J. Korean Math. Soc. **61** (2024), No. 5, pp. 1035–1050
https://doi.org/10.4134/JKMS.j230437
pISSN: 0304-9914 / eISSN: 2234-3008

TRANSLATION THEOREM FOR THE ANALYTIC FEYNMAN INTEGRAL ASSOCIATED WITH BOUNDED LINEAR OPERATORS ON ABSTRACT WIENER SPACES AND AN APPLICATION

Jae Gil Choi

Abstract. The Cameron–Martin translation theorem describes how Wiener measure changes under translation by elements of the Cameron–Martin space in an abstract Wiener space (AWS). Translation theorems for the analytic Feynman integrals also have been established in the literature. In this article, we derive a more general translation theorem for the analytic Feynman integral associated with bounded linear operators (B.L.OP.) on AWSs. To do this, we use a certain behavior which exists between the analytic Fourier–Feynman transform (FFT) and the convolution product (CP) of functionals on AWS. As an interesting application, we apply this translation theorem to evaluate the analytic Feynman integral of the functional

$$F(x) = \exp\left(-iq\int_0^T x(t)y(t)dt\right), \quad y \in C_0[0,T],\ q \in \mathbb{R}\setminus\{0\}$$

defined on the classical Wiener space $C_0[0,T]$.

1. Introduction and background

In order to provide our translation theorem for the "analytic Feynman integral associated with B.L.OP.s" on AWSs, we first follow the exposition of [7, 10, 11, 16, 17, 19].

Let $\mathbb{H}$ be a real infinite dimensional Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and associated norm $|\cdot|$, and let $\mathbb{B}$ be a real separable Banach space with norm $\|\cdot\|$. It is assumed that $\mathbb{H}$ is continuously, linearly, and densely embedded in $\mathbb{B}$ by a natural injection. Let ν be a centered Gaussian probability measure on $(\mathbb{B},\mathcal{B}(\mathbb{B}))$, where $\mathcal{B}(\mathbb{B})$ is the Borel σ-field of $\mathbb{B}$. The triple $(\mathbb{H},\mathbb{B},\nu)$ is called an

Received August 14, 2023; Revised January 28, 2024; Accepted May 8, 2024.
2020 *Mathematics Subject Classification.* Primary 46B09, 46G12; Secondary 28C20, 44A15, 42B10.
Key words and phrases. Abstract Wiener space, Cameron–Martin translation theorem, bounded linear operator, analytic Feynman integral, analytic Fourier–Feynman transform, convolution product.

©2024 Korean Mathematical Society

AWS if

$$\int_{\mathbb{B}} \exp\big(i(h,x)\big) d\nu(x) = \exp\bigg(-\frac{1}{2}|h|^2\bigg)$$

for any $h \in \mathbb{B}^*$, where $(\cdot,\cdot)$ denotes the natural dual paring ($\mathbb{B}^*$–$\mathbb{B}$ pairing), and where $\mathbb{B}^*$ is the topological dual of $\mathbb{B}$. Let $\mathbb{H}^*$ be the topological dual of $\mathbb{H}$. Then $\mathbb{B}^*$ is identified as a dense subspace of $\mathbb{H}^* \cong \mathbb{H}$ in the sense that, for all $y \in \mathbb{B}^*$ and $x \in \mathbb{H}$,

$$\langle y, x\rangle = (y,x). \tag{1.1}$$

Thus we have the triple

$$\mathbb{B}^* \subset \mathbb{H}^* \cong \mathbb{H} \subset \mathbb{B}. \tag{1.2}$$

The Hilbert space $\mathbb{H}$ is called the Cameron–Martin space in the AWS $\mathbb{B}$. For more details, see [14, 16, 19].

Let $\{e_n\}_{n=1}^{\infty}$ be a complete orthonormal set in $\mathbb{H}$ such that e_j's are in $\mathbb{B}^*$. For each $h \in \mathbb{H}$ and $x \in \mathbb{B}$, a stochastic linear functional $(h,x)^\sim$ is defined by

$$(h,x)^\sim = \begin{cases} \lim\limits_{n\to\infty} \sum_{j=1}^{n} \langle h, e_j\rangle (e_j, x), & \text{if the limit exists,} \\ 0, & \text{otherwise.} \end{cases} \tag{1.3}$$

By the definition of the stochastic linear functional $(\cdot,\cdot)^\sim$ and (1.1), it is clear that $(\theta,x)^\sim = (\theta,x)$ for all $\theta \in \mathbb{B}^*$ and $x \in \mathbb{B}$. It is well-known [11, 16, 17, 19] that for every non-zero h in $\mathbb{H}$, $(h,x)^\sim$ is a non-degenerate Gaussian random variable on $\mathbb{B}$ with mean 0 and variance $|h|^2$. The stochastic linear functional $(h,x)^\sim$ given by (1.3) is essentially independent of the choice of the complete orthonormal set used in its definition. Also, if both h and x are in $\mathbb{H}$, then Parseval's identity gives $(h,x)^\sim = \langle h,x\rangle$. Furthermore, $(h,\lambda x)^\sim = (\lambda h, x)^\sim = \lambda(h,x)^\sim$ for any $\lambda \in \mathbb{R}$, $h \in \mathbb{H}$ and $x \in \mathbb{B}$.

Remark 1.1. By Kallianpur and Bromley's study [16], the limit in (1.3) exists for ν-a.e. $x \in \mathbb{B}$, and for each $h \in \mathbb{H}(\cong \mathbb{H}^*)$, the Gaussian random variable $(h,\cdot)^\sim$ is in $L_2(\mathbb{B}, \mathcal{B}(\mathbb{B}), \nu)$. For a more detailed study, we also refer the reader to the reference [19, Section 1.4].

Given two Banach spaces $\mathbb{X}$ and $\mathbb{Y}$, let $\mathcal{L}(\mathbb{X},\mathbb{Y})$ denote the Banach space of all B.L.OP.s from $\mathbb{X}$ to $\mathbb{Y}$, and let $\mathcal{L}(\mathbb{X}) \equiv \mathcal{L}(\mathbb{X},\mathbb{X})$.

By the definition of the Banach space adjoint operator, given an operator $A \in \mathcal{L}(\mathbb{B})$, there exists a B.L.OP. $A^* : \mathbb{B}^* \to \mathbb{B}^*$ such that for all $\theta \in \mathbb{B}^*$ and $x \in \mathbb{B}$,

$$(A^*\theta)x = \theta(Ax). \tag{1.4}$$

By the structure of the dual paring and the triple (1.2), equation (1.4) can be rewritten by

$$(A^*\theta, x) = (\theta, Ax).$$

We are now ready to state the motivation and the aim of this article. The Cameron–Martin translation theorem on classical Wiener space was introduced in [2,3]. On the other hand, Cameron and Storvick [5,6] provided a translation theorem for the analytic Feynman integral of functionals on the classical Wiener space.

The Cameron–Martin translation theorem describes how Wiener measure changes under translation by certain elements of the Cameron–Martin space. The translation theorem and an analogue on AWSs can be found in [18,20].

