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FIXED POINTS FOR S-CONTRACTIONS OF TYPE E ON S-METRIC SPACES

Thangjam Bimol[1], N. Priyobarta[2], Yumnam Rohen[3] and Kumam Anthony Singh[4]

[1]Department of Mathematics, Jadonang Memorial College,
Noney, Manipur, India
e-mail: btsalun29@gmail.com

[2]Department of Mathematics, Dhanamanjuri University,
Manipur, 795001, India
e-mail: ningthoujampriyo9@gmail.com

[3]Department of Mathematics, Manipur University,
Canchipur, Manipur, 795003, India
e-mail: ymnehor2008@yahoo.com

[4]Department of Mathematics, Manipur University,
Canchipur, Manipur, 795003, India
e-mail: anthonykumam@manipuruniv.ac.in

Abstract. In this paper, we extend the concept of S-contractions of type E in an S-metric space. Further, by combining simulation function and S-contractions of type E, we examine the existence and uniqueness of fixed point in a complete S-metric space. Sufficient examples are provided and application to the solution of integral equation is also made.

1. Introduction and Preliminaries

The result of Banach fixed point [2] has been generalised in various directions in the last decades. Some of the important generalisations of Banach's result based on contraction condition are Kannan [9], Chatterjea [5], Alber

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[0]Corresponding author: N. Priyobarta(ningthoujampriyo9@gmail.com).

and Delabrier [1], etc. Recently, Fulga and Proca [7, 8] introduced the concept of E-contraction. The concept of E-contraction is further extended to S-contraction of type E by Fulga and Karapinar [6]. Sedghi et al. [15] introduced S-metric space by generalising metric space. Motivated by the results of [6] and [15], in this paper we introduce S-contractions of type E on S-metric spaces. Also, we use simulation function introduced by Khojasteh et al. [10] in order to obtain fixed points. For more information, one can see in [3, 4, 11, 12, 13, 14, 16, 17].

Definition 1.1. ([10]) A function $\sigma : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}$ is referred to as a simulation function if it verifies the following criteria:

(i) $\sigma(0,0) = 0$,

(ii) $\sigma(x,y) < y - x$ for every $x, y \in \mathbb{R}^+$,

(iii) if $\{x_n\}, \{y_n\}$ are two sequences defined on $(0,\infty)$ such that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n > 0$, then

$$\limsup_{n\to\infty} \sigma(x_n, y_n) < 0. \tag{1.1}$$

The collection of all simulation functions will be represented as $\mathcal{S}$. It is evident, as a result of axiom (ii), that

$$\sigma(x,x) < 0, \ \forall \ x > 0. \tag{1.2}$$

Consider Φ as the set of continuous functions $\phi : [0,\infty) \to [0,\infty)$ that adhere to the following criterion:

$$\phi(t) = 0 \quad \text{if and only if} \quad t = 0.$$

Suppose (X, d) is a metric space, and $\sigma \in \mathcal{S}$ represents a simulation function. We define a function $f : X \to X$ as an S-contraction with respect to σ (as defined in [10]) if the inequality

$$\sigma\big(\mathsf{d}(f\theta, f\vartheta), \mathsf{d}(\theta,\vartheta)\big) \geq 0 \quad \text{for every } \theta, \vartheta \in X \tag{1.3}$$

is satisfied.

Remark 1.2. Deriving from axiom (ii), it becomes apparent that

$$\mathsf{d}(f\theta, f\vartheta) \neq \mathsf{d}(\theta,\vartheta) \text{ holds true for all different } \theta, \vartheta \in X. \tag{1.4}$$

This implies that in cases where S functions as an S-contraction, it is not possible for S to be an isometry. Consequently, if a S-contraction S possesses a fixed point (when such a point exists), it is necessarily unique.

Theorem 1.3. ([10]) *In a complete metric space, each S-contraction has precisely one fixed point. Moreover, every sequence generated by the Picard iterative process converges, and its limit corresponds to the unique fixed point.*

In the year 2012, Sedghi and colleagues (Sedghi et al. [15]) presented the concept of S-metric space.

Definition 1.4. ([15]) Let $X \neq \phi$. An S-metric on X is a function $S : X \times X \times X \to [0, \infty)$ satisfying:

(i) $S(\theta, \vartheta, \delta) = 0$ if and only if $\theta = \vartheta = \delta$,

(ii) $S(\theta, \vartheta, z) \leq S(\theta, \theta, a) + S(\vartheta, \vartheta, a) + S(\delta, \delta, a)$ for all $\theta, \vartheta, \delta, a \in X$.

The pair (X, S) is referred to as an S-metric space.

