = \sum_{i=1}^{n} \lambda_i a_i$ where $\lambda_1, \lambda_2, \cdots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$), we have $\min_{1 \leq i \leq n} \psi(a_i, b) > 0$. Then for each $i = 1, 2, \cdots, n$,

$$\delta_0(b)\left[\inf_{u \in J(b)} Re\langle P(b) - u, b - a_i\rangle + g(b) - g(a_i)\right] + \sum_{j=1}^{n} \beta_j(b) Re\langle p_j, b - a_i\rangle > 0,$$

so that

$$\begin{aligned}
0 &= \psi(b, b) \\
&= \delta_0(b)\Big[\inf_{u \in J(b)} Re\langle P(\sum_{i=1}^{n} \lambda_i a_i) - u, b - \sum_{i=1}^{n} \lambda_i a_i\rangle \\
&\quad + g(b) - g(\sum_{i=1}^{n} \lambda_i a_i)\Big] + \sum_{j=1}^{n} \beta_j(b) Re\langle p_j, b - \sum_{i=1}^{n} \lambda_i a_i\rangle \\
&\geq \sum_{i=1}^{n} \lambda_i(\delta_0(b)\Big[\inf_{u \in J(b)} Re\langle P(a_i) - u, b - a_i\rangle \\
&\quad + g(b) - g(a_i)\Big] + \sum_{j=1}^{n} \beta_j(b) Re\langle p_j, b - a_i\rangle) \\
&> 0.
\end{aligned}$$

Hence we end up with a contrary result.

(3) Assume that $B \in \mathcal{F}(A)$, $a, b \in co(B)$ and $\{b_\beta\}_{\alpha \in \Gamma}$ is a net in A converging to b (respectively, weakly converging to b) with $\psi(ta + (1-t)b, b_\beta) \leq 0$ for all $\beta \in \Gamma$ and all $t \in [0, 1]$.

Case 1: $\delta_0(b) = 0$.

We see that $\delta_0(b_\beta) \geq 0$ for each $\alpha \in \Gamma$ and $\delta_0(b_\beta) \to 0$. Since $J(A)$ is strongly bounded and $\{b_\beta\}_{\alpha \in \Gamma}$ is a bounded net. So we conclude that

$$\limsup_{\beta} \left[\delta_0(b_\beta)(\min_{u \in J(b_\beta)} Re\langle P(b_\beta) - u, b_\beta - a\rangle + g(b_\beta) - g(a))\right] = 0. \tag{3.1}$$

Moreover,

$$\delta_0(b)\left[\min_{u \in J(b)} Re\langle P(b) - u, y - a\rangle + g(b) - g(a)\right] = 0.$$

Therefore,

$$\limsup_{\beta} \left[\delta_0(b_\beta)(\min_{u\in J(b_\beta)} Re\langle P(b_\beta)-u, b_\beta - a\rangle + g(b_\beta) - g(a))\right]$$
$$+\sum_{i=1}^{n}\delta_i(b)Re\langle p_i, b-a\rangle$$
$$=\sum_{i=1}^{n}\delta_i(b)Re\langle p_i, b-a\rangle$$
$$=\delta_0(b)\left[\min_{u\in J(b)} Re\langle P(b)-u, b-a\rangle + g(b)-g(a)\right]$$
$$+\sum_{i=1}^{n}\delta_i(b)Re\langle p_i, b-a\rangle. \tag{3.2}$$

If $t=1$ we have $\psi(a,b_\beta)\le 0$ for all $\beta\in\Gamma$, that is,

$$\delta_0(b_\beta)\left[\min_{u\in J(b_\beta)} Re\langle P(b_\beta)-u, b_\beta - a\rangle + g(b_\beta) - g(a)\right]$$
$$+\sum_{i=1}^{n}\delta_i(b_\beta)Re\langle p_i, b_\beta-a\rangle$$
$$\le 0, \quad \forall\alpha\in\Gamma. \tag{3.3}$$