Theorem 1.2. *Let $F \in L_1(\mathbb{B}, \mathcal{B}(\mathbb{B}), \nu)$ and let $x_0 \in \mathbb{H}$. Then it follows that*

$$\int_{\mathbb{B}} F(x + x_0) d\nu(x) = \exp\bigg(-\frac{1}{2}|x_0|^2 \bigg) \int_{\mathbb{B}} F(x) \exp\big((x_0, x)^{\sim} \big) d\nu(x).$$

There have been a tremendous number of papers in various useful applications of the translation theorem in connection with infinite dimensional analysis. Recently, Chang and Choi [7] developed the translation theorem for the abstract Wiener integral associated with B.L.OP.s on $\mathbb{B}$.

Theorem 1.3 ([7])**.** *Let A_1 and A_2 be operators in $\mathcal{L}(\mathbb{B})$, let $\theta \in \mathbb{B}^*$, and let F be a functional on $(\mathbb{B}, \mathcal{B}(\mathbb{B}), \nu)$ such that $F(A_1 x)$ is ν-integrable over $\mathbb{B}$. Then it follows that*

$$\begin{aligned} &\int_{\mathbb{B}} F(A_1 x + A_1 A_2^* \theta) d\nu(x) \\ &= \exp\bigg(-\frac{1}{2}|A_2^* \theta|^2 \bigg) \int_{\mathbb{B}} F(A_1 x) \exp\big((\theta, A_2 x) \big) d\nu(x). \end{aligned} \tag{1.5}$$

On the other hand, the analytic FFT is one of the most important transforms defined on infinite dimensional Banach spaces. The analytic FFT and its properties are similar in many respects to the Fourier transform defined for functions on Euclidean spaces. In [1,8,9], the authors defined an analytic FFT for functionals on $\mathbb{B}$ and studied the existence of the FFT and related topics. For an elementary survey of the analytic FFT, see the reference [23] and the references cited therein. Recently, in [10], Choi defined an analytic Feynman integral and an analytic FFT associated with B.L.OP.s on $\mathbb{B}$, and investigated many properties of the integral and the transform.

In this paper, we study a certain aspect of the analytic Feynman integral of functionals on $\mathbb{B}$. More precisely, we establish a translation theorem for the analytic Feynman integral associated with B.L.OP.s on $\mathbb{B}$, see Theorem 3.2 below. To provide a simple proof of our translation theorem for the analytic Feynman integral, we use the structures of the analytic FFT associated with B.L.OP.s and the CP with respect to B.L.OP.s on the AWS $\mathbb{B}$. That is, using a certain relation between the analytic FFT and the CP for functionals on $\mathbb{B}$, we establish a translation theorem for the analytic Feynman integral associated with B.L.OP.s on $\mathbb{B}$. As an interesting application, we apply this translation theorem to evaluate the analytic Feynman integral of the functional

$$F(x) = \exp\bigg(-iq \int_0^T x(t) y(t) dt \bigg), \quad y \in C_0[0,T],\ q \in \mathbb{R} \setminus \{0\}$$

defined on the classical Wiener space $(C_0[0,T], m_{\mathrm{w}})$.

2. Analytic Fourier–Feynman transform associated with bounded linear operators

In order to define an analytic Feynman integral of functionals on the AWS $\mathbb{B}$, we need the concept of the "scale-invariant measurability". Let $\mathcal{W}(\mathbb{B})$ be the class of ν-Carathéodory measurable subsets of $\mathbb{B}$. In order to study the translation structure for the analytic Feynman integral on $\mathbb{B}$, we will consider the complete probability space $(\mathbb{B}, \mathcal{W}(\mathbb{B}), \nu)$.

A subset S of $\mathbb{B}$ is said to be scale-invariant measurable (s.i.m.), see [11], provided ρS is $\mathcal{W}(\mathbb{B})$-measurable for every $\rho > 0$, and a s.i.m. subset N of $\mathbb{B}$ is said to be scale-invariant null provided $\nu(\rho N) = 0$ for every $\rho > 0$. A property that holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (s-a.e.). A functional F on $\mathbb{B}$ is said to be s.i.m. provided F is defined on a s.i.m. set and $F(\rho\,\cdot\,)$ is $\mathcal{W}(\mathbb{B})$-measurable for every $\rho > 0$. If two functionals F and G on $\mathbb{B}$ are equal s-a.e., i.e., for each $\rho > 0$, $\nu(\{x \in \mathbb{B} : F(\rho x) \neq G(\rho x)\}) = 0$, then we denote this equivalence relation between the functionals by $F \approx G$.

Given a ν-integrable functional F on $\mathbb{B}$ and an operator A in $\mathcal{L}(\mathbb{B})$, we will denote the abstract Wiener integral (associated with the operator A) of F by

$$\mathcal{I}_A[F] \equiv \mathcal{I}_{A,x}[F(Ax)] \equiv \int_{\mathbb{B}} F(Ax)d\nu(x).$$

Let $\mathbb{C}$, $\mathbb{C}_+$ and $\widetilde{\mathbb{C}}_+$ denote the set of complex numbers, complex numbers with positive real part and non-zero complex numbers with nonnegative real part, respectively.

Definition 2.1. Let $F : \mathbb{B} \to \mathbb{C}$ be a functional which is s-a.e. defined and s.i.m. such that given an operator $A \in \mathcal{L}(\mathbb{B})$, the abstract Wiener integral

$$J_F(A;\lambda) = \mathcal{I}_{A,x}[F(\lambda^{-1/2}Ax)] = \int_{\mathbb{B}} F(\lambda^{-1/2}Ax)d\nu(x)$$

exists as a finite number for all $\lambda > 0$. If there exists a function $J_F^*(A;\cdot)$ analytic on $\mathbb{C}_+$ such that $J_F^*(A;\lambda) = J_F(A;\lambda)$ for all $\lambda > 0$, then $J_F^*(A;\lambda)$ is defined to be the analytic Wiener integral (associated with the operator A) of F over $\mathbb{B}$ with parameter λ. For $\lambda \in \mathbb{C}_+$ we write

$$\mathcal{I}_A^{\mathrm{an}.w_\lambda}[F] \equiv \mathcal{I}_{A,x}^{\mathrm{an}.w_\lambda}[F(Ax)] \equiv \int_{\mathbb{B}}^{\mathrm{an}.w_\lambda} F(Ax)d\nu(x) = J_F^*(A;\lambda). \tag{2.1}$$

Let $q \neq 0$ be a real number, and let F be a s.i.m. functional whose analytic Wiener integral $\mathcal{I}_A^{\mathrm{an}.w_\lambda}[F]$ exists for all $\lambda \in \mathbb{C}_+$. If the following limit exists, we call it the analytic Feynman integral (associated with the operator A) of F

with parameter q, and we write

(2.2)
$$\mathcal{I}_A^{\mathrm{an}.f_q}[F] \equiv \mathcal{I}_{A,x}^{\mathrm{an}.f_q}[F(Ax)] \equiv \int_{\mathbb{B}}^{\mathrm{an}.f_q} F(Ax)d\nu(x) = \lim_{\lambda\to -iq} \mathcal{I}_{A,x}^{\mathrm{an}.w_\lambda}[F(Ax)]$$

where λ approaches $-iq$ through values in $\mathbb{C}_+$.