Definition 1.5. ([6]) A self-map S defined on a complete metric space (X, d) is classified as an S-contraction of type E with respect to σ if there exists $\sigma \in \mathcal{S}$ for which the following condition holds:

$$\sigma\left(\mathsf{d}(S\theta, S\vartheta), E(\theta, \vartheta)\right) \geq 0 \quad \text{for every } \theta, \vartheta \in X,$$

where

$$E(\theta, \vartheta) = \mathsf{d}(\theta, \vartheta) + |\mathsf{d}(\theta, S\theta) - \mathsf{d}(\vartheta, S\vartheta)|.$$

The set $\mathcal{C}_E(X)$ represents the collection of S-contractions of type E with respect to σ, which are defined on X.

Theorem 1.6. ([6]) *There exists a fixed point for every $S \in \mathcal{C}_E(X)$.*

Here, we extend the concept of S-contractions of type E in an S-metric space.

2. Main results

We will now present our primary findings. To achieve this, we initiate by introducing a new form of S-contraction.

Definition 2.1. A self-map denoted as f and defined on a complete S-metric space (X, S) is said to have S-contraction of type E_I with respect to σ if there exists $\sigma \in \mathcal{S}$ such that

$$\sigma\big(S(f\theta, f\vartheta, f\delta), E(\theta, \vartheta, \gamma)\big) \geq 0 \quad \text{for all } \theta, \vartheta, \gamma \in X, \tag{2.1}$$

where

$$\begin{aligned} E(\theta, \vartheta, \gamma) = S(\theta, \vartheta, \gamma) &+ |S(\theta, \theta, f\theta) - S(\vartheta, \vartheta, f\vartheta)| \\ &+ |S(\vartheta, \vartheta, f\vartheta) - S(\gamma, \gamma, f\gamma)|. \end{aligned} \tag{2.2}$$

Definition 2.2. A self-map denoted as f and defined on a complete S-metric space (X, S) is said to have S-contraction of type E_{II} with respect to σ if $\exists$ $\sigma \in \mathcal{S}$ such that

$$\sigma\big(S(f\theta, f\theta, f\vartheta), E(\theta, \theta, \vartheta)\big) \geq 0 \quad \text{for all } \theta, \vartheta \in X, \tag{2.3}$$

where

$$E(\theta,\theta,\vartheta)=S(\theta,\theta,\vartheta)+|S(\theta,\theta,f\theta)-S(\vartheta,\vartheta,f\vartheta)|\,.$$

Let $\mathcal{C}_E(X)$ denote the set of all S-contractions of type E with respect to σ defined on (X,S).

Theorem 2.3. *Every $f\in\mathcal{C}_E(X)$ possesses at least one fixed point.*

Proof. For any arbitrary θ_0 from the set X, we consider the constructive sequence θ_n contained within X. This sequence is defined as

$$\theta_{n+1}=f(\theta_n)=f^n(\theta_0)$$

for all $n\in\mathbb{N}$. Let's make the assumption that $\theta_{n+1}\neq\theta_n$ holds true for all natural numbers n. On the contrary, if the situation arises where $\theta_{n_0}=\theta_{n_0+1}$ for a certain $n_0\in\mathbb{N}$, then we have $f\theta_{n_0}=\theta_{n_0}$. This brings us to the conclusion of our proof, affirming that the point θ_{n_0} is indeed a fixed point of the function f. Consequently, $S(\theta_{n+1},\theta_{n+1},\theta_n)>0$ and from (2.1), it follows, for all $n\geq 1$, that

$$\begin{array}{rcl}0&\leq&\sigma\big(S(f\theta_n,f\theta_n,f\theta_{n-1}),E(\theta_n,\theta_n,\theta_{n-1})\big)\\ &=&\sigma\big(S(\theta_{n+1},\theta_{n+1},\theta_n),E(\theta_n,\theta_n,\theta_{n-1})\big)\\ &<&E(\theta_n,\theta_n,\theta_{n-1})-S(\theta_{n+1},\theta_{n+1},\theta_n).\end{array}\tag{2.4}$$

In conclusion, for all $n=1,2,3,\cdots$, we have

$$S(\theta_{n+1},\theta_{n+1},\theta_n)<E(\theta_n,\theta_n,\theta_{n-1}).\tag{2.5}$$

We take into account two situations in order to understand the inequality (2.5). For the first case, we assume that $S(\theta_{n+1},\theta_{n+1},\theta_n)\geq S(\theta_n,\theta_n,\theta_{n-1})$. The inequality (2.5) becomes

$$\begin{array}{rcl}S(\theta_{n+1},\theta_{n+1},\theta_n)&<&S(\theta_n,\theta_n,\theta_{n-1})+|S(\theta_n,\theta_n,\theta_{n+1})-S(\theta_{n-1},\theta_{n-1},\theta_n)|\\ &=&S(\theta_n,\theta_n,\theta_{n-1})+S(\theta_{n+1},\theta_{n+1},\theta_n)-S(\theta_n,\theta_n,\theta_{n-1})\\ &=&S(\theta_{n+1},\theta_{n+1},\theta_n).\end{array}$$

