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ON FRACTIONAL TIME-VARYING DELAY INTEGRODIFFERENTIAL EQUATIONS WITH MULTI-POINT MULTI-TERM NONLOCAL BOUNDARY CONDITIONS

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Abstract. In this paper, we study the existence and uniqueness of solutions for the fractional time-varying delay integrodifferential equation with multi-point multi-term nonlocal and fractional integral boundary conditions by using fixed point theorems. The fractional derivative considered here is in the Caputo sense. Examples are provided to illustrate the results.

1. Introduction

The mathematical modeling of systems and processes in the fields of physics, aerodynamics, polymer rheology, electrodynamics of complex medium, signal processing, tomography, chaotic dynamics, statistical physics, etc. involves

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derivatives of fractional order. In consequence, the subject of fractional differential equations has been gaining much importance and attention among researchers. Fractional-order models incorporate implicitly memory effects that are difficult to describe using classical calculus. For the basic theory and applications of fractional differential equations, one can refer to the books [5, 16, 26, 33, 35].

In recent years, the theory of existence and uniqueness of solutions for nonlinear fractional boundary value problems has attracted the attention of many researchers. For noteworthy papers dealing with fractional boundary value problems, see [1, 2, 12, 17, 20, 25, 31, 39, 40, 41, 45] and the references therein. Integral boundary conditions have various applications in applied fields such as blood flow problems, chemical engineering, underground water flow, population dynamics, cellular systems, mathematical chemistry and so on. Also many practically important problems lead to multipoint boundary value problems which arise in many areas of applied sciences; for example, in the theory of elastic stability, heat conduction, electric power networks, electric railway systems, telecommunication lines, to name but a few.

Delay is very often encountered in many real world systems and there has been widespread interest in the study of delay differential equations for many years. For example, in modeling HIV infection of $CD4^+$T-cells, the time delay describes the time between infection of $CD4^+$T-cells and emission of viral particles on a cellular level [43]. In biology, the population growth depends not only on the current population size but also on the size some times ago. The same situation occurs in many economic growth models. As a result, in recent years, fractional delay differential equations begin to arouse the attention of many researchers due to their applications in various fields.

Several papers have been devoted to the study of existence results for integer order delay differential equations [4, 6, 7, 19, 30, 38]. But in the case of fractional order delay differential equations comparatively less theory is developed and few of the works can be found in [3, 8, 10, 11, 13, 14, 23, 24, 34, 37, 42, 44] and the references therein.

The existence of mild solutions for a class of abstract fractional integrodifferential equation with nonlocal condition of the form

$$\begin{cases} {}^{C}D^{q}\big(u(t)+e(t,u(t))\big)=Au(t)+f\left(t,u(t),u(\alpha(t)),\int_0^t k(t,s,u(s),u(\beta(s)))\mathrm{d}s\right), \\ u(0)+g(u)=u_0, \end{cases}$$

was studied by Balachandran *et al.* [8], where ${}^{C}D^{q}$ is the Caputo fractional derivative of order $0 < q < 1$, $t \in J = [0,a]$, A is a closed linear unbounded operator in a Banach space X with dense domain $D(A)$, $u_0 \in X$ and f :

$J \times X^3 \to X$, $e : J \times X \to X$, $k : \Delta \times X^2 \to X$, $g : C(J;X) \to X$, $\alpha, \beta : J \to J$ are continuous with $\Delta = \{(t,s) : 0 \le s \le t \le a\}$.

Ouyang *et al.* [34] studied the existence of positive solutions to the nonlinear system of fractional order differential equations with delays of the form

$$\begin{cases} D^{\alpha_i}u_i(t) + f_i\big(t, u_1(\tau_{i1}(t)), ..., u_N(\tau_{iN}(t))\big) = 0,\ t \in (0,1), \\ u_i^{(j)}(0) = 0,\ j = 0,1,...,n_i-2,\ i = 1,2,...,N, \\ u_i^{(n_i-1)}(1) = \eta_i,\ i = 1,2,...,N, \end{cases}$$

where D^{α_i} is the standard Riemann-Liouville fractional derivative of order $\alpha_i \in (n_i - 1, n_i]$ for some integer $n_i > 1$, $\eta_i \ge 0$ for $i = 1,...,N$, $0 \le \tau_{ij}(t) \le t$ for $i,j = 1,2,...,N$, and f_i is a nonlinear function from $[0,1] \times \mathbb{R}_+^N$ to $\mathbb{R}_+ = [0,\infty)$.

The pantograph equation is a kind of delay differential equation where it gets the name from the work of Ockendon and Taylor [32] on the collection of current by the pantograph head of an electric locomotive. It arises in different fields such as electrodynamics, astrophysics, nonlinear dynamical systems, probability theory on algebraic structures, quantum mechanics, cell growth and so on. The pantograph equation is given by

$$\begin{cases} y'(t) = ay(t) + by(\lambda t),\ 0 \le t \le T, \\ y(0) = y_0, \end{cases}$$

where a, b, $y_0 \in C$ and $0 < \lambda < 1$. Properties of the analytic solution of this equation as well as numerical methods have been studied by several authors [27, 28]. Iserles and his coworkers [15, 21, 22] studied extensively on the ordinary pantograph equations. The multi-pantograph equation of the form

$$u'(t) = au(t) + \sum_{i=1}^{m} \mu_i(t)u(\lambda_i t) + f(t),\ t \ge 0,$$

where $a \in C; \mu_i(t)$ and $f(t)$ are analytical functions with $0 < \lambda_m < \lambda_{m-1} < \cdots < \lambda_1 < 1$, has been receiving considerable interest among the researchers.

The generalized nonlinear multi-pantograph equation of the form

$$\begin{cases} u'(t) = f(t, u(t), u(\lambda_1 t), \cdots, u(\lambda_m t)),\ 0 \le t \le T, \\ u(0) = u_0, \end{cases}$$

where $u_0 \in C$, f is a given function and $0 < \lambda_m < \lambda_{m-1} < \cdots < \lambda_1 < 1$, has been discussed by Liu and Li [29], whereas the nonlinear neutral pantograph equation

$$\begin{cases} u'(t) = F(t, u(t), u(\lambda t), u'(\lambda t)),\ t > 0, \\ u(0) = u_0, \end{cases}$$

was investigated by Sezer et al. [36]. Due to its interesting applications in many fields, recently, Balachandran et al. [9] studied the existence results for the fractional model of the pantograph equation

$$\begin{cases} {}^{C}D^{q}u(t) = f(t, u(t), u(\lambda t)), \ t \in J = [0, T], \\ u(0) = u_0, \end{cases}$$

where $0 < q < 1$, $0 < \lambda < 1$, $u_0 \in X$ and the function $f : J \times X \times X \to X$ is continuous.