The definitions of the analytic FFT and the CP [1, 8, 9, 23] on classical and abstract Wiener spaces are based on the structure of the analytic Feynman integral and the concept of the scale-invariant measurability. We now provide the definition of the L_1-analytic FFT associated with B.L.OP.s [10] on AWSs.

Definition 2.2. Let $F : \mathbb{B} \to \mathbb{C}$ be a s.i.m. functional such that for $A \in \mathcal{L}(\mathbb{B})$ and $y \in \mathbb{B}$, the following analytic Wiener integral

$$T_{\lambda,A}(F)(y) = \mathcal{I}_{A,x}^{\mathrm{an}.w_\lambda}[F(y + Ax)]$$

exists. Then for $q \in \mathbb{R} \setminus \{0\}$, the L_1-analytic FFT of F associated with the operator A and with parameter q, $T_{q,A}^{(1)}(F)$, is defined by the formula

$$T_{q,A}^{(1)}(F)(y) = \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} T_{\lambda,A}(F)(y) \tag{2.3}$$

for s-a.e. $y \in \mathbb{B}$, whenever this limit exists. That is to say,

$$T_{q,A}^{(1)}(F)(y) = \mathcal{I}_{A,x}^{\mathrm{an}.f_q}[F(y + Ax)] \tag{2.4}$$

for s-a.e. $y \in \mathbb{B}$.

We note that if the L_1-analytic FFT $T_{q,A}^{(1)}(F)$ exists for a B.L.OP. $A \in \mathcal{L}(\mathbb{B})$ and if $F \approx G$, then $T_{q,A}^{(1)}(G)$ exists and $T_{q,A}^{(1)}(G) \approx T_{q,A}^{(1)}(F)$.

Remark 2.3. (i) One can see that for each $A \in \mathcal{L}(\mathbb{B})$, $T_{q,A}^{(1)}(F) \approx T_{q,-A}^{(1)}(F)$, since

$$\int_{\mathbb{B}} F(x)d\nu(x) = \int_{\mathbb{B}} F(-x)d\nu(x).$$

Moreover, from equations (2.1), (2.2), (2.3) and (2.4), it follows that

$$\mathcal{I}_A^{\mathrm{an}.f_q}[F] = T_{q,A}^{(1)}(F)(0) \tag{2.5}$$

in the sense that if either side exists, then both sides exist and equality holds.

(ii) If A is the identity operator on $\mathbb{B}$, then this definition agrees with the previous definition of the (ordinary) analytic FFT [1, 8, 9].

The following definition of the CP is based on the definition of the CP studied in [15].

Definition 2.4. Let F and G be s.i.m. functionals on $\mathbb{B}$. For $\lambda \in \widetilde{\mathbb{C}}_+$ and $A_1, A_2 \in \mathcal{L}(\mathbb{B})$, we define their CP with respect to $\{A_1, A_2\}$ (if it exists) by

$$
\begin{aligned}
&(F * G)_\lambda^{(A_1,A_2)}(y) \\
(2.6)\quad &= \begin{cases} \int_{\mathbb{B}}^{\mathrm{an}.w_\lambda} F(y + A_1 x)G(y + A_2 x)d\nu(x), & \lambda \in \mathbb{C}_+ \\ \int_{\mathbb{B}}^{\mathrm{an}.f_q} F(y + A_1 x)G(y + A_2 x)d\nu(x), & \lambda = -iq,\ q \in \mathbb{R},\ q \neq 0. \end{cases}
\end{aligned}
$$

When $\lambda = -iq$, we denote $(F * G)_\lambda^{(A_1,A_2)}$ by $(F * G)_q^{(A_1,A_2)}$.

3. Translation theorem for analytic Feynman integral

In this section, we establish a translation theorem for the analytic Feynman integral of functionals F on AWS $(\mathbb{H}, \mathbb{B}, \nu)$.

Lemma 3.1. *Let F be a s.i.m. functional on $\mathbb{B}$ such that given a B.L.OP. $A_1 \in \mathcal{L}(\mathbb{B})$, $F(A_1(\rho x))$ is Wiener integrable over $\mathbb{B}$ for each $\rho > 0$. Assume that given a non-zero real q, the L_1-analytic FFT $T_{q,A_1}^{(1)}(F)$ of F exists. Then for each $\theta \in \mathbb{B}^*$ and any B.L.OP. $A_2 \in \mathcal{L}(\mathbb{B})$, it follows that*

$$T_{q,A_1}^{(1)}(F)(y + A_1 A_2^* \theta) = \exp\left(\frac{iq}{2}|A_2^*\theta|^2 + iq(\theta, y)\right)(F * R_{q,\theta})_q^{(A_1,A_2)}(y)$$

for s-a.e. $y \in \mathbb{B}$, where the functional $R_{q,\theta} : \mathbb{B} \to \mathbb{C}$ is given by

$$R_{q,\theta}(x) = \exp\big(- iq(\theta, x)\big).$$

Proof. By the assumption of the existence of $T_{q,A_1}^{(1)}(F)$, we may assume that the analytic Wiener integral (associated with the B.L.OP. A_1)

$$T_{\lambda,A_1}(F)(y) = \mathcal{I}_{A_1,x}^{\mathrm{an}.w_\lambda}[F(y + A_1 x)]$$

exists for all $\lambda \in \mathbb{C}_+$. Next, for $\lambda > 0$, let

$$G_y^\lambda(A_1 x) = F(y + \lambda^{-1/2} A_1 x). \tag{3.1}$$

Using (3.1) and (1.5) with F replaced with G_y^λ, it follows that for $\lambda > 0$,

$$
\begin{aligned}
&T_{\lambda,A_1}(F)(y + A_1 A_2^* \theta) \\
&= \mathcal{I}_{A_1,x}\big[F\big(y + A_1 A_2^*\theta + \lambda^{-1/2} A_1 x\big)\big] \\
&= \mathcal{I}_{A_1,x}\big[F\big(y + \lambda^{-1/2}(A_1 x + \lambda^{1/2} A_1 A_2^*\theta)\big)\big] \\
(3.2)\quad &= \mathcal{I}_{A_1,x}\big[G_y^\lambda(A_1 x + \lambda^{1/2} A_1 A_2^*\theta)\big] \\
&= \exp\left(- \frac{1}{2}|\lambda^{1/2} A_2^*\theta|^2\right) \int_{\mathbb{B}} G_y^\lambda(A_1 x) \exp\big((\lambda^{1/2}\theta, A_2 x)\big) d\nu(x) \\
&= \exp\left(- \frac{\lambda}{2}|A_2^*\theta|^2\right) \int_{\mathbb{B}} F(y + \lambda^{-1/2} A_1 x) \exp\big(\lambda^{1/2}(\theta, A_2 x)\big) d\nu(x) \\
&= \exp\left(- \frac{\lambda}{2}|A_2^*\theta|^2 + iq(\theta, y)\right)
\end{aligned}
$$

$$\times \int_{\mathbb{B}} F(y+\lambda^{-1/2}A_1x)\exp\big(-iq(\theta,y)+\lambda^{1/2}(\theta,A_2x)\big)d\nu(x).$$