Consequently,

$$\limsup_{\beta} \left[\delta_0(b_\beta)\min_{u\in J(b_\beta)} Re\langle P(b_\beta)-u, b_\beta - a\rangle + g(b_\beta) - g(a)\right]$$
$$+\liminf_{\beta}\left[\sum_{i=1}^{n}\delta_i(b_\beta)Re\langle p_i, b_\beta-a\rangle\right]$$
$$\le \limsup_{\beta} \left[\delta_0(b_\beta)\min_{u\in J(b_\beta)} Re\langle P(b_\beta)-u, b_\beta - a\rangle + g(b_\beta) - g(a)\right.$$
$$\left.+\sum_{i=1}^{n}\delta_i(b_\beta)Re\langle p_i, b_\beta-a\rangle\right]$$
$$\le 0.$$

Therefore,

$$\begin{aligned}&\limsup_{\beta}\left[\delta_0(b_\beta)\min_{u\in J(b_\beta)} Re\langle P(b_\beta)-u, b_\beta - a\rangle + g(b_\beta)-g(a)\right]\\ &\quad+\sum_{i=1}^{n}\delta_i(b)Re\langle p_i, b-a\rangle\\ &\le 0. \end{aligned} \tag{3.4}$$

So, by (3.2) and (3.4), we derive that $\Psi(a,b)\le 0$.

Case 2: $\delta_0(b)>0$.

As $\delta_0(b_\beta)\to\delta_0(b)$, there exists $\lambda\in\Gamma$ such that $\delta_0(b_\beta)>0$ for all $\beta\ge\lambda$. If $t=0$, we have $\psi(b,b_\beta)\le 0$ for all $\beta\in\Gamma$, that is, for all $\alpha\in\Gamma$,

$$\delta_0(b_\beta)\left[\inf_{u\in J(b_\beta)} Re\langle P(b_\beta)-u, b_\beta-b\rangle + g(b_\beta)-g(b)\right]+\sum_{i=1}^{n}\delta_i(b_\beta)Re\langle p_i, b_\beta-b\rangle\le 0,$$

Then

$$\begin{aligned}&\limsup_{\beta}\Big[\delta_0(b_\beta)(\inf_{u\in J(b_\beta)} Re\langle P(b_\beta)-u, b_\beta-b\rangle+g(b_\beta)-g(b))\\ &\quad+\sum_{i=1}^{n}\delta_i(b_\beta)Re\langle p_i, b_\beta-b\rangle\Big]\\ &\le 0.\end{aligned}\tag{3.5}$$

So, we have

$$\begin{aligned}&\limsup_{\beta}\left[\delta_0(b_\beta)(\inf_{u\in J(b_\beta)} Re\langle P(b_\beta)-u, b_\beta-b\rangle+g(b_\beta)-g(b))\right]\\ &\quad+\liminf_{\beta}\left[\sum_{i=1}^{n}\delta_i(b_\beta)Re\langle p_i, b_\beta-b\rangle\right]\\ &\le\limsup_{\beta}\Big[\delta_0(b_\beta)(\inf_{u\in J(b_\beta)} Re\langle P(b_\beta)-u, b_\beta-b\rangle+g(b_\beta)-g(b))\\ &\quad+\sum_{i=1}^{n}\delta_i(b_\beta)Re\langle p_i, b_\beta-b\rangle\Big]\\ &\le 0.\end{aligned}$$

Since $\liminf_\beta[\sum_{i=1}^{n}\delta_i(b_\beta)Re\langle p_i, b_\beta-b\rangle]=0$, we have

$$\limsup_{\beta}\left[\delta_0(b_\beta)(\min_{u\in J(b_\beta)} Re\langle P(b_\beta)-u, b_\beta-y\rangle+g(b_\beta)-g(b))\right]\le 0. \tag{3.6}$$

As $\delta_0(b_\beta)>0$ for all $\alpha\ge\lambda$, we conclude that

$$\delta_0(b)\limsup_{\beta}\left[\min_{u\in J(b_\beta)} Re\langle P(b_\beta)-u, b_\beta-y\rangle + g(b_\beta)-g(b))\right]$$
$$=\limsup_{\beta}\left[\delta_0(b_\beta)(\min_{u\in J(b_\beta)} Re\langle P(b_\beta)-u, b_\beta-y\rangle + g(b_\beta)-g(b))\right]. \qquad (3.7)$$