This leads to a contradiction. Hence, the subsequent case arises:

$$S(\theta_{n+1},\theta_{n+1},\theta_n)<S(\theta_n,\theta_n,\theta_{n-1}),\ \ \forall\ n=1,2,3,\cdots.\tag{2.6}$$

Thus, we can conclude that the sequence $\{S(\theta_n,\theta_n,\theta_{n-1})\}$ exhibits a non-increasing pattern and bounded below by 0. Consequently, the sequence

$\{S(\theta_n,\theta_n,\theta_{n-1})\}$ converges to some $S^* \geq 0$. Now

$$\begin{aligned}
&\lim_{n\to\infty} E(\theta_n,\theta_n,\theta_{n-1}) \qquad (2.7)\\
&= \lim_{n\to\infty}\big(S(\theta_n,\theta_n,\theta_{n-1}) + |S(\theta_n,\theta_n,\theta_{n+1}) - S(\theta_{n-1},\theta_{n-1},\theta_n)|\big)\\
&= \lim_{n\to\infty}\big(S(\theta_n,\theta_n,\theta_{n-1}) - S(\theta_n,\theta_n,\theta_{n+1}) + S(\theta_{n-1},\theta_{n-1},\theta_n)\big)\\
&= \lim_{n\to\infty}\big(2S(\theta_n,\theta_n,\theta_{n-1}) - S(\theta_{n+1},\theta_{n+1},\theta_n)\big)\\
&= S^*.
\end{aligned}$$

We claim that

$$S^* = \lim_{n\to\infty} S(\theta_n,\theta_n,\theta_{n-1}) = 0. \tag{2.8}$$

Imagine, in contrast, that $S^* > 0$. In this scenario, if we define

$$t_n = S(\theta_{n+1},\theta_{n+1},\theta_n)$$

and

$$s_n = E(\theta_n,\theta_n,\theta_{n-1}),$$

then we can deduce from the inequality (2.1) and condition (iii) that

$$\begin{aligned}
0 &\leq \limsup_{n\to\infty} \sigma\big(S(\theta_{n+1},\theta_{n+1},\theta_n), E(\theta_n,\theta_n,\theta_{n-1})\big)\\
&= \limsup_{n\to\infty} \sigma(t_n,s_n) \qquad (2.9)\\
&< 0.
\end{aligned}$$

This inconsistency demonstrates that $S^* = 0$.

Next, we will prove that $\{\theta_n\}$ is a Cauchy sequence. Imagine, in contrast, that the sequence $\{\theta_n\}$ is not a Cauchy sequence, then there exists subsequences $\{\theta_{\alpha(n)}\}$ and $\{\theta_{\beta(n)}\}$ of $\{\theta_n\}$ and a positive number $\varepsilon > 0$ such that $\alpha(n) > \beta(n) > n$ and

$$\begin{aligned}
S(\theta_{\alpha(n)},\theta_{\alpha(n)},\theta_{\beta(n)}) &\geq \varepsilon,\\
S(\theta_{\alpha(n)-1},\theta_{\alpha(n)-1},\theta_{\beta(n)}) &< \varepsilon, \ \forall\, n \in \mathbb{N}.
\end{aligned}$$

Therefore, by triangular inequality

$$\begin{aligned}
\varepsilon &\leq S(\theta_{\alpha(n)},\theta_{\alpha(n)},\theta_{\beta(n)})\\
&\leq 2S(\theta_{\alpha(n)},\theta_{\alpha(n)},\theta_{\alpha(n)-1}) + S(\theta_{\alpha(n)-1},\theta_{\alpha(n)-1},\theta_{\beta(n)})\\
&< 2S(\theta_{\alpha(n)},\theta_{\alpha(n)},\theta_{\alpha(n)-1}) + \varepsilon
\end{aligned}$$

and by (2.8), we get

$$\lim_{n\to\infty} S(\theta_{\alpha(n)},\theta_{\alpha(n)},\theta_{\alpha(n)-1}) = \varepsilon. \tag{2.10}$$

On the other hand, we can easily show that

$$\begin{aligned}&\big|S(\theta_{\alpha(n)-1},\theta_{\alpha(n)-1},\theta_{\beta(n)-1})-S(\theta_{\alpha(n)},\theta_{\alpha(n)},\theta_{\beta(n)})\big|\\&\leq 2\big(S(\theta_{\alpha(n)-1},\theta_{\alpha(n)-1},\theta_{\alpha(n)})+S(\theta_{\beta(n)-1},\theta_{\beta(n)-1},\theta_{\beta(n)})\big)\end{aligned}$$

and from (2.8), respectively (2.10)

$$\lim_{n\to\infty} S(\theta_{\alpha(n)-1},\theta_{\alpha(n)-1},\theta_{\beta(n)-1})=\varepsilon. \tag{2.11}$$