Motivated by the above works, in this paper, we discuss the existence and uniqueness of solutions for the following nonlinear time-varying delay integrodifferential equation of arbitrary order $q \in (n-1, n]$, $n \in \mathbb{N}$, $n \geq 2$, $t \in J = [0, 1]$, of the form

$$\begin{cases} {}^{C}D^{q}_{0+}x(t) = f\big(t, x(t), x(\alpha_1(t)), x(\alpha_2(t)), ..., x(\alpha_{r_1}(t)), Kx(t)\big), \\ \sum\limits_{i=1}^{m} a_i x(\zeta_i) = 0, \ 0 < \zeta_1 < \zeta_2 < ... < \zeta_m < 1, \\ x'(0) = x''(0) = ... = x^{(n-2)}(0) = 0, \\ x(1) = \sum\limits_{i=1}^{m} b_i (I^{p_i}_{0+}x)(\eta_i), \ 0 < \eta_1 < \eta_2 < ... < \eta_m < 1. \end{cases} \tag{1.1}$$

Here $Kx(t) = \int_0^t k\big(t, s, x(s), x(\beta_1(s)), x(\beta_2(s)), ..., x(\beta_{r_2}(s))\big) \mathrm{d}s$. The functions $f : J \times X^{r_1+2} \to X$, $k : \Omega \times X^{r_2+1} \to X$, $\alpha_i, \beta_j : J \to J$ are continuous with $0 \leq \alpha_i(t), \beta_j(t) \leq t$, $t \in J$, $i = 1, 2, ..., r_1$, $j = 1, 2, ..., r_2$, and a_i, b_i are suitably chosen real constants. $I^{p_i}_{0+}$ is the Riemann-Liouville fractional integral of order $p_i > 0$ for $i = 1, 2, ..., m$. Here $\Omega = \{(t, s) : 0 \leq s \leq t \leq 1\}$. $(X, \|\cdot\|)$ is a Banach space and $Z = C(J, X)$ denotes the Banach space of all continuous functions from $J \to X$ endowed with the topology of uniform convergence with the norm denoted by $\|\cdot\|_C$.

The paper is organized as follows: In Section 2, we introduce definitions, notations and some preliminary notions. In Section 3, we present our main results on existence and uniqueness of solutions using Krasnoselskii's, Schaefer's fixed point theorems and Banach contraction principle respectively. Examples are presented in Section 4 illustrating the applicability of the imposed conditions. No contributions exist in the literature, concerning the existence of solutions to the fractional time-varying delay integrodifferential equation with multi-point multi-term nonlocal and fractional integral boundary conditions in Banach spaces. Hence our aim in this paper is to fill this gap.

2. Preliminaries

In this section, we give some basic definitions, notations and lemmas [26] which will be used throughout the work.

Definition 2.1. The Riemann-Liouville fractional integral of a function $f \in L^1(\mathbb{R}^+)$ of order $q > 0$ is defined by

$$I_{0+}^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds,$$

provided the integral exists.

Definition 2.2. The Caputo fractional derivative of order $n-1 < q \leq n$ is defined by

$${}^C D_{0+}^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^{(n)}(s) ds,$$

where the function $f(t)$ has absolutely continuous derivatives upto order $(n-1)$. In particular, if $0 < q \leq 1$,

$${}^C D_{0+}^q f(t) = \frac{1}{\Gamma(1-q)} \int_0^t \frac{f'(s)}{(t-s)^q} ds,$$

where $f'(s) = Df(s) = \frac{df(s)}{ds}$.

For brevity of notation, I_{0+}^q is taken as I^q and ${}^C D_{0+}^q$ is taken as ${}^C D^q$.

Lemma 2.3. ([26]) *Let $p, q \geq 0$, $f \in L^1[a,b]$. Then $I^p I^q f(t) = I^{p+q} f(t) = I^q I^p f(t)$ and ${}^c D^q I^q f(t) = f(t)$ for all $t \in [a,b]$.*

Definition 2.4. A function $x(t) \in C(J, X)$ is said to be a solution of (1.1) if it satisfies the equation

$${}^C D^q x(t) = f\left(t, x(t), x(\alpha_1(t)), ..., x(\alpha_{r_1}(t)), Kx(t)\right), \; n-1 < q \leq n, \; t \in J,$$

and the boundary conditions

$$\begin{aligned} \sum_{i=1}^m a_i x(\zeta_i) &= 0, \; 0 < \zeta_1 < \zeta_2 < ... < \zeta_m < 1, \\ x'(0) &= x''(0) = ... = x^{(n-2)}(0) = 0, \\ x(1) &= \sum_{i=1}^m b_i (I^{p_i} x)(\eta_i), \; 0 < \eta_1 < \eta_2 < ... < \eta_m < 1. \end{aligned}$$

To study the nonlinear problem (1.1), we first consider the linear problem and obtain its solution.

Lemma 2.5. *For $f(t) \in C(J, X)$, the unique solution of the fractional boundary value problem*

$$\begin{cases} {}^{C}D^{q}x(t) = f(t),\ n-1 < q \le n,\ t \in J, \\ \sum\limits_{i=1}^{m} a_i x(\zeta_i) = 0,\ 0 < \zeta_1 < \zeta_2 < \dots < \zeta_m < 1, \\ x'(0) = x''(0) = \dots = x^{(n-2)}(0) = 0, \\ x(1) = \sum\limits_{i=1}^{m} b_i (I^{p_i} x)(\eta_i),\ 0 < \eta_1 < \eta_2 < \dots < \eta_m < 1, \end{cases} \tag{2.1}$$

is given by

$$x(t) = I^q f(t) + (A_4 - A_3 t^{n-1}) \sum_{i=1}^{m} a_i I^q f(\zeta_i) + (A_2 - A_1 t^{n-1}) \Big\{ \sum_{i=1}^{m} b_i I^{p_i+q} f(\eta_i) - I^q f(1) \Big\}, \tag{2.2}$$

where

$$A_1 = \frac{1}{A} \sum_{i=1}^{m} a_i,\ A_2 = \frac{1}{A} \sum_{i=1}^{m} a_i \zeta_i^{n-1},$$