As $\delta_0(b)>0$, by (3.6) and (3.7) we derive that

$$\limsup_{\beta}\left[\min_{u\in J(b_\beta)} Re\langle P(b_\beta)-u, b_\beta-y\rangle + g(b_\beta)-g(b))\right]\leq 0.$$

So, by condition (i), we have,

$$\limsup_{\beta}\left[\min_{u\in J(b_\beta)} Re\langle P(b)-u, b_\beta-x\rangle + g(b_\beta)-g(a))\right]\leq 0.$$

As J is a g-quasi-pseudo-monotone type III (respectively, sg-quasi-pseudo-monotone type III) operator, we have

$$\limsup_{\beta}[\min_{u\in J(b_\beta)} Re\langle P(b)-u, b_\beta-b\rangle + g(b_\beta)-g(b)]$$
$$\geq \min_{w\in J(b)} Re\langle P(a)-w, b-a\rangle + g(b)-g(a), \qquad (3.8)$$

for all $a\in A$. So, by assumption (ii), we have,

$$[\limsup_{\beta}(\min_{u\in J(b_\beta)} Re\langle P(b)-u, b_\beta-a\rangle + g(b_\beta)-g(a))]$$
$$\geq [\min_{w\in J(b)} Re\langle P(b)-w, b-a\rangle + g(b)-g(a)],$$

whenever (3.8) is true.

As $\delta_0(b)>0$, we can derive that

$$\delta_0(b)[\limsup_{\beta}(\min_{u\in J(b_\beta)} Re\langle P(b)-u, b_\beta-a\rangle + g(b_\beta)-g(a))]$$
$$\geq \delta_0(b)[\min_{w\in J(b)} Re\langle P(b)-w, b-a\rangle + g(b)-g(a)].$$

So,

$$\delta_0(b)[\limsup_{\beta}(\min_{u\in J(b_\beta)} Re\langle P(b)-u, b_\beta-a\rangle + g(b_\beta)-g(a))]$$
$$+\sum_{i=1}^{n}\delta_i(b)Re\langle p_i, b-a\rangle$$
$$\geq \delta_0(b)[\min_{w\in J(b)} Re\langle P(b)-w, b-a\rangle + g(b)-g(a)]$$
$$+\sum_{i=1}^{n}\delta_i(b)Re\langle p_i, b-a\rangle. \qquad (3.9)$$

If $t = 1$ we have $\psi(a, b_\beta) \leq 0$ for all $\beta \in \Gamma$, that is, for all $\alpha \in \Gamma$,

$$\begin{aligned}\delta_0(b_\beta)&\left[\min_{u\in J(b_\beta)} Re\langle P(b_\beta) - u, b_\beta - x\rangle + g(b_\beta) - g(a)\right]\\ &+ \sum_{i=1}^{n} \delta_i(b_\beta) Re\langle p_i, b_\beta - a\rangle\\ &\leq 0.\end{aligned}$$

Then

$$\begin{aligned}0 &\geq \limsup_\beta [\delta_0(b_\beta) \min_{u\in J(b_\beta)} Re\langle P(b_\beta) - u, b_\beta - a\rangle + g(b_\beta) - g(a)\\ &\quad + \sum_{i=1}^{n} \delta_i(b_\beta) Re\langle p_i, b_\beta - a\rangle]\\ &\geq \limsup_\beta [\delta_0(b_\beta) \min_{u\in J(b_\beta)} Re\langle P(b_\beta) - u, b_\beta - a\rangle + g(b_\beta) - g(a)]\\ &\quad + \liminf_\beta [\sum_{i=1}^{n} \delta_i(b_\beta) Re\langle p_i, b_\beta - a\rangle]\\ &= \delta_0(b)[\limsup_\beta \{\min_{u\in J(b_\beta)} Re\langle P(b_\beta) - u, b_\beta - a\rangle + g(b_\beta) - g(a)\}]\\ &\quad + \sum_{i=1}^{n} \delta_i(b) Re\langle p_i, b - a\rangle\\ &= \delta_0(b)[\limsup_\beta \{\min_{u\in J(b_\beta)} Re\langle P(b) - u, b_\beta - a\rangle + g(b_\beta) - g(a)\}]\\ &\quad + \sum_{i=1}^{n} \delta_i(b) Re\langle p_i, b - a\rangle\\ &\geq \delta_0(b)[\min_{w\in J(b)} Re\langle P(b) - w, b - a\rangle + g(b) - g(a)]\\ &\quad + \sum_{i=1}^{n} \delta_i(b) Re\langle p_i, b - a\rangle. \end{aligned} \tag{3.10}$$

Consequently, $\Psi(a, b) \leq 0$ has been derived.