And from equations (2.2), (2.8) and (2.10), it follows that

$$\begin{aligned}\lim_{n\to\infty} E(\theta_{\alpha(n)-1},\theta_{\alpha(n)-1},\theta_{\beta(n)-1}) &= \lim_{n\to\infty}\big\{S(\theta_{\alpha(n)-1},\theta_{\alpha(n)-1},\theta_{\beta(n)-1})\\&\quad+|S(\theta_{\alpha(n)-1},\theta_{\alpha(n)-1},f\theta_{\beta(n)-1})\\&\quad-S(\theta_{\beta(n)-1},\theta_{\beta(n)-1},f\theta_{\beta(n)-1})|\big\}\\&=\lim_{n\to\infty}\{S(\theta_{\alpha(n)-1},\theta_{\alpha(n)-1},\theta_{\beta(n)-1})\\&\quad+|S(\theta_{\alpha(n)-1},\theta_{\alpha(n)-1},\theta_{\alpha(n)})\\&\quad-S(\theta_{\beta(n)-1},\theta_{\beta(n)-1},\theta_{\beta(n)})|\big\}\\&=\varepsilon.\end{aligned} \tag{2.12}$$

Letting $t_n=S(\theta_{\alpha(n)},\theta_{\alpha(n)},\theta_{\beta(n)})$ and $s_n=E(\theta_{\alpha(n)-1},\theta_{\alpha(n)-1},\theta_{\beta(n)-1})$, we have $\lim_{n\to\infty} s_n=\lim_{n\to\infty} t_n=\varepsilon$ and combining with (iii)

$$\begin{aligned}0&\leq\limsup_{n\to\infty}\sigma\big(S(f\theta_{\alpha(n)-1},f\theta_{\alpha(n)-1},f\theta_{\beta(n)-1}),E(\theta_{\alpha(n)-1},\theta_{\alpha(n)-1},\theta_{\beta(n)-1})\big)\\&=\limsup_{n\to\infty}\sigma\big(S(\theta_{\alpha(n)},\theta_{\alpha(n)},\theta_{\beta(n)}),E(\theta_{\alpha(n)-1},\theta_{\alpha(n)-1},\theta_{\beta(n)-1})\big)\\&=\limsup_{n\to\infty}\sigma(t_n,s_n)\\&<0.\end{aligned} \tag{2.13}$$

This inconsistency demonstrates that $\varepsilon=0$, hence $\{\theta_n\}$ is Cauchy. Because of the completeness of the space (X,S), a point θ^* exists within the set X, such that

$$\lim_{n\to\infty}\theta_n=\theta^*=0. \tag{2.14}$$

Our next task is to prove that $\theta^*=f\theta^*$. Adopting a proof by contradiction approach, let's suppose that $S(\theta^*,\theta^*,f\theta^*)>0$. According to property (ii), for a sufficiently large $r\in\mathbb{N}$, it follows that

$$\begin{aligned}0&\leq\sigma\big(S(f\theta_r,f\theta_r,f\theta^*),E(\theta_r,\theta_r,\theta^*)\big)\\&=\sigma\big(S(\theta_{r+1},\theta_{r+1},f\theta^*),E(\theta_r,\theta_r,\theta^*)\big)\\&<E(\theta_r,\theta_r,\theta^*)-S(\theta_{r+1},\theta_{r+1},f\theta^*).\end{aligned} \tag{2.15}$$

Considering the sequences

$$t_r^*=S(\theta_{r+1},\theta_{r+1},f\theta^*)$$

and

$$s_r^* = E(\theta_r, \theta_r, \theta^*) = S(\theta_r, \theta_r, \theta^*) + |S(\theta_r, \theta_r, f\theta_r) - S(\theta^*, \theta^*, f\theta^*)|,$$

we find that

$$\lim_{r\to\infty} t_r^* = \lim_{r\to\infty} s_r^* = S(\theta^*, \theta^*, f\theta^*) > 0, \tag{2.16}$$

which implies together with (2.15)

$$0 \le \limsup_{r\to\infty} \sigma\big(S(f\theta_r, f\theta_r, f\theta^*), E(\theta_r, \theta_r, \theta^*)\big) < 0,$$

which is a contradiction. Thus, we have $S(\theta^*, \theta^*, f\theta^*) = 0$, that is, $f\theta^* = \theta^*$. $\square$