$$A_3 = \frac{1}{A} \Big(1 - \sum_{i=1}^{m} b_i \frac{\eta_i^{p_i}}{\Gamma(p_i+1)} \Big),\ A_4 = \frac{1}{A} \Big(1 - \sum_{i=1}^{m} b_i \frac{\eta_i^{p_i+n-1}\Gamma(n)}{\Gamma(p_i+n)} \Big)$$

with

$$A = \Big\{ \Big(\sum_{i=1}^{m} a_i \Big) \Big(\sum_{i=1}^{m} b_i \frac{\eta_i^{p_i+n-1}\Gamma(n)}{\Gamma(p_i+n)} - 1 \Big) + \Big(\sum_{i=1}^{m} a_i \zeta_i^{n-1} \Big) \Big(1 - \sum_{i=1}^{m} b_i \frac{\eta_i^{p_i}}{\Gamma(p_i+1)} \Big) \Big\} \neq 0.$$

Proof. For some vector constants $c_0, c_1, \dots, c_{n-1} \in X$, the general solution of (2.1) can be written as [26]

$$x(t) = I^q f(t) + c_0 + c_1 t + \dots + c_{n-1} t^{n-1}. \tag{2.3}$$

Using the boundary conditions $x'(0) = x''(0) = \dots = x^{(n-2)}(0) = 0$ in (2.3), we get

$$c_1 = c_2 = \dots = c_{n-2} = 0.$$

For the boundary condition $\sum\limits_{i=1}^{m} a_i x(\zeta_i) = 0$, we have

$$c_0 \sum_{i=1}^{m} a_i + c_{n-1} \sum_{i=1}^{m} a_i \zeta_i^{n-1} = -\sum_{i=1}^{m} a_i I^q f(\zeta_i). \tag{2.4}$$

Next, using the boundary condition $x(1) = \sum_{i=1}^{m} b_i(I^{p_i}x)(\eta_i)$ and Lemma 2.3 in (2.3), we have

$$c_0\left(1 - \sum_{i=1}^{m} b_i \frac{\eta_i^{p_i}}{\Gamma(p_i+1)}\right) - c_{n-1}\left(\sum_{i=1}^{m} b_i \frac{\eta_i^{p_i+n-1}\Gamma(n)}{\Gamma(p_i+n)} - 1\right)$$
$$= \sum_{i=1}^{m} b_i I^{p_i+q} f(\eta_i) - I^q f(1). \tag{2.5}$$

Solving (2.4) and (2.5) for c_0 and c_{n-1}, we have

$$c_0 = \frac{1}{A}\Bigg[\left(1-\sum_{i=1}^{m} b_i \frac{\eta_i^{p_i+n-1}\Gamma(n)}{\Gamma(p_i+n)}\right)\sum_{i=1}^{m} a_i I^q f(\zeta_i)$$
$$+\left(\sum_{i=1}^{m} a_i \zeta_i^{n-1}\right)\left\{\sum_{i=1}^{m} b_i I^{p_i+q} f(\eta_i) - I^q f(1)\right\}\Bigg],$$
$$c_{n-1} = \frac{-1}{A}\Bigg[\left(1-\sum_{i=1}^{m} b_i \frac{\eta_i^{p_i}}{\Gamma(p_i+1)}\right)\sum_{i=1}^{m} a_i I^q f(\zeta_i)$$
$$+\left(\sum_{i=1}^{m} a_i\right)\left\{\sum_{i=1}^{m} b_i I^{p_i+q} f(\eta_i) - I^q f(1)\right\}\Bigg].$$

Substituting the above values of $c_0, c_1, ..., c_{n-1}$ in (2.3), we get

$$x(t) = I^q f(t) + (A_4 - A_3 t^{n-1}) \sum_{i=1}^{m} a_i I^q f(\zeta_i)$$
$$+ (A_2 - A_1 t^{n-1})\left\{\sum_{i=1}^{m} b_i I^{p_i+q} f(\eta_i) - I^q f(1)\right\}.$$

Hence the proof is completed □

3. Main results

In view of Lemma 2.5, we transform (1.1) as

$$x = F(x), \tag{3.1}$$

where $F : Z \to Z$ is given by

$$(Fx)(t) = \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\left(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s)\right) \mathrm{d}s$$
$$+(A_4 - A_3 t^{n-1}) \sum_{i=1}^{m} a_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{q-1}}{\Gamma(q)}$$
$$\times f\left(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s)\right) \mathrm{d}s + (A_2 - A_1 t^{n-1})$$
$$\times \Bigg\{ \sum_{i=1}^{m} b_i \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} f\left(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s)\right) \mathrm{d}s$$
$$- \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f\left(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s)\right) \mathrm{d}s \Bigg\}, \qquad (3.2)$$

for $t \in J$.

Observe that the problem (1.1) has solutions if the operator equation (3.1) has fixed points. Assume that the following conditions hold:

(A1) There exist positive constants L_f and L_k such that

(i) $\|f(t, x_0, x_1, ..., x_{r_1+1}) - f(t, y_0, y_1, ..., y_{r_1+1})\| \le L_f \sum_{i=0}^{r_1+1} \|x_i - y_i\|$, $t \in J$, $x_i, y_i \in X$, $i = 0, 1, ..., r_1+1$,

(ii) $\|k(t, s, x_0, x_1, ..., x_{r_2}) - k(t, s, y_0, y_1, ..., y_{r_2})\| \le L_k \sum_{i=0}^{r_2} \|x_i - y_i\|$, $t, s \in J$, $x_i, y_i \in X$, $i = 0, 1, ..., r_2$.