On the other hand, by Definition 2.4, it follows that for $\lambda>0$,

$$\begin{aligned}
&(F*R_{q,\theta})_\lambda^{(A_1,A_2)}(y)\\
&=\int_{\mathbb{B}} F(y+\lambda^{-1/2}A_1x)R_{q,\theta}(y+\lambda^{-1/2}A_2x)d\nu(x)\\
&=\int_{\mathbb{B}} F(y+\lambda^{-1/2}A_1x)\exp\big(-iq(\theta,y+\lambda^{-1/2}A_2x)\big)d\nu(x)\\
&=\int_{\mathbb{B}} F(y+\lambda^{-1/2}A_1x)\exp\big(-iq(\theta,y)-iq\lambda^{-1/2}(\theta,A_2x)\big)d\nu(x).
\end{aligned}\tag{3.3}$$

Let

$$\mathcal{E}_1(\lambda;x)=\exp\big(-iq(\theta,y)+\lambda^{1/2}(\theta,A_2x)\big)$$

and

$$\mathcal{E}_2(\lambda;x)=\exp\big(-iq(\theta,y)-iq\lambda^{-1/2}(\theta,A_2x)\big)$$

for $\lambda\in\mathbb{C}_+$. Since $\mathcal{E}_1(\lambda;x)$ and $\mathcal{E}_2(\lambda;x)$ are analytic functions of $\lambda\in\mathbb{C}_+$ and $\lim_{\lambda\to-iq}\mathcal{E}_1(\lambda;x)=\lim_{\lambda\to-iq}\mathcal{E}_2(\lambda;x)$, there exist a connected and bounded neighbourhood Γ_q of $-iq$ in the right-half complex plane $\widetilde{\mathbb{C}}_+$ and a positive real number $M>0$ which satisfy $|\mathcal{E}_1(\lambda;x)|\le M$ and $|\mathcal{E}_2(\lambda;x)|\le M$ for all $\lambda\in\Gamma_q$. Furthermore, for each $\lambda>0$, $|\mathcal{E}_1(\lambda;x)|$ and $|\mathcal{E}_2(\lambda;x)|$ are ν-integrable functions of $x\in\mathbb{B}$. Thus, using Hölder's inequality, it follows that for $\lambda>0$,

$$\begin{aligned}
&\int_{\mathbb{B}}\Big|\big(\mathcal{E}_1(\lambda;x)-\mathcal{E}_2(\lambda;x)\big)F(y+\lambda^{-1/2}A_1x)\Big|d\nu(x)\\
&\le\int_{\mathbb{B}}\big|\mathcal{E}_1(\lambda;x)-1\big|\big|F(y+\lambda^{-1/2}A_1x)\big|d\nu(x)\\
&\quad+\int_{\mathbb{B}}\big|\mathcal{E}_2(\lambda;x)-1\big|\big|F(y+\lambda^{-1/2}A_1x)\big|d\nu(x)\\
&\le\big\|\mathcal{E}_1(\cdot;x)-1\big\|_{\infty,\Gamma_q\cap\mathbb{R}}\int_{\mathbb{B}}\big|F(y+\lambda^{-1/2}A_1x)\big|d\nu(x)\\
&\quad+\big\|\mathcal{E}_2(\cdot;x)-1\big\|_{\infty,\Gamma_q\cap\mathbb{R}}\int_{\mathbb{B}}\big|F(y+\lambda^{-1/2}A_1x)\big|d\nu(x)<\infty
\end{aligned}$$

where

$$\|\mathcal{E}_1(\cdot;x)-1\|_{\infty,\Gamma_q\cap\mathbb{R}}=\sup_{\lambda\in\Gamma_q\cap\mathbb{R}}|\mathcal{E}_1(\lambda;x)-1|$$

and

$$\|\mathcal{E}_2(\cdot;x)-1\|_{\infty,\Gamma_q\cap\mathbb{R}}=\sup_{\lambda\in\Gamma_q\cap\mathbb{R}}|\mathcal{E}_2(\lambda;x)-1|.$$

Hence, by the dominated convergence theorem, we see that

$$\lim_{\substack{\lambda\to-iq\\ \lambda\in\mathbb{C}_+}}\int_{\mathbb{B}}\Big|F(y+\lambda^{-1/2}A_1x)\mathcal{E}_1(\lambda;x)-F(y+\lambda^{-1/2}A_1x)\mathcal{E}_2(\lambda;x)\Big|d\nu(x)=0$$

and so that

$$\begin{aligned}
&\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}\int_{\mathbb{B}} F(y+\lambda^{-1/2}A_1x)\mathcal{E}_1(\lambda;x)d\nu(x)\\
&= \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}\int_{\mathbb{B}} F(y+\lambda^{-1/2}A_1x)\mathcal{E}_2(\lambda;x)d\nu(x)
\end{aligned} \tag{3.4}$$

for s-a.e. $y\in\mathbb{B}$. Applying this and using (2.3), (3.2), (3.3), (2.6) with G replaced with $R_{q,\theta}$, and (3.4), we finally have that

$$\begin{aligned}
&T^{(1)}_{q,A_1}(F)(y+A_1A_2^*\theta)\\
&= \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} T_{\lambda,A_1}(F)(y+A_1A_2^*\theta)\\
&= \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} \exp\bigg(-\frac{\lambda}{2}|A_2^*\theta|^2+iq(\theta,y)\bigg)\\
&\quad\times\int_{\mathbb{B}} F(y+\lambda^{-1/2}A_1x)\exp\big(-iq(\theta,y)+\lambda^{1/2}(\theta,A_2x)\big)d\nu(x)\\
&= \exp\bigg(\frac{iq}{2}|A_2^*\theta|^2+iq(\theta,y)\bigg)\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}\int_{\mathbb{B}} F(y+\lambda^{-1/2}A_1x)\mathcal{E}_1(\lambda;x)d\nu(x)\\
&= \exp\bigg(\frac{iq}{2}|A_2^*\theta|^2+iq(\theta,y)\bigg)\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}\int_{\mathbb{B}} F(y+\lambda^{-1/2}A_1x)\mathcal{E}_2(\lambda;x)d\nu(x)\\
&= \exp\bigg(\frac{iq}{2}|A_2^*\theta|^2+iq(\theta,y)\bigg)\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}}(F*R_{q,\theta})^{(A_1,A_2)}_{\lambda}(y)\\
&= \exp\bigg(\frac{iq}{2}|A_2^*\theta|^2+iq(\theta,y)\bigg)(F*R_{q,\theta})^{(A_1,A_2)}_{q}(y)
\end{aligned}$$

as desired. □

Using (2.6) with $y=0$ and with G replaced with $R_{q,\theta}$, and (2.5), we have the translation theorem for the analytic Feynman integral associated with B.L.OP.s.