Example 2.4. Let $X = [0, \frac{5}{3}] \cup \{2\}$ and $S : X \times X \times X \to \mathbb{R}$ by

$$S(\theta, \vartheta, \delta) = |\theta - \vartheta| + |\vartheta - \delta|.$$

Suppose that $\sigma : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is defined as $\sigma(s,t) = \frac{s}{2} - t$ and hence $\sigma \in \mathcal{S}$. Define a map $f : X \to X$ as follows

$$f(\theta) = \begin{cases} 1, & \text{if } \theta \in [0, \frac{5}{3}], \\ \frac{1}{3}, & \text{if } \theta = 2. \end{cases}$$

Notice that for $\theta = \vartheta = 2$ and $\delta = \frac{5}{3}$, we have

$$\begin{aligned} S\left(2, 2, \frac{5}{3}\right) &= |2-2| + \left|2 - \frac{5}{3}\right| = \frac{1}{3}, \\ S\left(f2, f2, f\frac{5}{3}\right) &= S\left(\frac{1}{3}, \frac{1}{3}, 1\right) = \left|\frac{1}{3} - \frac{1}{3}\right| + \left|\frac{1}{3} - 1\right| = \frac{2}{3}, \end{aligned}$$

and for these values, there is no $k_1 \in [0,1)$ such that

$$S\left(f2, f2, f\frac{5}{3}\right) = \frac{2}{3} \le k_1 \frac{1}{3} = k_1 S\left(2, 2, \frac{5}{3}\right).$$

Hence, the function f is not a contraction mapping. But, it exhibits S-contraction of type E_I. To validate our assertion, we must analyze two separate scenarios:

Case(i): $\delta = 2, \theta = \vartheta < 1$. Then we find that

$$S(\theta, \theta, 2) = 2 - \theta, S(\theta, \theta, f\theta) = 1 - \theta$$

and

$$S(2, 2, f2) = |2 - \frac{1}{3}| = \frac{5}{3}.$$

Also, we have

$$S(f\theta, f\theta, f2) = S(1, 1, \frac{1}{3}) = \frac{2}{3}.$$

Since

$$E(\theta,\theta,2) = 2-\theta+|1-\theta-\frac{5}{3}| = 2-\theta+\frac{3\theta+2}{3} = \frac{8}{3},$$

we have that

$$\begin{aligned}\sigma\big(S(f\theta,f\theta,f2),E(\theta,\theta,2)\big) &= \frac{1}{2}E(\theta,\theta,2)-S(f\theta,f\theta,f2)\\ &= \frac{1}{2}\cdot\frac{8}{3}-\frac{2}{3} > 0.\end{aligned}$$

Case(ii): If $\delta = 2, \theta = \vartheta \geq 1$, then

$$S(\theta,\theta,2) = 2-\theta,\ S(\theta,\theta,f\theta) = \theta-1$$

and

$$S(2,2,f2) = |2-\frac{1}{3}| = \frac{5}{3}.$$

Also, we have

$$S(f\theta,f\theta,f2) = S(1,1,\frac{1}{3}) = \frac{2}{3}.$$

As a result, we have

$$E(\theta,\theta,2) = 2-\theta+|\theta-1-\frac{5}{3}| = 2-\theta+\frac{8-3\theta}{3} = \frac{14-6\theta}{3}$$

and

$$\begin{aligned}\sigma\big(S(f\theta,f\theta,f2),E(\theta,\theta,2)\big) &= \frac{1}{2}E(\theta,\theta,2)-S(f\theta,f\theta,f2)\\ &= \frac{14-6\theta}{6}-\frac{2}{3} = \frac{5-3\theta}{3}\\ &\geq 0.\end{aligned}$$

Our deduction leads us to the conclusion that f is a S-contraction of type E_I. Moreover, all the criteria of Theorem 2.3 are satisfied and $\theta = 1$ is a fixed point of f. Finally, it's worth noting that the uniqueness of the fixed point is a consequence of Remark 1.2.

Example 2.5. Let $X = \{1,3,4,5\}$ and $S : X\times X\times X \to \mathbb{R}$ defined by

$$S(\theta,\vartheta,\delta) = |\theta-\vartheta|+|\vartheta-\delta|.$$

Let $f : X \to X$ be defined as $f1 = f3 = f4 = 3, f5 = 1$ and $\sigma(t,s) = \frac{1}{2}s-t$. Then it can easily calculate that

$$\begin{aligned} S(3,3,4) = S(4,4,5) &= 1,\ S(3,3,5) = S(1,1,3) = 2,\\ S(1,1,4) = 3,\ S(1,1,5) &= 4,\\ S(f3,f3,f4) = S(f1,f1,f3) &= S(f1,f1,f4) = 0,\\ S(f3,f3,f5) = S(f4,f4,f5) &= S(f1,f1,f5) = 2.\end{aligned}$$

Also, we have

$$E(1,1,4) = E(1,1,3) \ = \ E(4,4,5) \ = \ 4,$$
$$E(1,1,5) = E(3,3,5) \ = \ 6 \ \text{ and } \ E(3,3,4) \ = \ 2.$$

First of all we show that f is not a contraction mapping. This can be illustrated by considering the values $\theta = \vartheta = 4$ and $\delta = 5$. In this scenario, it's impossible to identify a real constant $k_2 \in [0,1)$ that would satisfy the condition $S(f4, f4, f5) = 2 \leq k_2 S(4,4,5)$. As a result, the function f is not a contraction mapping.