(A2) There exist continuous nondecreasing function $\psi : [0, \infty) \to (0, \infty)$ and $\mu \in L^1(J, \mathbb{R}^+)$ such that, for each $(t, x_0, x_1, ..., x_{r_1+1}) \in J \times X^{r_1+2}$,

$$\|f(t, x_0, x_1, ..., x_{r_1+1})\| \le \mu(t)\psi(\|x\|).$$

(A3) Let $\Delta_1 = L_f\big\{(r_1+1)\theta_1 + L_k(r_2+1)\theta_2\big\} < 1$, where

$$\theta_1 = \frac{1 + |A_2| + |A_1|}{\Gamma(q+1)} + (|A_4| + |A_3|)\rho_1 + (|A_2| + |A_1|)\rho_3$$

and

$$\theta_2 = \frac{1 + |A_2| + |A_1|}{\Gamma(q+2)} + (|A_4| + |A_3|)\rho_2 + (|A_2| + |A_1|)\rho_4$$

with

$$\rho_1 = \sum_{i=1}^{m} |a_i| \frac{\zeta_i^q}{\Gamma(q+1)}, \quad \rho_2 = \sum_{i=1}^{m} |a_i| \frac{\zeta_i^{q+1}}{\Gamma(q+2)},$$

and

$$\rho_3 = \sum_{i=1}^{m} |b_i| \frac{\eta_i^{p_i+q}}{\Gamma(p_i+q+1)}, \quad \rho_4 = \sum_{i=1}^{m} |b_i| \frac{\eta_i^{p_i+q+1}}{\Gamma(p_i+q+2)}.$$

We prove the existence of solutions to (1.1) by applying Krasnoselskii's fixed point theorem.

Lemma 3.1. ([18]) (Krasnoselskii Theorem) *Let S be a nonempty closed convex subset of a Banach space X. Let $\mathcal{P}, \mathcal{Q}$ be two operators such that*

(i) $\mathcal{P}x + \mathcal{Q}y \in S$, *whenever* $x, y \in S$,
(ii) $\mathcal{P}$ *is compact and continuous,*
(iii) $\mathcal{Q}$ *is a contraction mapping.*

Then there exists $z \in S$ such that $z = \mathcal{P}z + \mathcal{Q}z$.

Theorem 3.2. *Suppose that the assumptions* (A1) *and* (A2) *hold with*

$$\begin{aligned} L = L_f \Big\{ &(r_1+1)\left[\frac{|A_2|+|A_1|}{\Gamma(q+1)} + (|A_4|+|A_3|)\rho_1 + (|A_2|+|A_1|)\rho_3\right] \\ &+ L_k(r_2+1)\left[\frac{|A_2|+|A_1|}{\Gamma(q+2)} + (|A_4|+|A_3|)\rho_2 + (|A_2|+|A_1|)\rho_4\right]\Big\} \\ &< 1. \end{aligned} \tag{3.3}$$

Then the boundary value problem (1.1) *has at least one solution on J.*

Proof. Consider $B_r = \{x \in Z : \|x\| \le r\}$. Now, for $t \in J$, we decompose F as $F_1 + F_2$ on B_r, where

$$\begin{aligned} (F_1x)(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f\left(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s)\right) \mathrm{d}s, \\ (F_2x)(t) &= (A_4 - A_3t^{n-1}) \sum_{i=1}^{m} a_i \int_0^{\zeta_i} \frac{(\zeta_i - s)^{q-1}}{\Gamma(q)} f(s, x(s), x(\alpha_1(s)), \cdots, \\ &\quad x(\alpha_{r_1}(s)), Kx(s))\mathrm{d}s \\ &\quad + (A_2 - A_1t^{n-1}) \Big\{ \sum_{i=1}^{m} b_i \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} f(s, x(s), x(\alpha_1(s)), \cdots, \\ &\quad x(\alpha_{r_1}(s)), Kx(s))\mathrm{d}s \\ &\quad - \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f\left(s, x(s), x(\alpha_1(s)), \cdots, x(\alpha_{r_1}(s)), Kx(s)\right) \mathrm{d}s \Big\}. \end{aligned}$$

Choose $r > \|\mu\|_{L^1} \psi(r) \theta_1$. For $x, y \in B_r$, we find that

$$
\begin{aligned}
&\|F_1x + F_2y\| \\
&\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\|f\left(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s)\right)\|\mathrm{d}s + (|A_4|+|A_3|)\sum_{i=1}^m |a_i| \\
&\quad \times \int_0^{\zeta_i}\frac{(\zeta_i - s)^{q-1}}{\Gamma(q)}\|f\left(s, y(s), y(\alpha_1(s)), ..., y(\alpha_{r_1}(s)), Ky(s)\right)\|\mathrm{d}s + (|A_2|+|A_1|) \\
&\quad \times \left\{\sum_{i=1}^m |b_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)}\|f\left(s, y(s), y(\alpha_1(s)), ..., y(\alpha_{r_1}(s)), Ky(s)\right)\|\mathrm{d}s\right. \\
&\quad \left. + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\|f\left(s, y(s), y(\alpha_1(s)), ..., y(\alpha_{r_1}(s)), Ky(s)\right)\|\mathrm{d}s\right\} \\
&\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\mu(s)\psi(\|x\|)ds + (|A_4|+|A_3|)\sum_{i=1}^m |a_i|\int_0^{\zeta_i}\frac{(\zeta_i - s)^{q-1}}{\Gamma(q)}\mu(s)\psi(\|y\|)\mathrm{d}s \\
&\quad + (|A_2|+|A_1|)\left\{\sum_{i=1}^m |b_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)}\mu(s)\psi(\|y\|)\mathrm{d}s\right. \\
&\quad \left. + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\mu(s)\psi(\|y\|)\mathrm{d}s\right\} \\
&\leq \mu\|_{L^1}\psi(r)\left[\frac{1+|A_2|+|A_1|}{\Gamma(q+1)} + (|A_4|+|A_3|)\sum_{i=1}^m |a_i|\frac{\zeta_i^q}{\Gamma(q+1)}\right. \\
&\quad \left. + (|A_2|+|A_1|)\sum_{i=1}^m |b_i|\frac{\eta_i^{p_i+q}}{\Gamma(p_i+q+1)}\right] \\
&\leq \mu\|_{L^1}\psi(r)\theta_1 \\
&\leq r.
\end{aligned}
$$

Thus $F_1x + F_2y \in B_r$. Next we prove that F_2 is a contraction.

