Nonlinear Functional Analysis and Applications
Vol. 29, No. 3 (2024), pp. 739-751
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2024.29.03.06
http://nfaa.kyungnam.ac.kr/journal-nfaa

FRACTIONAL ORDER OF DIFFERENTIAL INCLUSION GOVERNED BY AN INVERSE STRONGLY AND MAXIMAL MONOTONE OPERATOR

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Abstract. In this paper, we study the existence and uniqueness of solutions for a class of fractional differential inclusion including a maximal monotone operator in real space with an initial condition. The main results of the existence and uniqueness are obtained by using resolvent operator techniques and multivalued fixed point theory.

1. Introduction

The existence and uniqueness of solutions for a class of fractional differential inclusions including a maximal monotone operator is a complex topic in

[0]Received October 22, 2023. Revised March 13, 2024. Accepted March 20, 2024.
[0]2020 Mathematics Subject Classification: 34A08, 47H05, 47J22.
[0]Keywords: Fractional differential inclusion, maximal monotone operator, fixed point theorem, strongly monotone operator, maximally $\beta-$inverse strongly monotone operator.
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mathematics. It involves studying the behavior of fractional differential equations with inclusion terms and the properties of maximal monotone operators. To determine the existence and uniqueness of solutions in this context, it is necessary to consider various mathematical techniques and theories specific to fractional calculus and monotone operators. These may include fixed point theorems, variational methods, and functional analysis. It is important to note that providing a comprehensive answer to this question would require a detailed analysis of the specific fractional differential inclusion and the properties of the maximal monotone operator involved. Differential inclusions of fractional order have played a major role in physical, chemical and electronic [1, 3, 4, 7, 13]. These differential inclusions are used to describe the nonlinear viscoelastic behavior of some viscoelastic materials

In this paper, we consider the following fractional differential inclusion with initial condition

$$(\mathcal{P}) \qquad \begin{cases} D_{0^+}^{\alpha} x(t) \in -Ax(t), & \text{a.e. } t \in (0,T], \\ t^{1-\alpha} x(t)\big|_{t=0} = x_0, & x_0 \in \mathbb{R}^*, \end{cases}$$

where $T > 0$, $A : \mathbb{R} \rightrightarrows \mathbb{R}$ is a maximal monotone operator (set-valued mapping) and $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$. Research regarding the differential inclusion in situations where the normal derivative ($\alpha = 1$) has been studied in [2, 14].

Within this paper, we shall establish both the presence and singularity of the solution to the posed problem by employing the Banach fixed point theorem, specifically when the operator A is an inverse strongly monotone. To show the existence and uniqueness of solutions of $(\mathcal{P})$, for all $\lambda > 0$, we consider the following problem:

$$(\mathcal{P}_\lambda) \qquad \begin{cases} D_{0^+}^{\alpha} x(t) = -A_\lambda\left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right), \text{ a.e. } t \in (0,T], \\ t^{1-\alpha} x(t)\big|_{t=0} = x_0 \in \mathbb{R}^*, \end{cases}$$

where $T > 0$, A_λ is a Yosida approximation of A (see, Definition 2.9) and $D_{0^+}^{\alpha}$ is a Riemann-Liouville fractional derivative of order $\alpha \in (0,1)$.

2. Preliminaries

In this section, we recall some basic notions. Let $T > 0$, denote by $C([0,T],\mathbb{R})$ the Banach space of all continuous functions from $[0,T]$ into $\mathbb{R}$ with the norm:

$$x \longrightarrow \|x\|_\infty = \sup\{|x(t)| : t \in [0,T]\}.$$

For all $\lambda > 0$, we consider the space

$$C_\lambda([0,T],\mathbb{R}) := \left\{x : [0,T] \to \mathbb{R} \mid t^\lambda x \in C([0,T],\mathbb{R})\right\}$$

with the norm : $x \longrightarrow \|x\|_{C_\lambda} = \sup_{t\in[0,T]} \left|t^\lambda x(t)\right|$. Clearly $\left(C_\lambda([0,T],\mathbb{R}), \|\cdot\|_{C_\lambda}\right)$, then it is a Banach space.