In the following steps, we will establish that function f satisfies the conditions of being an S-contraction of type E_I. To achieve this, we will systematically analyze all possible cases:

For $\theta = \vartheta = 1, \delta = 3$, we have

$$\sigma\big(S(f1, f1, f3), E(1,1,3)\big) = \sigma(0,4) = \frac{4}{2} - 0 = 2.$$

For $\theta = \vartheta = 1, \delta = 4$, we have

$$\sigma\big(S(f1, f1, f4), E(1,1,4)\big) = \sigma(0,4) = \frac{4}{2} - 0 = 2.$$

For $\theta = \vartheta = 1, \delta = 5$, we have

$$\sigma\big(S(f1, f1, f5), E(1,1,5)\big) = \sigma(2,6) = \frac{6}{2} - 2 = 1.$$

For $\theta = \vartheta = 3, \delta = 4$, we have

$$\sigma\big(S(f3, f3, f4), E(3,3,4)\big) = \sigma(0,2) = \frac{2}{2} - 0 = 1.$$

For $\theta = \vartheta = 3, \delta = 5$, we have

$$\sigma\big(S(f3, f3, f5), E(3,3,5)\big) = \sigma(2,4) = \frac{4}{2} - 2 = 0.$$

For $\theta = \vartheta = 4, \delta = 5$, we have

$$\sigma\big(S(f4, f4, f5), E(4,4,5)\big) = \sigma(2,4) = \frac{4}{2} - 2 = 0.$$

Evidently, $f \in C_E(X)$.

Furthermore, all the requirements stated in Theorem 2.3 are attained, and $\theta = 3$ is a fixed point of the function f. As demonstrated in the previous example, the uniqueness of the fixed point is derived from Remark 1.2.

Example 2.6. Let $X = [0, \frac{1}{2}] \cup \{\frac{3}{4}\}$ and define

$$S(\theta, \vartheta, \delta) = \begin{cases} \max\{\theta, \vartheta, \delta\}, & \text{if not } \theta = \vartheta = \delta, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$f(\theta)=\begin{cases}\frac{\theta}{1+\theta}, \text{ if } \theta\in[0,\frac{1}{4})\cup(\frac{1}{4},1],\\ \\ \frac{1}{2}, \text{ if } \theta=\frac{3}{4}.\end{cases}$$

Now, we will show that f is an S-contraction of type E_I for $\sigma(t,s)=\frac{s}{s+1}-t$.

Case(i): For $0\le r\le\vartheta\le\theta\le\frac{1}{2}$, we have

$$S(\theta,\vartheta,\delta)=\max\{\theta,\vartheta,\delta\}=\theta,$$

$$S(f\theta,f\vartheta,f\delta)=\max\{\frac{\theta}{\theta+1},\frac{\vartheta}{\vartheta+1},\frac{\delta}{\delta+1}\}=\frac{\theta}{\theta+1},$$

$$S(\theta,\theta,f\theta)=\max\{\theta,\frac{\theta}{\theta+1}\}=\theta,$$

$$S(\vartheta,\vartheta,f\vartheta)=\max\{\vartheta,\frac{\vartheta}{\vartheta+1}\}=\vartheta,$$

$$S(\delta,\delta,f\delta)=\max\{\delta,\frac{\delta}{\delta+1}\}=\delta.$$

So, we have

$$E(\theta,\vartheta,\delta)=\theta+|\theta-\vartheta|+|\vartheta-\delta|=2\theta-\delta$$

and

$$\begin{aligned}\sigma\big(S(f\theta,f\vartheta,f\delta),E(\theta,\vartheta,\delta)\big)&=\frac{E(\theta,\vartheta,\delta)}{1+E(\theta,\vartheta,\delta)}-S(f\theta,f\vartheta,f\delta)\\&=\frac{2\theta-\delta}{1+2\theta-\delta}-\frac{\theta}{\theta+1}\\&=\frac{\theta-\delta}{(2\theta-\delta+1)}(\theta+1)\\&>0.\end{aligned}$$

Clearly, the above observation remains applicable in the cases where $0\le\theta\le\vartheta\le\delta\le\frac{1}{2}$ and $0\le\theta\le\delta\le\vartheta\le\frac{1}{2}$.