$$
\begin{aligned}
&\|F_2x - F_2y\| \\
&\leq (|A_4|+|A_3|)\sum_{i=1}^m |a_i|\int_0^{\zeta_i} \frac{(\zeta_i - s)^{q-1}}{\Gamma(q)}\|f\left(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s)\right) \\
&\quad - f\left(s, y(s), y(\alpha_1(s)), ..., y(\alpha_{r_1}(s)), Ky(s)\right)\|\mathrm{d}s + (|A_2|+|A_1|) \\
&\quad \times \left\{\sum_{i=1}^m |b_i|\int_0^{\eta_i}\frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)}\|f\left(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s)\right)\right. \\
&\quad - f\left(s, y(s), y(\alpha_1(s)), ..., y(\alpha_{r_1}(s)), Ky(s)\right)\|\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} |f\left(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s)\right) \\
&- f\left(s, y(s), y(\alpha_1(s)), ..., y(\alpha_{r_1}(s)), Ky(s)\right) \| \mathrm{d}s \Big\} \\
\leq\ & (|A_4|+|A_3|) \sum_{i=1}^{m} |a_i| \int_0^{\zeta_i} \frac{(\zeta_i - s)^{q-1}}{\Gamma(q)} L_f \big[\|x(s)-y(s)\| + \|x(\alpha_1(s))-y(\alpha_1(s))\| \\
&+ \cdots + \|x(\alpha_{r_1}(s)) - y(\alpha_{r_1}(s))\| + \|Kx(s) - Ky(s)\| \big] \mathrm{d}s + (|A_2| + |A_1|) \\
&\times \Big\{ \sum_{i=1}^{m} |b_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} L_f \big[\|x(s) - y(s)\| + \|x(\alpha_1(s)) - y(\alpha_1(s))\| \\
&+ \cdot + \|x(\alpha_{r_1}(s)) - y(\alpha_{r_1}(s))\| + \|Kx(s) - Ky(s)\| \big] \mathrm{d}s + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \\
&\times L_f \big[\|x(s) - y(s)\| + \|x(\alpha_1(s)) - y(\alpha_1(s))\| + ... + \|x(\alpha_{r_1}(s)) - y(\alpha_{r_1}(s))\| \\
&+ \|Kx(s) - Ky(s)\| \big] \mathrm{d}s \Big\} \\
\leq\ & (|A_4| + |A_3|) \sum_{i=1}^{m} |a_i| \int_0^{\zeta_i} \frac{(\zeta_i - s)^{q-1}}{\Gamma(q)} L_f \big[(r_1+1)\|x-y\| \\
&+ \| \int_0^s k(s, \tau, x(\tau), x(\beta_1(\tau)), ..., x(\beta_{r_2}(\tau))) \mathrm{d}\tau \\
&- \int_0^s k(s, \tau, y(\tau), y(\beta_1(\tau)), ..., y(\beta_{r_2}(\tau))) \mathrm{d}\tau \| \big] \mathrm{d}s \\
&+ (|A_2| + |A_1|) \Big\{ \sum_{i=1}^{m} |b_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} L_f \big[(r_1+1)\|x-y\| \\
&+ \| \int_0^s k(s, \tau, x(\tau), x(\beta_1(\tau)), ..., x(\beta_{r_2}(\tau))) \mathrm{d}\tau \\
&- \int_0^s k(s, \tau, y(\tau), y(\beta_1(\tau)), ..., y(\beta_{r_2}(\tau))) \mathrm{d}\tau \| \big] \mathrm{d}s + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \\
&\times L_f \big[(r_1+1)\|x-y\| + \| \int_0^s k(s, \tau, x(\tau), x(\beta_1(\tau)), ..., x(\beta_{r_2}(\tau))) \mathrm{d}\tau \\
&- \int_0^s k(s, \tau, y(\tau), y(\beta_1(\tau)), ..., y(\beta_{r_2}(\tau))) \mathrm{d}\tau \| \big] \mathrm{d}s \Big\} \\
\leq\ & (|A_4| + |A_3|) \sum_{i=1}^{m} |a_i| \int_0^{\zeta_i} \frac{(\zeta_i - s)^{q-1}}{\Gamma(q)} L_f \big[(r_1+1)\|x-y\| \\
&\qquad + L_k (r_2+1) \|x-y\| s \big] \mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&+(|A_2|+|A_1|)\Bigg\{\sum_{i=1}^{m}|b_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{p_i+q-1}}{\Gamma(p_i+q)}L_f\big[(r_1+1)\|x-y\| \\
&\qquad\qquad + L_k(r_2+1)\|x-y\|s\big]\mathrm{d}s \\
&+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}L_f\big[(r_1+1)\|x-y\|+L_k(r_2+1)\|x-y\|s\big]\mathrm{d}s\Bigg\} \\
\le\ & L_f\Bigg\{(r_1+1)\bigg[\frac{|A_2|+|A_1|}{\Gamma(q+1)}+(|A_4|+|A_3|)\rho_1+(|A_2|+|A_1|)\rho_3\bigg] \\
&+L_k(r_2+1)\bigg[\frac{|A_2|+|A_1|}{\Gamma(q+2)}+(|A_4|+|A_3|)\rho_2+(|A_2|+|A_1|)\rho_4\bigg]\Bigg\}\|x-y\| \\
\le\ & L\|x-y\|.
\end{aligned}
$$

Hence F_2 is a contraction. Continuity of f and k implies that the operator F_1 is continuous. Also F_1 is uniformly bounded on B_r as

$$
\begin{aligned}
\|(F_1x)(t) &\le \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\|f\left(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s)\right)\|\mathrm{d}s \\
&\le \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\mu(s)\psi(\|x\|)\mathrm{d}s \\
&\le \frac{\|\mu\|_{L^1}\psi(r)}{\Gamma(q+1)}.
\end{aligned}
$$

To prove that the operator F_1 is compact, it remains to show that F_1 is equicontinuous. Now, for any $t_1,t_2\in J$ with $t_1<t_2$ and $x\in B_r$, we have

$$
\begin{aligned}
&\|(F_1x)(t_2)-(F_1x)(t_1)\| \\
\le\ & \int_0^{t_1}\frac{\big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\big]}{\Gamma(q)}\|f(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s))\|\mathrm{d}s \\
&+\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}\|f\left(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s)\right)\|\mathrm{d}s \\
\le\ & \int_0^{t_1}\frac{\big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\big]}{\Gamma(q)}\mu(s)\psi(\|x\|)\mathrm{d}s+\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}\mu(s)\psi(\|x\|)\mathrm{d}s \\
\le\ & \psi(r)\bigg[\int_0^{t_1}\frac{\big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\big]}{\Gamma(q)}\mu(s)\mathrm{d}s+\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}\mu(s)\mathrm{d}s\bigg].
\end{aligned}
$$

As $t_2\to t_1$, the right hand side of the above inequality tends to zero independent of $x\in B_r$. Thus F_1 is equicontinuous. By Arzela-Ascoli's Theorem, F_1 is compact on B_r. Hence, by the Krasnoselskii fixed point theorem, there exists a fixed point $x\in Z$ such that $Fx=x$ which is a solution to the fractional boundary value problem (1.1). □

The next existence result is based on Schaefer's fixed point theorem.