By $L^1([0,T],\mathbb{R})$, we denote the space of all Lebegue-integrable functions from $[0,T]$ into $\mathbb{R}$ with the norm

$$x \longrightarrow \|x\|_{L^1} = \int_0^T |x(t)|\, dt.$$

Let $AC([0,T],\mathbb{R})$ be the space of functions x which are absolutly continuous on $[0,T]$. It is known see [16, p.338] that $AC([0,T],\mathbb{R})$ coincides with the space of primitives of Lebesgue summable functions.

$$x \in AC([0,T],\mathbb{R}) \iff x(t) = c + \int_0^t \varphi(s)\, ds \ \left(\varphi(.) \in L^1([0,T],\mathbb{R})\right).$$

For $n \in \mathbb{N}^*$ we denote by $AC^n([0,T],\mathbb{R})$ the space of real-valued functions x which have continuous derivatives up to order $n-1$ on $[0,T]$ such that $x^{(n-1)} \in AC([0,T],\mathbb{R})$. In particular, $AC^1([0,T],\mathbb{R}) = AC([0,T],\mathbb{R})$.

Definition 2.1. ([15]) The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $x \in L^1([0,T],\mathbb{R})$ is defined by

$$I_{0^+}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} x(s)\, ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function defined by $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1} e^{-t} dt$.

Definition 2.2. ([15]) For a function $x \in AC^n([0,T],\mathbb{R})$, the Riemann–Liouville fractional derivative of order $\alpha > 0$ of x, is defined by

$$D_{0^+}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1} x(s)\, ds, \quad t \in [0,T],$$

where $n = [\alpha]+1$ ($[\alpha]$ denotes the integer part of the real number α).

Lemma 2.3. ([15]) *The general solution of linear fractional differential equation*

$$D_{0^+}^\alpha x(t) = 0, \quad t > 0$$

is given by

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} ...... + c_n t^{\alpha-n}, \tag{2.1}$$

where $c_i \in \mathbb{R}$, $i = 1, 2, ..., n$.

Lemma 2.4. ([15]) *For* $t > 0$, *we have*

$$I_{0^+}^\alpha t^{\beta-1} \quad = \tfrac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} t^{\beta+\alpha-1}, \quad \alpha, \beta > 0. \tag{2.2}$$

Lemma 2.5. *The function $x_\lambda(\cdot)$ solves the problem $(\mathcal{P}_\lambda)$, if and only if it is a solution of the integral equation:*

$$x_\lambda(t)=x_0t^{\alpha-1}-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}A_\lambda\left(x_\lambda(s)-\lambda D_{0+}^\alpha x_\lambda(s)\right)ds,\quad t\in(0,T]. \tag{2.3}$$

Proof. Suppose the function $x_\lambda(\cdot)$ satisfies the problem $(\mathcal{P}\lambda)$. Then applying I_{0+}^α to both sides of $D_{0+}^\alpha x_\lambda(t)=-A_\lambda\left(x_\lambda(t)-\lambda D_{0+}^\alpha x_\lambda(t)\right)$, we find

$$I_{0+}^\alpha D_{0+}^\alpha x_\lambda(t)=-I_{0+}^\alpha A_\lambda\left(x_\lambda(t)-\lambda D_{0+}^\alpha x_\lambda(t)\right).$$

In view of Lemma 2.3, we get

$$x_\lambda(t)=c_1t^{\alpha-1}-\frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}A_\lambda\left(x_\lambda(s)-\lambda D_{0+}^\alpha x_\lambda(s)\right)ds. \tag{2.4}$$

The condition $t^{1-\alpha}x_\lambda(t)\big|_{t=0}=x_o$ implies that

$$c_1=x_0. \tag{2.5}$$

Substituting (2.5) in (2.4) we get the integral equation (2.3). The converse can be proved by direct computations. The proof is completed. $\square$

A multifunction and a set-valued map are related concepts in mathematics, particularly in the field of functional analysis and set theory see, [5, 6].

Definition 2.6. ([6]) Let X and Y be two sets. $F:X\rightrightarrows Y$ or $F:X\to\mathcal{P}(Y)$ (where $\mathcal{P}(Y)=2^Y$ denotes set of all possible subsets of Y) is a multifunction or set-valued map, if to each element x of X, we associate a subset $F(x)\subset Y$.

The domain of a multifunction $F:X\rightrightarrows Y$ is the subset of X denoted by

$$\mathrm{dom}F:=\{x\in X:F(x)\neq\phi\}.$$

The graph of a multifunction $F:X\rightrightarrows Y$ is the subset of $X\times Y$ define by

$$\mathrm{graph}F:=\{(x;y)\in X\times Y:x\in\mathrm{dom}A\text{ and }y\in F(x)\}.$$

We call the reverse of F, the set multivaued map $F^{-1}:Y\rightrightarrows X$ such that

$$x\in F^{-1}(y)\iff y\in F(x).$$

A monotone operator and a maximal monotone operator are concepts from convex analysis and functional analysis, often used in the study of convex optimization and variational inequalities see [10, 11, 12, 17].

In the following, we consider $\mathbb{X}$ a real Hilbert space and $A:\mathbb{X}\rightrightarrows\mathbb{X}$ a set-valued map.