Case(ii): For $0\le\theta\le\vartheta\le\frac{1}{2}$ and $\delta=\frac{3}{4}$, we have

$$S(\theta,\vartheta,\delta)=\max\{\theta,\vartheta,\delta\}=\frac{3}{4}$$

and

$$S(f\theta,f\vartheta,f\delta)=\max\{\frac{\theta}{\theta+1},\frac{\vartheta}{\vartheta+1},\frac{1}{2}\}=\frac{1}{2},$$

$$S(\theta,\theta,f\theta)=\max\{\theta,\theta,\frac{\theta}{\theta+1}\}=\theta, S(\vartheta,\vartheta,f\vartheta)=\max\{\vartheta,\vartheta,\frac{\vartheta}{\vartheta+1}\}=\vartheta$$

and

$$S(\delta,\delta,f\delta)=\max\{\delta,\delta,\frac{\delta}{\delta+1}\}=\delta=\frac{3}{4}.$$

So, we have

$$\begin{aligned}E(\theta,\vartheta,\delta) &= S(\theta,\vartheta,\delta)+|S(\theta,\theta,f\theta)-S(\vartheta,\vartheta,f\vartheta)| \\ &\quad +|S(\vartheta,\vartheta,f\vartheta)-S(\delta,\delta,f\delta)| \\ &= \frac{3}{4}+|\theta-\vartheta|+|\vartheta-\frac{3}{4}| \\ &= \frac{3}{4}+\vartheta-\theta+\frac{3}{4}-\vartheta \;=\; \frac{3}{2}-\theta.\end{aligned}$$

Consequently

$$\sigma(\frac{1}{2},\frac{3}{2}-\theta)=\frac{\frac{3}{2}-\theta}{\frac{5}{2}-\theta}-\frac{1}{2}=\frac{1-2\theta}{5-2\theta}>0.$$

Case(iii): For $0\leq\delta\leq\frac{1}{2}$ and $\theta=\vartheta=\frac{3}{4}$, it is similar to Case(ii).

In every scenario, it is evident that f belongs to the set $C_E(X)$. This concludes the demonstration, leading us to the deduction that f has a fixed point at $\theta=0$. Referring to Remark 1.2, this fixed point of f is unique.

3. Consequences and application

Within this section, we present a corollary and delve into an instance where the main outcome finds application, allowing for the depiction of a solution to an integral equation.

Corollary 3.1. *Let $f:X\to X$ be defined on a complete S-metric space (X,S). If there exist $\mu_1,\mu_2\in\Phi$ with $\mu_1(s)<s\leq\mu_2(s)$ for all $s>0$, such that for all $\theta,\vartheta,\delta\in X$, the following inequality is fulfilled*

$$\mu_2\left(S(f\theta,f\vartheta,f\delta)\right)\leq\mu_1\left(E(\theta,\vartheta,\delta)\right),$$

where

$$E(\theta,\vartheta,\delta)=S(\theta,\vartheta,\delta)+|S(\theta,\theta,f\theta)-S(\vartheta,\vartheta,f\vartheta)|+|S(\vartheta,\vartheta,f\vartheta)-S(\delta,\delta,f\delta)|.$$

Then f has a unique fixed point.

Proof. Take $\sigma(t,s)=\sigma_1(t,s)$ in Example 2.4 and apply Theorem 2.3. □

By opting the function σ provided in Example 2.4 and utilizing Theorem 2.3, we can derive additional corollaries similar to Corollary 3.1. So, we skip list of these corollaries through this analogy.

Consider the set $X=C(I,\mathbb{R})$ represents the collection of all continuous functions on $I=[0,1]$ endowed with an S-metric

$$S(\theta,\vartheta,\gamma)=||\theta-\vartheta||+||\vartheta-\gamma||=\sup\{|\theta(s)-\vartheta(s)|+|\vartheta(s)-\gamma(s)|:s\in I\},$$

for all $\theta, \vartheta, \gamma \in X$. Consequently, the pair (X, S) establishes an S-metric space that is complete. Now, we will delve into the analysis of the integral equation.

$$\theta(s) = \xi(s) + \int_0^1 K(s,x)\eta(x,\theta(x))dx, \ \ s \in [0,1]. \tag{3.1}$$

Consider the continuous functions $\eta : I \times \mathbb{R} \to \mathbb{R}$ and $\xi : I \to \mathbb{R}$, as well as a function $K : I \times I \to \mathbb{R}^+$ such that $K(s,.) \in L^1(I)$ for each $s \in [0,1]$. We address the mapping $f : X \to X$, which is defined as follows:

$$f(\theta)(s) = \xi(s) + \int_0^1 K(s,x)\eta(x,\theta(x))dx, \ \ s \in [0,1]. \tag{3.2}$$