Lemma 3.3. ([18]) (Schaefer Theorem) *Let X be a Banach space. Assume that $F : X \to X$ is a completely continuous operator and the set*

$$S(F) = \{x \in X : x = \lambda Fx \text{ for some } 0 < \lambda < 1\}$$

is bounded. Then F has a fixed point in X.

Theorem 3.4. *Suppose that there exists constant $M^* > 0$ such that, for all $x_i \in X$, $i = 0, 1, ..., r_1 + 1$,*

$$\|f(t, x_0, x_1, ..., x_{r_1+1})\| \le M^*, \quad t \in J.$$

Then the boundary value problem (1.1) *has at least one solution on J.*

Proof. Let $B_r = \{x \in Z : \|x\| \le r\}$. The continuity of F defined by (3.2) follows from the continuity of f and k. Now we show that F maps bounded sets into bounded sets in B_r.

$$\begin{aligned}
&\|(Fx)(t)\| \\
&\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f\left(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s)\right)\| \mathrm{d}s \\
&\quad + (|A_4| + |A_3|) \sum_{i=1}^m |a_i| \int_0^{\zeta_i} \frac{(\zeta_i - s)^{q-1}}{\Gamma(q)} \\
&\quad \times \|f\left(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s)\right)\| \mathrm{d}s \\
&\quad + (|A_2| + |A_1|) \Bigg\{ \sum_{i=1}^m |b_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i + q)} \\
&\quad \times \|f\left(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s)\right)\| \mathrm{d}s \\
&\quad + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \|f\left(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s)\right)\| \mathrm{d}s \Bigg\} \\
&\le M^* \left[\frac{1 + |A_2| + |A_1|}{\Gamma(q+1)} + (|A_4| + |A_3|)\rho_1 + (|A_2| + |A_1|)\rho_3 \right] \\
&\le M^* \theta_1.
\end{aligned}$$

Hence F maps bounded sets into bounded sets in B_r.

Next we show that F maps bounded sets into equicontinuous sets in B_r. For that, let $t_1, t_2 \in J$ with $t_1 < t_2$. Then, for $x \in B_r$,

$$
\begin{aligned}
&\|(Fx)(t_2)-(Fx)(t_1)\\
&\leq \int_0^{t_1}\frac{\left[(t_2-s)^{q-1}-(t_1-s)^{q-1}\right]}{\Gamma(q)}\|f(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s))\|\mathrm{d}s\\
&\quad+\int_{t_1}^{t_2}\frac{(t_2-s)^{q-1}}{\Gamma(q)}\|f\left(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s)\right)\|\mathrm{d}s\\
&\quad+|A_3|(t_2^{n-1}-t_1^{n-1})\sum_{i=1}^{m}|a_i|\int_0^{\zeta_i}\frac{(\zeta_i-s)^{q-1}}{\Gamma(q)}\\
&\quad\times\|f\left(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s)\right)\|\mathrm{d}s\\
&\quad+|A_1|(t_2^{n-1}-t_1^{n-1})\bigg\{\sum_{i=1}^{m}|b_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{p_i+q-1}}{\Gamma(p_i+q)}\\
&\quad\times\|f\left(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s)\right)\|\mathrm{d}s\\
&\quad+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\|f\left(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s)\right)\|\mathrm{d}s\bigg\}.
\end{aligned}
$$

As $t_2 \to t_1$, the right hand side of the above inequality tends to zero independent of $x \in B_r$. Thus F maps bounded sets into equicontinuous sets in B_r. By Arzela-Ascoli's Theorem, F is completely continuous. Now it remains to show that the set

$$S = \{x \in Z : x = \lambda F(x),\ \text{for some } 0 < \lambda < 1\}$$

is bounded. Let $x \in S$, then $x = \lambda F(x)$ for some $0 < \lambda < 1$. Thus, for each $t \in J$, we have

$$
\begin{aligned}
\|x(t)\| &= \lambda(Fx)(t)\|\\
&\leq \int_0^t\frac{(t-s)^{q-1}}{\Gamma(q)}\|f\left(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s)\right)\|\mathrm{d}s\\
&\quad+(|A_4|+|A_3|)\sum_{i=1}^{m}|a_i|\int_0^{\zeta_i}\frac{(\zeta_i-s)^{q-1}}{\Gamma(q)}\\
&\quad\times\|f\left(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s)\right)\|\mathrm{d}s\\
&\quad+(|A_2|+|A_1|)\bigg\{\sum_{i=1}^{m}|b_i|\int_0^{\eta_i}\frac{(\eta_i-s)^{p_i+q-1}}{\Gamma(p_i+q)}\\
&\quad\times\|f\left(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s)\right)\|\mathrm{d}s\\
&\quad+\int_0^1\frac{(1-s)^{q-1}}{\Gamma(q)}\|f\left(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s)\right)\|\mathrm{d}s\bigg\}
\end{aligned}
$$

$$\leq M^* \left[\frac{1 + |A_2| + |A_1|}{\Gamma(q+1)} + (|A_4| + |A_3|)\rho_1 + (|A_2| + |A_1|)\rho_3 \right]$$
$$\leq M^* \theta_1.$$

This shows that S is bounded and as a consequence of Schaefer's fixed point theorem, we deduce that F has a fixed point which is a solution to the fractional boundary value problem (1.1). □

The next uniqueness result is based on Banach contraction principle.

Theorem 3.5. *Assume that the hypotheses* (*A*1) *and* (*A*3) *hold. Then the boundary value problem* (1.1) *has a unique solution on* J.

Proof. Let

$$M_1 = \sup_{t \in J} \|f(t, 0, ..., 0)\|$$

and

$$M_2 = \sup_{t,s \in J} \|k(t, s, 0, ..., 0)\|.$$

Then, consider

$$B_r = \{x \in Z : \|x\| \leq r\},$$

where $r \geq \frac{\Delta_2}{1-\Delta_1}$ with $\Delta_2 = L_f M_2 \theta_2 + M_1 \theta_1$ and Δ_1 is given by the assumption (A3).