Definition 2.7. ([6]) The operator $A:\mathbb{X}\rightrightarrows\mathbb{X}$ is said to be monotone, if $\langle y_1-y_2,x_1-x_2\rangle\geq 0$ for all $(x_i,y_i)\in\mathrm{graph}\,A, i=1,2$, where, $\langle.,.\rangle$ represents the inner product.

Definition 2.8. ([6]) A monotone operator A is called maximal, if there is no other monotone operator whose graph strictly contains the graph of A.

Maximal monotone multifunctions find applications in various areas, including convex analysis, optimization, game theory, and economics. They are particularly useful when dealing with problems that involve non-differentiable or non-convex functions, as they allow for a more flexible modeling of relationships between variables. Understanding maximal monotone multifunctions involves a deep understanding of set-valued mappings, convexity, and functional analysis. It's a sophisticated mathematical concept used in advanced optimization and variational inequality problems. The resolvent operator and the Yosida approximation are concepts closely related to maximal monotone operators in convex analysis and optimization.

Definition 2.9. ([5]) Let $A : \mathbb{X} \rightrightarrows \mathbb{X}$ be a maximal monotone operator. For $\lambda > 0$ the operator $J_\lambda^A : \mathbb{X} \to \mathrm{dom}A \subset \mathbb{X}$ defined by

$$J_\lambda^A = (\mathrm{I_d} + \lambda A)^{-1}$$

is called the resolvent of A or the resolvent operator of A. and for $\lambda > 0$ the operator $A_\lambda : \mathbb{X} \to \mathrm{dom}A \subset \mathbb{X}$ defined by

$$A_\lambda = \frac{1}{\lambda}\left(\mathrm{I_d} - J_\lambda^A\right)$$

is called theYosida approximation of A, where $\mathrm{I_d}$ is the identity operator in $\mathbb{X}$.

Proposition 2.10. ([5]) *Let $A : \mathbb{X} \rightrightarrows \mathbb{X}$ be a maximal monotone operator. Then for all $\lambda > 0$ we have the resolvent J_λ^A is a maximal monotone, single-valued and A_λ is Lipchitz with constant $\frac{1}{\lambda}$.*

Proposition 2.11. ([6]) *Let $A : \mathbb{X} \rightrightarrows \mathbb{X}$ be a maximal monotone operator. Then, we have*

(1) For all $\lambda, \mu > 0$ and for all $x \in \mathbb{X}$, $J_\lambda^A(x) = J_\mu^A\left(\frac{\mu}{\lambda}x + \left(1 - \frac{\mu}{\lambda}\right)J_\lambda^A(x)\right).$
(2) For all $\lambda, \mu > 0$, $(A_\lambda)_\mu = A_{\lambda+\mu}$.
(3) For all $\lambda > 0$ and for all $y \in \mathbb{X}$, $A_\lambda(y) \in A\left(J_\lambda^A(y)\right).$
(4) For all $\lambda > 0$, $\mathrm{dom}J_\lambda^A = \mathrm{dom}A_\lambda = \mathbb{X}$.

Theorem 2.12. ([5]) *Let $A : \mathbb{X} \rightrightarrows \mathbb{X}$ be a maximal monotone. Then for all $\lambda > 0$, $\mathrm{I_d} + \lambda A$ is bijective, in other terms for each $y \in \mathbb{X}$, there is a unique $x \in \mathrm{dom}\, A$ such that $y \in x + \lambda A(x)$.*

Definition 2.13. ([9]) Let $A : \mathbb{X} \rightrightarrows \mathbb{X}$ be an operator and $\beta > 0$.

(1) We say that A is $\beta-$strongly monotone if for all $(x, u) \in \mathrm{graph}A$, $(y, v) \in \mathrm{graph}A$, $\langle x - y, u - v\rangle \geq \beta \parallel x - y \parallel^2$.

(2) $A : \mathbb{X} \rightrightarrows \mathbb{X}$ is called maximally $\beta-$strongly monotone operator if there is no $\beta-$strongly monotone operator $B : \mathbb{X} \rightrightarrows \mathbb{X}$ such that $\operatorname{graph}A \subset \operatorname{graph}B$, that is, for every $(x,u) \in \mathbb{X}\times\mathbb{X}$, $(x,u) \in \operatorname{graph}A$ if and only if for all $(y,v) \in \operatorname{graph}A, \langle x-y, u-v\rangle \geq \beta \parallel x-y \parallel^2$.

Definition 2.14. ([9]) Let $A : \mathbb{X} \rightrightarrows \mathbb{X}$ be an operator and $\beta > 0$.