(4) As A is a compact (respectively, weakly compact) subset of the Hausdorff topological vector space G, it is also closed. Now, if we take $K = A$, then for any $a_0 \in K = A$ we have $\psi(a_0, y) > 0$ for all $b \in A \setminus K (= A \setminus A = \emptyset)$. So, the assumption (4) of Theorem 1.1 in [3] has been verified to be satisfied.

We can complete the last part of this proof by seeing the poof of the first step of Theorem 1 in [4]. Consequently, the following has been derived:

$$\sup_{a\in L(\hat{b})}\left[\inf_{u\in J(\hat{b})} Re\langle P(\hat{b})-u,\hat{b}-a\rangle + g(\hat{b})-g(a)\right]\leq 0.$$

So, if we apply Theorem 2.1 in [7, pp.2418-2420](see also Aubin [1, pp.40-41]) as we did in the third step of Theorem 1 in [4], it can be derived that there exist a point $\hat{w}\in J(\hat{b})$ such that $Re\langle P(\hat{b})-\hat{w},\hat{b}-a\rangle \leq g(a)-g(\hat{b})$ for all $a\in L(\hat{b})$.

Looking into the above proof, we get the conclusion that the need for G to be LC is required when and only when the separation theorem is applied to the case $b\notin L(b)$. Therefore, if $L: A\to 2^A$ is the constant mapping $L(a)=A$ $\forall a\in A$, G is not required to be LC.

In conclusion, we say that if $J\equiv 0$, to derive that for all $a\in A$, $b\mapsto \psi(a,b)$ is lower semi-continuous (respectively, wlsc), Lemma 3.1 in [5] will not be required and the weaker continuity condition on $\langle\cdot\,,\,\cdot\rangle$ that $\forall f\in H$, the map $a\mapsto\langle f,a\rangle$ is continuous (respectively, weakly continuous) on A is sufficient. Consequently, we have completed the proof of this theorem. □

Below, we present the 2nd result of this paper:

Theorem 3.2. *Let G be a locally convex Hausdorff topological vector space over Ψ, A be a nonempty compact convex subset of G and H be a vector space (VS) over Ψ. Let $\langle\cdot\,,\cdot\rangle : H\times G\to\Psi$ be a bilinear functional such that $\langle\cdot\,,\cdot\rangle$ separates points in H and for all $f\in F$ and the map $x\longmapsto\langle f,x\rangle$ be continuous on G. Consider H with the strong topology $\delta\langle H,G\rangle$. Assume that*

(1) *$L: A\to 2^A$ is a continuous map such that each $L(a)$ is closed and convex;*
(2) *$g: A\to\mathbb{R}$ is convex and continuous;*
(3) *$J: A\to 2^H$ is a g-quasi-pseudo-monotone type III (respectively, sg-quasi-peudo-monotone type III) operator and is an upper semi-continuous mapping such that each $J(a)$ is strongly, that is, $\delta\langle H,G\rangle$-compact and convex (respectively, weakly, that is, $\sigma\langle H,G\rangle$-compact and convex) and $J(a)$ is strongly bounded, that is, bounded in the strong topology of H;*
(4) *$P: A\to 2^H$ is a continuous linear map and is therefore single-valued for each $a\in A$; also for each $b\in\Sigma=\{b\in A:\sup_{a\in L(b)}\inf_{u\in J(b)} Re\langle P(b)-u,b-a\rangle+g(b)-g(a)]>0\}$, there exists a point x in $L(b)$ with $\inf_{u\in J(b)} Re\langle P(b)-u,b-a\rangle+g(b)-g(a)>0$.*