Theorem 3.2. *Let A_1 and F be as in* Lemma 3.1. *Assume that given a non-zero real q, the analytic Feynman integral associated with the operator A_1, $\mathcal{I}^{\text{an}.f_q}_{A_1,x}[F(A_1x)]$, exists. Then for each $\theta\in\mathbb{B}^*$ and any $A_2\in\mathcal{L}(\mathbb{B})$,*

$$\begin{aligned}
&\mathcal{I}^{\text{an}.f_q}_{A_1,x}[F(A_1x+A_1A_2^*\theta)]\\
&= \exp\bigg(\frac{iq}{2}|A_2^*\theta|^2\bigg)\int_{\mathbb{B}}^{\text{an}.f_q} F(A_1x)\exp\big(-iq(\theta,A_2x)\big)d\nu(x).
\end{aligned} \tag{3.5}$$

Remark 3.3. It is of interest to note that if $F \equiv 1$, then equation (3.5) yields an analytic Feynman integration formula: given an operator $A \in \mathcal{L}(\mathbb{B})$,

$$\mathcal{I}_{A,x}^{\text{an.}f_q}\big[\exp\big(-iq(A^*\theta, x)\big)\big] = \exp\bigg(-\frac{iq}{2}|A^*\theta|^2\bigg). \tag{3.6}$$

Choosing A_1 and A_2 to be the identity operator I on $\mathbb{B}$, the assertion in the following corollary holds true. This result subsumes a similar result obtained in [1, 5, 6].

Corollary 3.4. *Assume that given a non-zero real q, the analytic Feynman integral $\mathcal{I}_{I,x}^{\text{an.}f_q}[F(x)]$ exists. Then for each $x_0 \in \mathbb{B}^*$,*

$$\begin{aligned}&\mathcal{I}_{I,x}^{\text{an.}f_q}[F(x+x_0)]\\ &\equiv \int_{\mathbb{B}}^{\text{an.}f_q} F(x+x_0)d\nu(x)\\ &= \exp\bigg(\frac{iq}{2}|x_0|^2\bigg)\int_{\mathbb{B}}^{\text{an.}f_q} F(x)\exp\big(-iq(x_0,x)\big)d\nu(x).\end{aligned} \tag{3.7}$$

Taking F to be a cylinder functional, we have the following evaluation formula for the analytic Feynman integral, which plays a key role in order to generalize the Wiener integration formula [24]. The analytic Feynman integral version of the Wiener integration formula will be provided Section 4 below.

Theorem 3.5. *Given a vector β in $\mathbb{B}^*$ and a Lebesgue integrable function f on $\mathbb{R}$, let $F : \mathbb{B} \to \mathbb{C}$ be given by $F(x) = f((\beta, x))$ for s-a.e. $x \in \mathbb{B}$. If the analytic Feynman integral $\mathcal{I}_{A_1,x}^{\text{an.}f_q}[F(A_1x)]$ associated with an operator $A_1 \in \mathcal{L}(\mathbb{B})$ exists, then for each $\theta \in \mathbb{B}^*$ and any $A_2 \in \mathcal{L}(\mathbb{B})$,*

$$\begin{aligned}&\int_{\mathbb{B}}^{\text{an.}f_q} f((A_1^*\beta, x))\exp\big(-iq(A_2^*\theta, x)\big)d\nu(x)\\ &= \exp\bigg(-\frac{iq}{2}|A_2^*\theta|^2\bigg)\mathcal{I}_{A_1,x}^{\text{an.}f_q}\big[f((A_1^*\beta, x) + (A_1^*\beta, A_2^*\theta))\big].\end{aligned} \tag{3.8}$$

4. On the classical Wiener space

Let $\mathbb{B} = C_0[0, T]$ be the classical Wiener space (i.e., the Banach space of all real-valued continuous functions x on the interval $[0, T]$ with $x(0) = 0$) with the uniform norm $\|x\| = \sup_{t\in[0,T]}|x(t)|$. The classical Wiener measure m_{w} characterized by

$$m_{\text{w}}(\{x : x(t) \le a\}) = \frac{1}{\sqrt{2\pi t}}\int_{-\infty}^{a}\exp\bigg(-\frac{u^2}{2t}\bigg)du$$

for $t \in (0, T]$. Let $\mathbb{H} = C_0'[0, T]$ be the Cameron–Martin space in $C_0[0, T]$, namely, the space of all functions $h \in C_0[0, T]$ such that h is absolutely continuous and the derivative $Dh \equiv dh/dt$ is of class $L_2[0, T]$. The inner product on

$\mathbb{H} = C_0'[0,T]$ is given by

$$\langle h_1, h_2\rangle_{C_0'} = \int_0^T Dh_1(t)Dh_2(t)dt.$$

It is well-known that $\mathbb{B}^* \equiv C_0^*[0,T]$ can be identified as the space

$$C_0^*[0,T] = \big\{\theta \in C_0'[0,T] : D\theta \text{ is a right continuous function of bounded variation on } [0,T]\big\}.$$

Then $(C_0'[0,T], C_0[0,T], m_{\rm w})$ is one of the most important examples of AWSs. For more details, see [11,19].

It is well-known that $\{x(t) : (x,t) \in C_0[0,T] \times [0,T]\}$ is a standard Wiener process on the probability space $(C_0[0,T], \mathcal{B}(C_0[0,T]), m_{\rm w})$ where $\mathcal{B}(C_0[0,T])$ denotes the Borel σ-field on $C_0[0,T]$. Let $\mathcal{W}(C_0[0,T])$ denote the class of all Wiener measurable subsets. It is also well-known that the $m_{\rm w}$-Carathéodory completion $\sigma(\mathcal{B}(C_0[0,T]))$ is equal to the σ-field $\mathcal{W}(C_0[0,T])$. We note that $(C_0[0,T], \mathcal{W}(C_0[0,T]), m_{\rm w})$ is a complete probability space.

Remark 4.1. Let U be the unitary operator from $L_2[0,T]$ onto $C_0'[0,T]$ given by $Uv(t) = \int_0^t v(s)ds$ for $v \in L_2[0,T]$. Let RCBV$[0,T]$ be the space of real-valued functions on $[0,T]$ which are right continuous and are of bounded variation on $[0,T]$. Then we see that $C_0^*[0,T] = \{Uv : v \in \text{RCBV}[0,T]\}$. For any $h \in C_0'[0,T]$ and $g \in C_0^*[0,T]$, let the operation $\odot$ between $C_0'[0,T]$ and $C_0^*[0,T]$ be defined by $h \odot g = U(DhDg)$. Then $(C_0^*[0,T], \odot)$ is a commutative algebra with the identity function $e : [0,T] \to \mathbb{R}$ given by $e(t) = t$.

Note that if $\{g_n\}_{n=1}^\infty$ is a complete orthonormal set of functions in $C_0'[0,T]$, each of whose derivatives is in RCBV$[0,T]$, then the sequence $\{Dg_n\}_{n=1}^\infty$ is a complete orthonormal set of functions in $L_2[0,T]$ and the stochastic linear functional $(h,x)^\sim$ given by (1.3) on $C_0'[0,T]\times C_0[0,T]$ equals the Paley–Wiener–Zygmund stochastic integral $\int_0^T Dh(t)dx(t)$ for each $h \in C_0'[0,T]$ and $m_{\rm w}$-a.e. $x \in C_0[0,T]$, see [21,22].