(1) We say that A is $\beta-$inverse strongly monotone if for all $(x,u) \in \operatorname{graph}A$, $(y,v) \in \operatorname{graph}A, \langle x-y, u-v\rangle \geq \beta \parallel u-v \parallel^2$.
(2) A $\beta-$inverse strongly monotone operator $A : \mathbb{X} \rightrightarrows \mathbb{X}$ is called maximally $\beta-$inverse strongly monotone operator if there is no $\beta-$inverse strongly monotone operator $B : \mathbb{X} \rightrightarrows \mathbb{X}$ such that $\operatorname{graph}A \subset \operatorname{graph}B$, that is, for every $(x,u) \in \mathbb{X}\times\mathbb{X}$, $(x,u) \in \operatorname{graph}A$ if and only if $(y,v) \in \operatorname{graph}A, \langle x-y, u-v\rangle \geq \beta \parallel u-v \parallel^2$.

Proposition 2.15. ([8]) *If A is a $\beta-$strongly monotone, then J_λ^A is $\frac{1}{1+\lambda\beta}$-Lipschitz.*

Proposition 2.16. ([8]) *If A is maximally $\beta-$inverse strongly monotone, then A_λ is $\frac{1}{\beta+\lambda}$-Lipschitz.*

Now, we will state the Banach fixed point theorem.

Theorem 2.17. ([14]) *Let $\mathbb{X}$ be a Banach space with a contraction mapping $T : \mathbb{X} \to \mathbb{X}$. Then T admits a unique fixed-point.*

3. Main results

3.1. **Existence and uniqueness of solutions to differential inclusions with inverse strongly monotone operator.** In this subsection, we consider $A : \mathbb{R} \rightrightarrows \mathbb{R}$ is a maximally $\beta-$inverse strongly monotone operator.

Theorem 3.1. *If $\beta > \frac{\Gamma(\alpha)T^\alpha}{\Gamma(2\alpha)}$ and for all $\lambda > 0$, there exists a unique solution of problem $(\mathcal{P}_\lambda)$ in $C_{1-\alpha}([0,T], \mathbb{R})$.*

Proof. Let $\lambda > 0$. For our problem, we define the function $f : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ by

$$f(u,v) = -A_\lambda(u - \lambda v).$$

Then, we should note that f is well-defined because $A_\lambda : \mathbb{R} \to \mathbb{R}$ is singular and $\operatorname{dom}A_\lambda = \mathbb{R}$.
We define the operator $\Phi : C_{1-\alpha}([0,T], \mathbb{R}) \to C_{1-\alpha}([0,T], \mathbb{R})$ by

$$(\Phi x)(t) = x_0 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)\, ds, \quad t \in (0,T],$$

where $g : [0,T] \to \mathbb{R}$ is a function satisfying the functional equation

$$g\left(t\right) = -A_{\lambda}\left(x\left(t\right) - \lambda g\left(t\right)\right).$$

By Lemma 2.5, the fixed points of operator Φ are solutions of $\left(\mathcal{P}_{\lambda}\right)$.

In order to prove that Φ accepts a fixed point, We will follow the next steps:

Step 1. The operator Φ is well-define, that's to say: for every $x\left(\cdot\right) \in C_{1-\alpha}\left(\left[0,T\right],\mathbb{R}\right)$ and $t \in (0,T]$ the integral $\frac{1}{\Gamma(\alpha)}\int_0^t \left(t-s\right)^{\alpha-1} g\left(s\right)ds$ belongs to $C_{1-\alpha}\left(\left[0,T\right],\mathbb{R}\right)$.
For each $t \in (0,T]$, we have

$$\begin{aligned}\left|g\left(t\right)\right| - \left|f\left(0,0\right)\right| &\leq \left|g\left(t\right) - f\left(0,0\right)\right|, \text{ where } f\left(0,0\right) = A_{\lambda}\left(0\right), \\ &\leq \left|-A_{\lambda}\left(x\left(t\right) - \lambda g\left(t\right)\right) + A_{\lambda}\left(0\right)\right| \\ &\leq \frac{1}{\lambda+\beta}\left|x\left(t\right)\right| + \frac{\lambda}{\lambda+\beta}\left|g\left(t\right)\right|,\end{aligned}$$

because A_{λ} is $\frac{1}{\lambda+\beta}-$Lipschitz. After doing some simple arithmetic operations of the above, we get

$$\left|g\left(t\right)\right| \leq \frac{1}{\beta}\left|x\left(t\right)\right| + \frac{\lambda+\beta}{\beta}\left|A_{\lambda}\left(0\right)\right|.$$