Now we show that $FB_r \subset B_r$, where $F : Z \to Z$ is defined by (3.2). For $x \in B_r$, we have

$$\begin{aligned}
\|(Fx)(t)\| \leq & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s))\| \mathrm{d}s \\
& + (|A_4| + |A_3|) \sum_{i=1}^m |a_i| \int_0^{\zeta_i} \frac{(\zeta_i - s)^{q-1}}{\Gamma(q)} \\
& \times \|f(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s))\| \mathrm{d}s \\
& + (|A_2| + |A_1|) \left\{ \sum_{i=1}^m |b_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} \right. \\
& \times \|f(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s))\| \mathrm{d}s \\
& \left. + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \|f(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s))\| \mathrm{d}s \right\}
\end{aligned}$$

$$\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big[\|f\,(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s))$$

$$- f(s, 0, ..., 0)\| + \|f(s, 0, ..., 0)\|\big] \mathrm{d}s + (|A_4| + |A_3|)$$

$$\times \sum_{i=1}^{m} |a_i| \int_0^{\zeta_i} \frac{(\zeta_i - s)^{q-1}}{\Gamma(q)} \big[\|f\,(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s))$$

$$- f(s, 0, ..., 0)\| + \|f(s, 0, ..., 0)\|\big] \mathrm{d}s + (|A_2| + |A_1|)$$

$$\times \bigg\{ \sum_{i=1}^{m} |b_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i + q)} \big[\|f\,(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s))$$

$$- f(s, 0, ..., 0)\| + \|f(s, 0, ..., 0)\|\big] \mathrm{d}s + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}$$

$$\times \big[\|f\,(s, x(s), x(\alpha_1(s)), ..., x(\alpha_{r_1}(s)), Kx(s)) - f(s, 0, ..., 0)\|$$

$$+ \|f(s, 0, ..., 0)\|\big] \mathrm{d}s \bigg\}$$

$$\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \big\{L_f \big[(r_1 + 1)\|x\| + \|Kx(s)\|\big] + M_1\big\} \mathrm{d}s + (|A_4| + |A_3|)$$

$$\times \sum_{i=1}^{m} |a_i| \int_0^{\zeta_i} \frac{(\zeta_i - s)^{q-1}}{\Gamma(q)} \big\{L_f \big[(r_1 + 1)\|x\| + \|Kx(s)\|\big] + M_1\big\} \mathrm{d}s$$

$$+ (|A_2| + |A_1|) \bigg\{ \sum_{i=1}^{m} |b_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i + q)} \big\{L_f \big[(r_1 + 1)\|x\|$$

$$+ \|Kx(s)\|\big] + M_1\big\} \mathrm{d}s$$

$$+ \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \big\{L_f \big[(r_1 + 1)\|x\| + \|Kx(s)\|\big] + M_1\big\} \mathrm{d}s \bigg\}$$

$$\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \bigg\{ L_f \big[(r_1 + 1)r + \int_0^s \big(\|k(s, \tau, x(\tau), x(\beta_1(\tau)), ..., x(\beta_{r_2}(\tau)))$$

$$- k(s, \tau, 0, ..., 0)\| + \|k(s, \tau, 0, ..., 0)\|\big) d\tau\big] + M_1 \bigg\} \mathrm{d}s$$

$$+ (|A_4| + |A_3|) \sum_{i=1}^{m} |a_i| \int_0^{\zeta_i} \frac{(\zeta_i - s)^{q-1}}{\Gamma(q)} \bigg\{ L_f \big[(r_1 + 1)r$$

$$+ \int_0^s \big(\|k(s, \tau, x(\tau), x(\beta_1(\tau)), ..., x(\beta_{r_2}(\tau))) - k(s, \tau, 0, ..., 0)\|$$

$$+ \|k(s, \tau, 0, ..., 0)\|\big) d\tau\big] + M_1 \bigg\} \mathrm{d}s + (|A_2| + |A_1|) \bigg\{ \sum_{i=1}^{m} |b_i| \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i + q)}$$

$$
\begin{aligned}
&\times \Big\{ L_f \big[(r_1+1) r + \int_0^s \big(\| k(s,\tau,x(\tau),x(\beta_1(\tau)),...,x(\beta_{r_2}(\tau))) - k(s,\tau,0,...,0) \| \\
&\quad + \| k(s,\tau,0,...,0) \| \big) d\tau \big] + M_1 \Big\} \mathrm{d}s + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \Big\{ L_f \big[(r_1+1) r \\
&\quad + \int_0^s \big(\| k(s,\tau,x(\tau),x(\beta_1(\tau)),...,x(\beta_{r_2}(\tau))) - k(s,\tau,0,...,0) \| \\
&\quad + \| k(s,\tau,0,...,0) \| \big) d\tau \big] + M_1 \Big\} \mathrm{d}s \Big\} \\
&\le r L_f \{ (r_1+1)\theta_1 + L_k (r_2+1)\theta_2 \} + L_f M_2 \theta_2 + M_1 \theta_1 \\
&\le \Delta_1 r + \Delta_2 \le r.
\end{aligned}
$$

This shows that $FB_r \subset B_r$. Next, for $x, y \in Z$ and $t \in J$, we obtain

$$
\begin{aligned}
&\| (Fx)(t) - (Fy)(t) \| \\
&\le \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \| f(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s)) \\
&\quad - f(s,y(s),y(\alpha_1(s)),...,y(\alpha_{r_1}(s)),Ky(s)) \| \mathrm{d}s + (|A_4|+|A_3|) \sum_{i=1}^m |a_i| \\
&\quad \times \int_0^{\zeta_i} \frac{(\zeta_i - s)^{q-1}}{\Gamma(q)} \| f(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s)) \\
&\quad - f(s,y(s),y(\alpha_1(s)),...,y(\alpha_{r_1}(s)),Ky(s)) \| \mathrm{d}s + (|A_2|+|A_1|) \Big\{ \sum_{i=1}^m |b_i| \\
&\quad \times \int_0^{\eta_i} \frac{(\eta_i - s)^{p_i+q-1}}{\Gamma(p_i+q)} \| f(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s)) \\
&\quad - f(s,y(s),y(\alpha_1(s)),...,y(\alpha_{r_1}(s)),Ky(s)) \| \mathrm{d}s + \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \\
&\quad \times \| f(s,x(s),x(\alpha_1(s)),...,x(\alpha_{r_1}(s)),Kx(s)) \\
&\quad - f(s,y(s),y(\alpha_1(s)),...,y(\alpha_{r_1}(s)),Ky(s)) \| \mathrm{d}s \Big\} \\
&\le L_f \{ (r_1+1)\theta_1 + L_k (r_2+1)\theta_2 \} \| x - y \| \le \Delta_1 \| x - y \|.
\end{aligned}
$$