Let $A_{\rm w} : C_0'[0,T] \to C_0'[0,T]$ be the linear operator defined by

$$A_{\rm w}h(t) = \int_0^t h(s)ds.$$

Then, we see that the Hilbert space adjoint operator $A_{\rm w}^*$ of $A_{\rm w}$ is given by

$$A_{\rm w}^*h(t) = th(T) - \int_0^t h(s)ds = \int_0^t (h(T) - h(s))ds,$$

and the linear operator $S_{\rm w} = A_{\rm w}^*A_{\rm w} = A_{\rm w}^*(A_{\rm w}^*)^*$ is given by

$$S_{\rm w}h(t) = \int_0^T \min\{s,t\}h(s)ds. \tag{4.1}$$

Furthermore, we see that $S_{\rm w}$ is a self-adjoint, compact operator on $C_0'[0,T]$ and that

$$\langle h_1, S_{\rm w}h_2\rangle_{C_0'} = \langle A_{\rm w}h_1, A_{\rm w}h_2\rangle_{C_0'} = \int_0^T h_1(t)h_2(t)dt$$

for all $h_1, h_2 \in C_0'[0,T]$. Hence $S_{\rm w}$ is positive definite.

The notational conveniences in the observations (i) and (ii) below will be very useful in the development of our application of the translation theorem.

(i) By the definitions of the operators $A_{\rm w}$, $A_{\rm w}^*$ and $S_{\rm w}$, one can see that the restrictions $A_{\rm w}|_{C_0^*[0,T]}$, $A_{\rm w}^*|_{C_0^*[0,T]}$ and $S_{\rm w}|_{C_0^*[0,T]}$ are in $\mathcal{L}(C_0^*[0,T])$. We shall use the same symbols $A_{\rm w}$, $A_{\rm w}^*$ and $S_{\rm w}$ for the restrictions $A_{\rm w}|_{C_0^*[0,T]}$, $A_{\rm w}^*|_{C_0^*[0,T]}$ and $S_{\rm w}|_{C_0^*[0,T]}$ respectively.

(ii) Also, one can see that the extension operators

$$\widetilde{A}_{\rm w},\ \widetilde{A}_{\rm w}^*,\ \widetilde{S}_{\rm w} : C_0[0,T] \to C_0[0,T]$$

of $A_{\rm w}$, $A_{\rm w}^*$ and $S_{\rm w}$, respectively, are in the space $\mathcal{L}(C_0[0,T])$. Also, we shall again use the same symbols $A_{\rm w}$, $A_{\rm w}^*$ and $S_{\rm w}$, respectively, for the extension operators.

Remark 4.2. Note that for each $h \in C_0'[0,T]$, h is a continuous function of bounded variation on $[0,T]$. Thus, it follows that $C_0'[0,T] \subset C_0^*[0,T]$ in view of set inclusion structure, and the operator $\widetilde{S}_{\rm w}\big|_{C_0'[0,T]} \equiv S_{\rm w} : C_0'[0,T] \to C_0'[0,T]$ given by (4.1) is a trace class operator of $C_0'[0,T]$ in view of [19, Theorem 1.4.6, p.83]. In fact, the trace of $S_{\rm w}$ is given by $\mathrm{Tr}S_{\rm w} = \frac{1}{2}T^2$. Also, one can see easily that the range of the operator $S_{\rm w} : C_0[0,T] \to C_0[0,T]$ is a subset of $C_0^*[0,T]$, namely, $S_{\rm w} \in \mathcal{L}(C_0[0,T], C_0^*[0,T])$.

The following examples are simple consequences of our translation theorem.

Example 4.3. Letting $\mathbb{B} = C_0[0,T]$ and choosing A_1 and A_2 to be the operator $A_{\rm w}^*$, we have the following analytic Feynman integration formula from (3.5): for a s.i.m. functional F on $C_0[0,T]$, it follows that

(4.2)

$$\begin{aligned}
&\mathcal{I}^{\text{an}.f_q}_{A_{\rm w}^*,x}\big[F(A_{\rm w}^*x + S_{\rm w}\theta)\big]\\
&\equiv \mathcal{I}^{\text{an}.f_q}_{A_{\rm w}^*,x}\big[F(A_{\rm w}^*x + A_{\rm w}^*(A_{\rm w}^*)^*\theta)\big]\\
&\overset{*}{=} \exp\bigg(\frac{iq}{2}|(A_{\rm w}^*)^*\theta|^2_{C_0'}\bigg)\int^{\text{an}.f_q}_{C_0[0,T]} F(A_{\rm w}^*x)\exp\big(-iq(\theta, A_{\rm w}^*x)\big)dm_{\rm w}(x)\\
&= \exp\bigg(\frac{iq}{2}|A_{\rm w}\theta|^2_{C_0'}\bigg)\int^{\text{an}.f_q}_{C_0[0,T]} F(A_{\rm w}^*x)\exp\big(-iq(A_{\rm w}\theta, x)\big)dm_{\rm w}(x)\\
&= \exp\bigg(\frac{iq}{2}\int_0^T(\theta(t))^2dt\bigg)\int^{\text{an}.f_q}_{C_0[0,T]} F(A_{\rm w}^*x)\exp\bigg(-iq\int_0^T\theta(t)dx(t)\bigg)dm_{\rm w}(x),
\end{aligned}$$

where by $\overset{*}{=}$ we mean that if either side exists, both sides exist and equality holds. Similarly, letting $\mathbb{B} = C_0[0,T]$ and choosing $A_1 = A_{\rm w}$ and $A_2 = A_{\rm w}^*$ in

(3.5), it also follows that

$$\begin{aligned}&\mathcal{I}^{\mathrm{an}.f_q}_{A_{\mathrm{w}},x}\big[F(A_{\mathrm{w}}x+A_{\mathrm{w}}A_{\mathrm{w}}\theta)\big]\\ &\overset{*}{=} \exp\bigg(\frac{iq}{2}\int_0^T(\theta(t))^2dt\bigg)\\ &\quad\times\int^{\mathrm{an}.f_q}_{C_0[0,T]}F(A_{\mathrm{w}}x)\exp\bigg(-iq\int_0^T\theta(t)dx(t)\bigg)dm_{\mathrm{w}}(x).\end{aligned}\tag{4.3}$$

Letting $F\equiv 1$, equation (4.2) yields the well-known analytic Feynman integration formula: for any function θ in $C_0^*[0,T]$,

$$\int^{\mathrm{an}.f_q}_{C_0[0,T]}\exp\bigg(-iq\int_0^T\theta(t)dx(t)\bigg)dm_{\mathrm{w}}(x)=\exp\bigg(-\frac{iq}{2}\int_0^T(\theta(t))^2dt\bigg).\tag{4.4}$$

Also, equation (4.3) with $F\equiv 1$ yields equation (4.4).

Setting $x_0(t)=\int_0^t\theta(s)ds$ in equation (3.7) with $F\equiv 1$, we also have equation (4.4).

Example 4.4. Letting $A_2=A_{\mathrm{w}}$, equation (3.6) yields the following analytic Feynman integration formula: for any function θ in $C_0^*[0,T]$,

$$\begin{aligned}&\int^{\mathrm{an}.f_q}_{C_0[0,T]}\exp\bigg(-iq\int_0^T(\theta(T)-\theta(t))dx(t)\bigg)dm_{\mathrm{w}}(x)\\ &=\exp\bigg(-\frac{iq}{2}\int_0^T(\theta(T)-\theta(t))^2dt\bigg),\end{aligned}$$

which can be obtained from (3.7) with $x_0(t)=\int_0^t(\theta(T)-\theta(s))ds$.