Then, $\left|g\left(t\right)\right| \leq \frac{1}{\beta}\left|x\left(t\right)\right| + c$, where $c = \frac{\lambda+\beta}{\beta}\left|A_{\lambda}\left(0\right)\right|$.
For every $x\left(\cdot\right) \in C_{1-\alpha}\left(\left[0,T\right],\mathbb{R}\right)$, we have

$$\begin{aligned}\left|\frac{t^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_0^t \left(t-s\right)^{\alpha-1} g\left(s\right)ds\right| &\leq \frac{t^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_0^t \left(t-s\right)^{\alpha-1}\left|g\left(s\right)\right|ds, \\ &\leq \frac{t^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_0^t \left(t-s\right)^{\alpha-1}\left(\frac{1}{\beta}\left|x\left(s\right)\right| + c\right)ds, \\ &\leq \frac{t^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_0^t \left(t-s\right)^{\alpha-1}s^{\alpha-1}\left(\frac{1}{\beta}\sup_{t\in[0,T]}\left|t^{1-\alpha}x\left(t\right)\right|\right)ds \\ &\quad + \frac{t^{1-\alpha}}{\Gamma\left(\alpha\right)}\int_0^t \left(t-s\right)^{\alpha-1}cds, \\ &\leq t^{1-\alpha}\left(\frac{1}{\beta}\left\|x\right\|_{C_{1-\alpha}}\right)\frac{1}{\Gamma\left(\alpha\right)}\int_0^t \left(t-s\right)^{\alpha-1}s^{\alpha-1}ds \\ &\quad + \frac{ct}{\Gamma\left(\alpha+1\right)} \\ &\leq t^{1-\alpha}\left(\frac{1}{\beta}\left\|x\right\|_{C_{1-\alpha}}\right)\left(I_{0^+}^{\alpha}t^{\alpha-1}\right)\left(t\right) + \frac{ct}{\Gamma\left(\alpha+1\right)}.\end{aligned}$$

By using (2.2), we get

$$\left|\frac{t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} g(s)\,ds\right| \le \frac{T^{\alpha}\Gamma(\alpha)}{\beta\Gamma(2\alpha)}\|x\|_{C_{1-\alpha}} + \frac{cT}{\Gamma(\alpha+1)},$$

that is to say that the integral exists and belongs to $C_{1-\alpha}([0,T],\mathbb{R})$.

Step 2. Let $x(.), y(.) \in C_{1-\alpha}([0,T],\mathbb{R})$. Then for $t \in (0,T]$, we have

$$(\Phi x)(t) - (\Phi y)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(g(s)-h(s))\,ds,$$

where $g(.), h(.) \in C_{1-\alpha}([0,T],\mathbb{R})$ such that

$$g(t) = -A_{\lambda}(x(t) - \lambda g(t)), h(t) = -A_{\lambda}(y(t) - \lambda h(t)).$$

Since $A : \mathbb{R} \rightrightarrows \mathbb{R}$ is a maximally $\beta-$inverse strongly monotone operator, then A_{λ} is $\frac{1}{\lambda+\beta}-$Lipschitz and we get

$$\begin{aligned}|g(t) - h(t)| &= |-A_{\lambda}(x(t) - \lambda g(t)) + A_{\lambda}(y(t) - \lambda h(t))| \\ &\le \frac{1}{\lambda+\beta}|x(t) - y(t)| + \frac{\lambda}{\lambda+\beta}|g(t) - h(t)|.\end{aligned}$$

So, $|g(t) - h(t)| - \frac{\lambda}{\lambda+\beta}|g(t) - h(t)| \le \frac{1}{\lambda+\beta}|x(t) - y(t)|$. We get,

$$|g(t) - h(t)| \le \frac{1}{\beta}|x(t) - y(t)|.$$

For $t \in (0,T]$, we have

$$\begin{aligned}|(\Phi x)(t) - (\Phi y)(t)| &\le \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}|g(s) - h(s)|\,ds \\ &\le \frac{1}{\beta\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} s^{\alpha-1}\left|s^{1-\alpha}(x(s) - y(s))\right| ds \\ &\le \frac{1}{\beta}\|x-y\|_{C_{1-\alpha}}\left(I_{0^+}^{\alpha} t^{\alpha-1}\right)(t) \\ &\le \frac{\Gamma(\alpha)\,t^{2\alpha-1}}{\beta\Gamma(2\alpha)}\|x-y\|_{C_{1-\alpha}}.\end{aligned}$$

Therefore, we have

$$\left|t^{1-\alpha}((\Phi x)(t) - (\Phi y)(t))\right| \le \frac{\Gamma(\alpha)\,t^{\alpha}}{\beta\Gamma(2\alpha)}\|x-y\|_{C_{1-\alpha}},$$

which implies that

$$\|\Phi x - \Phi y\|_{C_{1-\alpha}} \le \frac{\Gamma(\alpha)\,T^{\alpha}}{\beta\Gamma(2\alpha)}\|x-y\|_{C_{1-\alpha}}.$$

Since $\beta > \frac{\Gamma(\alpha)T^{\alpha}}{\Gamma(2\alpha)}$, thus $\frac{\Gamma(\alpha)T^{\alpha}}{\beta\Gamma(2\alpha)} < 1$. So Φ is a contraction.