Moreover, suppose that the following conditions are satisfied:

(i) *for each* $B \in \mathcal{F}(A)$ *and each* $x, b \in co(B)$ *and any net* $\{b_\beta\}_{\alpha\in\Gamma}$ *in* A *converging to* y *we have*

$$\limsup_\beta \left[\inf_{u\in J(b_\beta)} Re\langle P(b) - u, b_\beta - b\rangle + g(b_\beta) - g(b)\right] \leq 0$$

whenever

$$\limsup_\beta \left[\inf_{u\in J(b_\beta)} Re\langle P(b) - u, b_\beta - a\rangle + g(b_\beta) - g(a)\right] \leq 0$$

and

(ii) $\limsup_\beta[\inf_{w\in J(b_\beta)} Re\langle P(b) - w, b_\beta - a\rangle + g(b_\beta) - g(a)]$ $\geq \inf_{w\in J(b)} Re\langle P(b) - w, b - a\rangle + g(b) - g(a)$ *whenever* $\limsup_\beta[\inf_{w\in J(b_\beta)} Re\langle P(b) - w, b_\beta - b\rangle + g(b_\beta) - g(b)]$ $\geq \inf_{w\in J(b)} Re\langle P(b) - w, b - a\rangle + g(b) - g(a)$ *for all* $a \in A$.

Then there exists a point $\hat{b} \in A$ *such that*

(i)' $\hat{b} \in S(\hat{b})$ *and*

(ii)' *there exist a point* $\hat{w} \in J(\hat{b})$ *with* $Re\langle P(\hat{b}) - \hat{w}, \hat{b} - a\rangle \leq g(a) - g(\hat{b})$ *for all* $a \in S(\hat{b})$.

Moreover, if $L(a) = A$ *for all* $a \in A$, G *is not required to be locally convex.*

Proof. The proof of the above theorem is similar to the proof of Theorem 2 in [4]. □

Before we end this paper, we would like to point out that the new results in this paper should have potential applications in nonlinear analysis and related applications, e.g., see Aubin [1], Yuan [11] and references wherein.

References

[1] J.P. Aubin, *Applied Functional Analysis*, Wiley-Interscience, New York, 1979.

[2] M.S.R. Chowdhury, *The surjectivity of upper-hemi-continuous and pseudo-monotone type II operators in reflexive Banach Spaces*, Ganit: J. Bangladesh Math. Soc., **20** (2000), 45-53.

[3] M.S.R. Chowdhury and K.-K. Tan, *Generalization of Ky Fan's minimax inequality with applications to generalized variational inequalities for pseudo-monotone operators and fixed point theorems*, J. Math. Anal. Appl., **204** (1996), 910-929.

[4] M.S.R. Chowdhury and K.-K. Tan, *Application of upper hemi-continuous operators on generalized bi-quasi-variational inequalities in locally convex topological vector spaces*, Positivity, **3** (1999), 333-344.

[5] M.S.R. Chowdhury and K.-K. Tan, *Generalized bi-quasi-variational inequalities for quasi-pseudo-monotone type II operators on compact sets*, Cent. Eur. J. Math., **8(1)** (2010), 158-169.

[6] K. Fan, *A minimax inequality and applications*, in "Inequalities, III" (O. Shisha, Ed.), pp.103-113, Academic Press, San Diego, 1972.

[7] H. Kneser, *Sur un theórème fundamental de la théorie des jeux*, C. R. Acad. Sci. Paris, **234** (1952), 2418–2420.

[8] M.-H. Shih and K.-K. Tan, *Generalized quasivariational inequalities in locally convex topological vector spaces*, J. Math. Anal. Appl., **108** (1985), 333–343.

[9] M.-H. Shih and K.-K. Tan, *Generalized bi-quasi-variational inequalities*, J. Math. Anal. Appl., **143** (1989), 66–85.

[10] W. Takahashi, *Nonlinear variational inequalities and fixed point theorems*, J. Math. Soc. Japan, **28** (1976), 168-181.

[11] George X.Z. Yuan, *KKM Theory and Applicatons in Nonlinear Analysis*, Marcel Dekker, Inc., New York, 1999.