5. An application

In [24], Yeh applied Fernique's theorem [13, 19] to calculate the Wiener integral

$$\int_{C_0[0,T]}\exp\bigg(\eta^2\int_0^T x(t)y(t)dt\bigg)dm_{\mathrm{w}}(x),\ \ -\frac{\pi}{2}\le\eta\le\frac{\pi}{2}.\tag{5.1}$$

More precisely, Yeh first used a generalized translation result [4, Theorem I] and the immediate consequence (a refinement of Fernique's theorem):

$$\int_{C_0[0,T]}\exp\bigg(i\mu\int_0^T(x(t))^2dt\bigg)dm_{\mathrm{w}}(x)=\big(\sec(i\mu)^{1/2}\big)^{1/2},\ \ \mu\in\mathbb{R}.$$

But, Yeh's procedure to evaluate the Wiener integral (5.1), as well as the application of [4, Theorem I], is very difficult and complicated.

Using our translation theorem, we in this section will provide an explicit form of the analytic Feynman integral which is an analytic Feynman integral version of the Wiener integration formula (5.1); namely,

$$\int_{C_0[0,T]}^{\mathrm{an}.f_q} \exp\Big(-iq\int_0^T x(t)y(t)dt\Big) dm_{\mathrm{w}}(x) \text{ with } y \in C_0[0,T].$$

Given a function f in $L_1(\mathbb{R})$ and $\beta \in C_0^*[0,T]\setminus\{0\}$, let $F(x) = f((\beta,x)^\sim)$ for s-a.e. $x \in C_0[0,T]$. Then by the change of variable theorem, we have, for each $\rho > 0$ and any $A \in \mathcal{L}(\mathbb{B})$,

$$\tag{5.2} \begin{aligned} \int_{C_0[0,T]} F(\rho Ax)dm_{\mathrm{w}}(x) &= \int_{C_0[0,T]} f(\rho(A^*\beta,x))dm_{\mathrm{w}}(x) \\ = \Big(2\pi\rho^2|A^*\beta|^2_{C_0'}\Big)^{-1/2} &\int_{\mathbb{R}} f(u)\exp\Big(-\frac{u^2}{2\rho^2|A^*\beta|^2_{C_0'}}\Big)du. \end{aligned}$$

Since the exponential function $H(\rho;u) = \exp(-u^2/2\rho^2|A^*\beta|^2_{C_0'})$ is in $C_0(\mathbb{R})$, the space of bounded continuous functions on $\mathbb{R}$ that vanish at infinity, the Wiener integral (5.2) exists as a finite number. In fact, for all $\lambda \in \widetilde{\mathbb{C}}_+$, $|H(\lambda^{-1/2};u)| \le 1$ and $H(\lambda^{-1/2};u)$ is an analytic function of $\lambda \in \mathbb{C}_+$. Applying the Morera theorem, one can show that the Lebesgue integral

$$\int_{\mathbb{R}} f(u)\exp\Big(-\frac{\lambda u^2}{2|A^*\beta|^2_{C_0'}}\Big)du$$

in the last expression of (5.2) with $\rho > 0$ replaced with $\lambda^{-1/2}$ is an analytic function of $\lambda \in \mathbb{C}_+$. Thus, in view of Definition 2.1 and by the dominated convergence theorem, it follows that for all non-zero real q, the analytic Feynman integral associated with the operator $A \in \mathcal{L}(C_0[0,T])$ in the following equation exists and is calculated by

$$\tag{5.3} \begin{aligned} \mathcal{I}_{A,x}^{\mathrm{an}.f_q}[f((A^*\beta,x))] &= \lim_{\substack{\lambda\to -iq \\ \lambda\in\mathbb{C}_+}} \int_{\mathbb{R}} f(u)\exp\Big(-\frac{\lambda u^2}{2|A^*\beta|^2_{C_0'}}\Big)du \\ &= \Big(\frac{-iq}{2\pi|A^*\beta|^2_{C_0'}}\Big)^{1/2}\int_{\mathbb{R}} f(u)\exp\Big(\frac{iqu^2}{2|A^*\beta|^2_{C_0'}}\Big)du. \end{aligned}$$

Next, applying equations (3.8) and (5.3), it follows that for all B.L.OP.s A_1 and A_2 in $\mathcal{L}(C_0[0,T])$,

$$\tag{5.4} \begin{aligned} &\int_{C_0[0,T]}^{\mathrm{an}.f_q} f((A_1^*\beta,x))\exp\big(-iq(A_2^*\theta,x)\big)dm_{\mathrm{w}}(x) \\ &= \Big(\frac{-iq}{2\pi|A_1^*\beta|^2_{C_0'}}\Big)^{1/2}\exp\Big(-\frac{iq}{2}|A_2^*\theta|^2_{C_0'}\Big) \\ &\quad\times\int_{\mathbb{R}} f(u)\exp\Big(\frac{iq(u-(A_1^*\beta,A_2^*\theta))^2}{2|A_1^*\beta|^2_{C_0'}}\Big)du. \end{aligned}$$

Letting $A_1 = I$ (the identity operator), and $A_2 = A_{\mathrm{w}}$ in (5.4) yields the formula

(5.5)

$$\begin{aligned}
&\int_{C_0[0,T]}^{\mathrm{an}.f_q} f((\beta,x)) \exp\big(-iq(A_{\mathrm{w}}^*\theta,x)\big)dm_{\mathrm{w}}(x)\\
&= \bigg(\frac{-iq}{2\pi|\beta|_{C_0'}^2}\bigg)^{1/2} \exp\bigg(-\frac{iq}{2}|A_{\mathrm{w}}^*\theta|_{C_0'}^2\bigg) \int_{\mathbb{R}} f(u)\exp\bigg(\frac{iq(u-(\beta,A_{\mathrm{w}}^*\theta))^2}{2|\beta|_{C_0'}^2}\bigg)du\\
&= \bigg(\frac{-iq}{2\pi|\beta|_{C_0'}^2}\bigg)^{1/2} \exp\bigg(-\frac{iq}{2}|A_{\mathrm{w}}^*\theta|_{C_0'}^2 + \frac{iq(A_{\mathrm{w}}\beta,\theta)^2}{2|\beta|_{C_0'}^2}\bigg)\\
&\quad\times \int_{\mathbb{R}} f(u)\exp\bigg(\frac{iq}{2|\beta|_{C_0'}^2}u^2 - \frac{iq(A_{\mathrm{w}}\beta,\theta)}{|\beta|_{C_0'}^2}u\bigg)du.
\end{aligned}$$