As a consequence of Banach fixed point theorem, if we get that Φ has a unique fixed point which is a unique solution of the problem $(\mathcal{P}_\lambda)$. $\square$

Example 3.2. Let $A : \mathbb{R} \rightrightarrows \mathbb{R}$ be a maximal monoton operator defined as $A(x) = \{2x\}$. We consider the following problem

$$\begin{cases} -D_{0^+}^{\alpha} x(t) \in -A(x(t)), & \text{a.e. } t \in (0,T], \quad \alpha \in (0,1), \\ t^{1-\alpha} x(t)\big|_{t=0} = x_0, & x_0 \in \mathbb{R}^*. \end{cases} \tag{3.1}$$

The operator $A(\cdot)$ is $2-$inverse strongly monotone operator. The fractional inclusion (3.1) is given, by the following linear initial value problem

$$\begin{cases} D_{0^+}^{\alpha} x(t) = -2x(t), & \text{a.e. } t \in (0,T], \quad \alpha \in (0,1), \\ t^{1-\alpha} x(t)\big|_{t=0} = x_0, & x_0 \in \mathbb{R}^*. \end{cases}$$

The unique solution has the form [15, 18],

$$x(t) = \Gamma(\alpha) x_0 t^{\alpha-1} E_{\alpha,\alpha}(-2t^{\alpha}),$$

where $E_{\alpha,\alpha}(\cdot)$ is Mittag–Leffler function [15] defined by

$$E_{\alpha,\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma((k+1)\alpha)}.$$

In the next proposition, we prove that the solution of problem $(\mathcal{P}_\lambda)$ is not depend by λ. For all $\lambda > 0$, we define the set,

$$E_\lambda = \{x \in C_{1-\alpha}([0,T], \mathbb{R}) \mid D_{0^+}^{\alpha} x(t) = -A_\lambda(x(t) - \lambda D_{0^+}^{\alpha} x(t)), t \in [0,T]\}.$$

Proposition 3.3. *For all $\lambda, \mu > 0$, we have $E_\lambda = E_\mu$.*

Proof. Let $\lambda > 0$. Then, we have

$$\begin{aligned} x \in E_\lambda \iff\ & D_{0^+}^{\alpha} x(t) = -A_\lambda\left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right), t \in [0,T] \\ \iff\ & D_{0^+}^{\alpha} x(t) = -\tfrac{1}{\lambda}\left(\mathrm{id} - J_\lambda^A\right)\left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right), t \in [0,T] \\ \iff\ & -\lambda D_{0^+}^{\alpha} x(t) = x(t) - \lambda D_{0^+}^{\alpha} x(t) \\ & -J_\lambda^A\left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right), t \in [0,T] \\ \iff\ & x(t) = J_\lambda^A\left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right), t \in [0,T]. \end{aligned} \tag{3.2}$$

Using the Proposition 2.11, equivalence (3.2) becomes as follows

$$\begin{aligned} x \in E_\lambda \iff\ & x(t) = J_\mu^A\left(\tfrac{\mu}{\lambda}\left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right) + \left(1 - \tfrac{\mu}{\lambda}\right) x(t)\right) \\ \iff\ & x(t) = J_\mu^A\left(x(t) - \mu D_{0^+}^{\alpha} x(t)\right), t \in [0,T] \\ \iff\ & D_{0^+}^{\alpha} x(t) = -A_\mu\left(x(t) - \mu D_{0^+}^{\alpha} x(t)\right), t \in [0,T] \\ \iff\ & x \in E_\mu. \end{aligned}$$

This completes the proof. $\square$

In the following hypothesis, we will show that problems $(\mathcal{P}_\lambda)$ and $(\mathcal{P})$ have the same set of solutions.