Theorem 3.2. *The integral equation* (3.1) *possesses a unique solution within the set X when the subsequent conditions are satisfied:*

(a_1) *there is a $\mu \in \Phi$ such that for each $s > 0$, $\mu(s) < s$ satisfying*

$$\begin{aligned} 0 &\leq |\eta(x,\theta_1(x)) - \eta(x,\theta_2(x))| \\ &\leq \mu\big(|\theta_1(x) - \theta_2(x)| + \big||\theta_1(x) - f(\theta_1)(x)| - |\theta_2(x) - f(\theta_2)(x)|\big|\big) \end{aligned}$$

for all $x \in I$ and $\theta_1, \theta_2 \in X$.

(a_2) *followed by*

$$\sup_{s\in I} \int_0^1 K(s,x)dx \leq 1.$$

Proof. It should be noted that any fixed point of (3.1) is also a solution for the (3.1). It can be deduced from (a_1) and (a_2) that

$$\begin{aligned} |f(\theta_1)(s) - f(\theta_2)(s)| &= \left| \int_0^1 K(s,x)[\eta(x,\theta_1(x)) - \eta(x,\theta_2(x))]dx \right| \\ &\leq \int_0^1 K(s,x)\big|\eta(x,\theta_1(x)) - \eta(x,\theta_2(x))\big|dx \\ &\leq \int_0^1 K(s,x)\mu\big(|\theta_1(x) - \theta_2(x)| + \big||\theta_1(x) \\ &\quad - f(\theta_1)(x)| - |\theta_2(x) - f(\theta_2)(x)|\big|\big)dx \\ &\leq \mu\big(E(\theta_1,\theta_1,\theta_2)\big), \end{aligned}$$

where $E(\theta_1,\theta_1,\theta_2) = ||\theta_1 - \theta_2|| + \big|||\theta_1 - f\theta_1|| - ||\theta_2 - f\theta_2||\big|$. Hence, we can deduce that

$$||f\theta_1 - f\theta_2|| \leq \mu\big(||\theta_1 - \theta_2|| + \big|||\theta_1 - f\theta_1|| - ||\theta_2 - f\theta_2||\big|\big).$$

Therefore, we have

$$\sigma\big(S(f\theta_1, f\theta_1, f\theta_2), E(\theta_1,\theta_1,\theta_2)\big) = \mu\big(E(\theta_1,\theta_1,\theta_2)\big) - S(f\theta_1, f\theta_1, f\theta_2) \geq 0.$$

This leads to the inference that all the conditions stated in Corollary 3.1 are fulfilled, and consequently, so are the conditions of Theorem 2.3. As a result, the operator f possesses a unique fixed point, exclusively representing solution to integral equation (3.1) within the domain X. □

Example 3.3. To illustrate Theorem 3.2, we examine the following integral equation as an example:

$$\theta(x) = \frac{1}{1+s^4} + \frac{1}{3}\int_0^1 \frac{x\sin 2x}{12(1+s^2)}\frac{|\theta|}{1+|\theta|}dx,\ \ s \in [0,1]. \tag{3.3}$$

This equation is derived from equation (3.1) by selecting $\xi(s) = \frac{1}{1+s^4}$, $K(s,x) = \frac{x}{2(1+s^2)}$, and $\eta(s,\theta) = \frac{|\theta|\sin 2s}{6(1+|\theta|)}$.

Consider a self-mapping f defined as follows:

$$f(\theta)(s) = \xi(s) + \int_0^1 K(s,x)\eta(x,\theta(x))dx,\ \ s \in [0,1], \tag{3.4}$$

taking $\mu(s) = \frac{s}{2}$, we get that

$$\begin{aligned} \big|\eta(s,\theta_1) - \eta(s,\theta_2)\big| &= \left|\frac{\sin 2s}{6}\frac{|\theta_1|}{1+|\theta_1|} - \frac{\sin 2s}{6}\frac{|\theta_2|}{1+|\theta_2|}\right| \\ &\le \frac{1}{6}|\theta_1-\theta_2| \le \mu\left(\big||\theta_1-\theta_2| + |\theta_1-\theta_2| - |\theta_1-\theta_2|\big|\right) \\ &= \mu\big(E(\theta_1,\theta_1,\theta_2)\big). \end{aligned}$$

On the other hand,

$$\sup_{s\in I}\int_0^1 K(s,x)dx = \sup_{s\in I}\int_0^1 \frac{x}{2(1+s^2)}dx = \frac{1}{4} < 1.$$

Therefore, we can deduce that equation (3.3) possesses a unique solution within the set $C(I,\mathbb{R})$.

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