Here Δ_1 depends only on the parameters involved in the problem. By assumption $(A3)$, $\Delta_1 < 1$ and therefore F is a contraction. Hence, by the Banach contraction principle, the problem (1.1) has a unique solution on J. □

4. Examples

Consider the following fractional boundary value problem

$$
\begin{aligned}
{}^{C}D^{9/2}x(t) &= \frac{t}{3}\frac{|x(t)|+|x(\frac{t^2}{2})|+|x(t^3)|+|x(\frac{t}{2})|}{1+|x(t)|+|x(\frac{t^2}{2})|+|x(t^3)|+|x(\frac{t}{2})|}\\
&\quad+\frac{1}{3}\int_0^t\frac{e^{-s}}{4}\frac{|x(\frac{\sqrt{s}}{9})|+|x(\frac{\sin s}{2})|}{1+|x(\frac{\sqrt{s}}{9})|+|x(\frac{\sin s}{2})|}ds,\ t\in[0,1], \qquad (4.1)\\
&\sum_{i=1}^{4}a_i x(\zeta_i)=0,\ 0<\zeta_1<\zeta_2<\zeta_3<\zeta_4<1,\\
&x'(0)=x''(0)=x'''(0)=0,\\
&x(1)=\sum_{i=1}^{4}b_i(I^{p_i}x)(\eta_i),\ 0<\eta_1<\eta_2<\eta_3<\eta_4<1.
\end{aligned}
$$

Here $X=\mathbb{R}$, $q=\frac{9}{2}$, $n=5$, $m=4$,

$a_1=27$, $a_2=-11$, $a_3=32$, $a_4=55$, $\zeta_1=\frac{1}{6}$, $\zeta_2=\frac{1}{4}$, $\zeta_3=\frac{1}{3}$, $\zeta_4=\frac{1}{2}$,

$b_1=-19$, $b_2=15$, $b_3=53$, $b_4=75$, $\eta_1=\frac{1}{9}$, $\eta_2=\frac{1}{7}$, $\eta_3=\frac{1}{5}$, $\eta_4=\frac{1}{3}$,

$p_1=\frac{1}{8}$, $p_2=\frac{1}{6}$, $p_3=\frac{1}{4}$, $p_4=\frac{1}{2}$.

From the above given data, we see that $A=-391.097603$, $A_1=-0.263361$, $A_2=-0.009743$, $A_3=0.213051$, $A_4=-0.001827$, $\rho_1=0.051368$, $\rho_2=0.00451$, $\rho_3=0.00293$, $\rho_4=0.000155$, $\theta_1=0.03616$ and $\theta_2=0.005433$.

(i) From (4.1), we have

$$
\begin{aligned}
&f(t,x(t),...,x(\alpha_{r_1}(t)),Kx(t))\\
&=\frac{t}{3}\frac{|x(t)|+|x(\frac{t^2}{2})|+|x(t^3)|+|x(\frac{t}{2})|}{1+|x(t)|+|x(\frac{t^2}{2})|+|x(t^3)|+|x(\frac{t}{2})|}+\frac{1}{3}Kx(t),
\end{aligned}
$$

where, $Kx(t)=\displaystyle\int_0^t\frac{e^{-s}}{4}\frac{|x(\frac{\sqrt{s}}{9})|+|x(\frac{\sin s}{2})|}{1+|x(\frac{\sqrt{s}}{9})|+|x(\frac{\sin s}{2})|}ds,$

$\alpha_1(t)=t^2/2$, $\alpha_2(t)=t^3$, $\alpha_3(t)=t/2, \beta_1(s)=\sqrt{s}/9$, $\beta_2(s)=\sin s/2$. The condition $(A1)$ is satisfied with $L_f=\frac{1}{3}$ and $L_k=\frac{1}{4}$. The condition $(A2)$ is satisfied with $\mu(t)=1$ and $\psi(\|x\|)=\frac{5}{12}$. Computing the value of L, we have $L=0.023231<1$, thereby satisfying the condition (3.3). Thus all the assumptions of the Theorem 3.2 are satisfied. Hence the problem (4.1) has at least one solution on J.

(ii) Now we take
$f\left(t, x(t), ..., x(\alpha_{r_1}(t)), Kx(t)\right)$

$$= \frac{1}{7(1+t)^2} \frac{|x(t)| + |x(\frac{t}{2})| + |x(\frac{t^3}{2})| + |x(t^3)|}{1 + |x(t)| + |x(\frac{t}{2})| + |x(\frac{t^3}{2})| + |x(t^3)|} + Kx(t) \text{ in (4.1)},$$

where $Kx(t) = \int_0^t \frac{e^{-s}}{5} \frac{|x(s)| + |x(s^2)| + |x(\sin s)|}{1 + |x(s)| + |x(s^2)| + |x(\sin s)|} ds,$
$\alpha_1(t) = t/2$, $\alpha_2(t) = t^3/2$, $\alpha_3(t) = t^3$, $\beta_1(s) = s^2$, $\beta_2(s) = \sin s$.
Clearly

$$\|f\left(t, x(t), x(\alpha_1(t)), ..., x(\alpha_{r_1}(t)), Kx(t)\right)\| \leq M^*$$

with $M^* = \frac{12}{35}$. Thus all the assumptions of the Theorem 3.4 are satisfied. Hence the problem (4.1) with the given function f has at least one solution on J.

(iii) Taking
$f\left(t, x(t), ..., x(\alpha_{r_1}(t)), Kx(t)\right)$

$$= \frac{e^{-t}}{1+e^t} \frac{|x(t)| + |x(\frac{t}{5})| + |x(\sin t)|}{1 + |x(t)| + |x(\frac{t}{5})| + |x(\sin t)|} + \frac{1}{2} Kx(t) \text{ in (4.1)},$$

we have
$Kx(t) = \int_0^t \frac{t}{3} \frac{|x(s)| + |x(s^3)|}{1 + |x(s)| + |x(s^3)|} ds$, $\alpha_1(t) = t/5$, $\alpha_2(t) = \sin t$, $\beta_1(s) = s^3$.
The condition $(A1)$ is satisfied with $L_f = \frac{1}{2}$ and $L_k = \frac{1}{3}$. Computing the value of Δ_1, we have $\Delta_1 = 0.056051 < 1$, thereby satisfying the condition $(A3)$. Thus all the assumptions of the Theorem 3.5 are satisfied. Hence the problem (4.1) with the given function f has a unique solution on J.

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