Proposition 3.4. *Let $A : \mathbb{R} \rightrightarrows \mathbb{R}$ be a maximal monotone operator. Then, for all $\lambda > 0$ and $t \in [0,T]$, we have*

$$D_{0+}^{\alpha} x(t) \in -Ax(t) \quad \Leftrightarrow \quad D_{0+}^{\alpha} x(t) = -A_\lambda \left(x(t) - \lambda D_{0+}^{\alpha} x(t)\right).$$

Proof. For all $\lambda > 0$ and $t \in [0,T]$, we have

$$\begin{aligned} D_{0+}^{\alpha} x(t) \in -Ax(t) &\iff -\lambda D_{0+}^{\alpha} x(t) \in \lambda Ax(t) \\ &\iff x(t) - \lambda D_{0+}^{\alpha} x(t) \in (\mathrm{id} + \lambda A) x(t) \\ &\iff x(t) = J_\lambda^A \left(x(t) - \lambda D_{0+}^{\alpha} x(t)\right) \\ &\iff -\lambda D_{0+}^{\alpha} x(t) = -\lambda D_{0+}^{\alpha} x(t) + x(t) \\ &\qquad -J_\lambda^A \left(x(t) - \lambda D_{0+}^{\alpha} x(t)\right) \\ &\iff D_{0+}^{\alpha} x(t) = -A_\lambda \left(x(t) - \lambda D_{0+}^{\alpha} x(t)\right). \end{aligned}$$

$\square$

Theorem 3.5. *If $\beta > \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} T^\alpha$, then problem $(\mathcal{P})$ has unique solution in the space $C_{1-\alpha}([0,T], \mathbb{R})$.*

Proof. If $\beta > \frac{\Gamma(\alpha)}{\Gamma(2\alpha)} T^\alpha$, then from Theorem 3.1, the problem $(\mathcal{P}_\lambda)$ has unique solution $x(\cdot)$ in the space $C_{1-\alpha}([0,T], \mathbb{R})$. From Proposition 3.4, we get $x(\cdot)$ is a solution of $(\mathcal{P})$ in the space $C_{1-\alpha}([0,T], \mathbb{R})$. $\square$

3.2. **Existence and uniqueness of solutions to differential inclusions with maximal monotone operator.** In this subsection, we consider $A : \mathbb{R} \rightrightarrows \mathbb{R}$ is a maximally monotone operator. We will prove the existence of the solution and its uniqueness in the next case.

For all $\varepsilon > 0$, we consider the following fractional differential equation

$$\begin{cases} D_{0+}^{\alpha} x(t) = -A_\varepsilon (x(t)) \text{ a.e. } t \in (0,T], \ \alpha \in (0,1), \\ t^{1-\alpha} x(t)\big|_{t=0} = x_0 \in \mathbb{R}^*. \end{cases} \tag{3.3}$$

To show the existence and uniqueness of solutions of (3.3), we have proved an equivalence with the following disturbed problem

$$(\mathcal{P}_{\lambda,\varepsilon}) \qquad \begin{cases} D_{0+}^{\alpha} x(t) = -A_{\lambda+\varepsilon} \left(x(t) - \lambda D_{0+}^{\alpha} x(t)\right) \quad \text{a.e. } t \in (0,T], \ \alpha \in (0,1), \\ t^{1-\alpha} x(t)\big|_{t=0} = x_0 \in \mathbb{R}^*, \end{cases}$$

for all $\lambda, \varepsilon > 0$.

Theorem 3.6. *If $\varepsilon > \frac{\Gamma(\alpha) T^\alpha}{\Gamma(2\alpha)}$, then for all $\lambda > 0$, there exists a unique solution of problem $(\mathcal{P}_{\lambda,\varepsilon})$ in $C_{1-\alpha}([0,T], \mathbb{R})$.*

Proof. We define the operator $\Phi : C_{1-\alpha}([0,T],\mathbb{R}) \to C_{1-\alpha}([0,T],\mathbb{R})$ by

$$(\Phi x)(t) = x_0 t^{\alpha-1} - \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds,\ t \in [0,T],$$

where $g : [0,T] \to \mathbb{R}$ is a function satisfying the functional equation

$$g(t) = A_{\lambda+\varepsilon}\left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right).$$

By Lemma 2.5, the fixed points of operator Φ are solutions of $(\mathcal{P}_{\lambda,\varepsilon})$.

Since $A : \mathbb{R} \rightrightarrows \mathbb{R}$ is a maximal monotone operator, $A_{\lambda+\varepsilon} : \mathbb{R} \to \mathbb{R}$ is $\frac{1}{\lambda+\varepsilon}-$Lipschitz, singular and $\operatorname{dom} A_{\lambda+\varepsilon} = \mathbb{R}$, we can follow the same steps in Theorem 3.1, or $\varepsilon > \frac{\Gamma(\alpha)T^{\alpha}}{\Gamma(2\alpha)}$, that is, $\frac{\Gamma(\alpha)T^{\alpha}}{\varepsilon\Gamma(2\alpha)} < 1$ thus, Φ is a contraction. By Banach fixed point theorem, we get that Φ has a unique fixed point which is a unique solution of the problem $(\mathcal{P}_{\lambda,\varepsilon})$. □

In the next proposition, we prove that the solution of problem $(\mathcal{P}_{\lambda,\varepsilon})$ is not depend by λ. For all $\lambda, \varepsilon > 0$, we define the set

$$E_{\lambda,\varepsilon} = \left\{x \in C_{1-\alpha}([0,T],\mathbb{R}) \mid D_{0^+}^{\alpha} x(t) = -A_{\lambda+\varepsilon}\left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right) t \in [0,T]\right\}.$$

Proposition 3.7. *For all $\lambda, \mu > 0$, we have $E_{\lambda,\varepsilon} = E_{\mu,\varepsilon}$.*

Proof. Let $\lambda, \mu > 0$. We have,

$$\begin{aligned} x \in E_{\lambda,\varepsilon} \iff & -(\lambda+\varepsilon) D_{0^+}^{\alpha} x(t) = x(t) - \lambda D_{0^+}^{\alpha} x(t) \\ & \qquad - J_{\lambda+\varepsilon}^{A}\left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right),\ t \in [0,T] \\ \iff & J_{\lambda+\varepsilon}^{A}\left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right) = x(t) + \varepsilon D_{0^+}^{\alpha} x(t),\ t \in [0,T]. \end{aligned}$$

Using the Proposition 2.11, we get $x \in E_{\lambda,\varepsilon}$ is equivalent to

$$\begin{aligned} & J_{\mu+\varepsilon}^{A}\left(\tfrac{\mu+\varepsilon}{\lambda+\varepsilon}\left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right) + \left(1 - \tfrac{\mu+\varepsilon}{\lambda+\varepsilon}\right) J_{\lambda+\varepsilon}^{A}\left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right)\right) \\ & = x(t) + \varepsilon D_{0^+}^{\alpha} x(t). \end{aligned} \tag{3.4}$$

So (3.4) is equivalent to,

$$\begin{aligned} & J_{\mu+\varepsilon}^{A}\left(\tfrac{\mu+\varepsilon}{\lambda+\varepsilon}\left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right) + \left(1 - \tfrac{\mu+\varepsilon}{\lambda+\varepsilon}\right)\left(x(t) + \varepsilon D_{0^+}^{\alpha} x(t)\right)\right) \\ & = x(t) + \varepsilon D_{0^+}^{\alpha} x(t),\ t \in [0,T] \\ & \iff J_{\mu+\varepsilon}^{A}\left(x(t) - \mu D_{0^+}^{\alpha} x(t)\right) = x(t) + \varepsilon D_{0^+}^{\alpha} x(t),\ t \in [0,T] \\ & \iff x \in E_{\mu,\varepsilon}. \end{aligned}$$

This completes the proof. □

Theorem 3.8. *If $\varepsilon > \frac{\Gamma(\alpha)T^{\alpha}}{\Gamma(2\alpha)}$, then there exists a unique solution of problem* (3.3) *in $C_{1-\alpha}([0,T],\mathbb{R})$.*

Proof. If $\varepsilon > \frac{\Gamma(\alpha)T^\alpha}{\Gamma(2\alpha)}$, by Theorem 3.6, the problem $(\mathcal{P}_{\lambda,\varepsilon})$ has unique solution in $C_{1-\alpha}([0,T],\mathbb{R})$. Since A_ε is a maximal monotone operator, by Proposition 2.11 and Proposition 3.4, we find

$$\begin{array}{l} D_{0^+}^{\alpha} x(t) = -A_{\lambda+\varepsilon}\left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right) \\ \Longleftrightarrow D_{0^+}^{\alpha} x(t) = -\left(A_\varepsilon\right)_\lambda \left(x(t) - \lambda D_{0^+}^{\alpha} x(t)\right) \\ \Longleftrightarrow D_{0^+}^{\alpha} x(t) = -A_\varepsilon x(t). \end{array}$$

Then for all $\varepsilon > \frac{\Gamma(\alpha)T^\alpha}{\Gamma(2\alpha)}$, the problem (3.3) has unique solution in $C_{1-\alpha}([0,T],\mathbb{R})$. □

Conclusion: If the goal of the paper is to prove the existence and uniqueness of solutions for a problem without imposing any conditions, it indicates that the authors are aiming for a very general result that applies broadly. This can be a challenging task, as proving existence and uniqueness without any conditions often requires a deep understanding of the mathematical structures involved and potentially involves more abstract or advanced techniques.

Acknowledgments: The authors would like to thank the anonymous referee for his valuable comments.

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