Let $\beta(t) = t$ for $t \in [0,T]$, and for a continuous function y of bounded variation on $[0,T]$, let $\theta(t) = \int_0^t y(s)ds$. Then $\theta \in C_0^*[0,T]$ and

$$\exp\big(-iq(A_{\mathrm{w}}^*\theta,x)\big) = \exp\bigg(-iq\int_0^T x(t)y(t)dt\bigg)$$

for all $x \in C_0[0,T]$ and $q \in \mathbb{R}\setminus\{0\}$. For notational convenience, let

$$\begin{aligned}
Y(t) &\equiv \theta(t) = \int_0^t y(s)ds,\\
Z(t) &\equiv \int_0^t Y(t)dt = A_{\mathrm{w}}\theta(t),
\end{aligned}$$

and

$$\|Y\|^2 \equiv \int_0^T (Y(t))^2dt = |A_{\mathrm{w}}\theta|_{C_0'}^2.$$

Then it follows that

$$|A_{\mathrm{w}}^*\theta|_{C_0'}^2 = \int_0^T (\theta(T)-\theta(t))^2dt = T(Y(T))^2 - 2Y(T)Z(T) + \|Y\|^2 \tag{5.6}$$

and

$$(A_{\mathrm{w}}\beta,\theta) = \langle A_{\mathrm{w}}\beta,\theta\rangle_{C_0'} = \int_0^T ty(t)dt = TY(T) - Z(T). \tag{5.7}$$

Substituting (5.6) and (5.7) into (5.5), it follows that

$$\begin{aligned}
&\int_{C_0[0,T]}^{\mathrm{an}.f_q} f(x(T))\exp\bigg(-iq\int_0^T x(t)y(t)dt\bigg)dm_{\mathrm{w}}(x)\\
&= \bigg(\frac{-iq}{2\pi T}\bigg)^{1/2}\exp\bigg(-\frac{iq}{2}\bigg(\|Y\|^2 - \frac{Z(T)^2}{T}\bigg)\bigg)\\
&\quad\times\int_{\mathbb{R}} f(u)\exp\bigg(\frac{iq}{2T}u^2 - iq\bigg(Y(T) - \frac{Z(T)}{T}\bigg)u\bigg)du
\end{aligned}$$

and

$$\begin{aligned}&\int_{C_0[0,T]}^{\mathrm{an}.f_q}\exp\bigg(-iq\int_0^T x(t)y(t)dt\bigg)dm_{\mathrm{w}}(x)\\&=\exp\bigg(-\frac{iq}{2}\Big(T(Y(T))^2-2Y(T)Z(T)+\|Y\|^2\Big)\bigg)\\&=\exp\bigg(-\frac{iq}{2}|A_{\mathrm{w}}^*\theta|_{C_0'}^2\bigg)\\&=\exp\bigg(-\frac{iq}{2}\int_0^T\bigg(\int_t^T y(s)ds\bigg)^2dt\bigg),\end{aligned}$$

which are the main results in [24]. In [12], Chung and Kang calculated this problem via the concept of the conditional abstract Wiener integral.

Acknowledgements. The author would like to express his gratitude to the editor and the referees for their valuable comments and suggestions which have improved the original paper.

References

[1] J. M. Ahn, K. S. Chang, B. S. Kim, and I. Yoo, *Fourier-Feynman transform, convolution and first variation*, Acta Math. Hungar. **100** (2003), no. 3, 215–235. https://doi.org/10.1023/A:1025041525913

[2] R. H. Cameron and R. E. Graves, *Additive functionals on a space of continuous functions. I*, Trans. Amer. Math. Soc. **70** (1951), 160–176. https://doi.org/10.2307/1990530

[3] R. H. Cameron and W. T. Martin, *Transformations of Wiener integrals under translations*, Ann. of Math. (2) **45** (1944), 386–396. https://doi.org/10.2307/1969276

[4] R. H. Cameron and W. T. Martin, *Transformations of Wiener integrals under a general class of linear transformations*, Trans. Amer. Math. Soc. **58** (1945), 184–219. https://doi.org/10.2307/1990282

[5] R. H. Cameron and D. A. Storvick, *A translation theorem for analytic Feynman integrals*, Trans. Amer. Math. Soc. **125** (1966), 1–6. https://doi.org/10.2307/1994583

[6] R. H. Cameron and D. A. Storvick, *A new translation theorem for the analytic Feynman integral*, Rev. Roumaine Math. Pures Appl. **27** (1982), no. 9, 937–944.

[7] S. J. Chang and J. G. Choi, *Evaluation formula for Wiener integral of polynomials in terms of natural dual pairings on abstract Wiener spaces*, Bull. Korean Math. Soc. **59** (2022), no. 5, 1093–1103. https://doi.org/10.4134/BKMS.b210542

[8] K. S. Chang, B. S. Kim, and I. Yoo, *Fourier-Feynman transform, convolution and first variation of functionals on abstract Wiener space*, Integral Transform. Spec. Funct. **10** (2000), no. 3-4, 179–200. https://doi.org/10.1080/10652460008819285

[9] K. S. Chang, T. S. Song, and I. Yoo, *Analytic Fourier-Feynman transform and first variation on abstract Wiener space*, J. Korean Math. Soc. **38** (2001), no. 2, 485–501.

[10] J. G. Choi, *Analytic Fourier-Feynman transforms associated with bounded linear operators on abstract Wiener spaces*, J. Math. Anal. Appl. **521** (2023), no. 1, Paper No. 126952, 16 pp. https://doi.org/10.1016/j.jmaa.2022.126952

[11] D. M. Chung, *Scale-invariant measurability in abstract Wiener spaces*, Pacific J. Math. **130** (1987), no. 1, 27–40. http://projecteuclid.org/euclid.pjm/1102690291

[12] D. M. Chung and S. J. Kang, *Evaluation formulas for conditional abstract Wiener integrals*, Stochastic Anal. Appl. **7** (1989), no. 2, 125–144. https://doi.org/10.1080/07362998908809173

[13] X. Fernique, *Intégrabilité des vecteurs gaussiens*, C. R. Acad. Sci. Paris Sér. A-B **270** (1970), A1698–A1699.

[14] L. Gross, *Abstract Wiener spaces*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 1, 31–42, Univ. California Press, Berkeley, CA, 1967.

[15] M.-K. Im, U.-C. Ji, and Y.-J. Park, *Relations among the first variation, the convolutions and the generalized Fourier-Gauss transforms*, Bull. Korean Math. Soc. **48** (2011), no. 2, 291–302. https://doi.org/10.4134/BKMS.2011.48.2.291

[16] G. Kallianpur and C. C. Bromley, *Generalized Feynman integrals using analytic continuation in several complex variables*, Stochastic Analysis and Applications, 217–267, Adv. Probab. Related Topics, 7, Dekker, New York, 1984.

[17] G. Kallianpur, D. Kannan, and R. L. Karandikar, Analytic and sequential Feynman integrals on abstract Wiener and Hilbert spaces, and a Cameron Martin formula, Ann. Inst. H. Poincaré Probab. Statist. **21** (1985), no. 4, 323–361.

[18] J. Kuelbs, *Abstract Wiener spaces and applications to analysis*, Pacific J. Math. **31** (1969), 433–450. http://projecteuclid.org/euclid.pjm/1102977879

[19] H.-H. Kuo, *Gaussian Measures in Banach Spaces*, Lecture Notes in Mathematics, Vol. 463, Springer, Berlin, 1975.

[20] U. G. Lee and J. G. Choi, *An extension of the Cameron-Martin translation theorem via Fourier-Hermite functionals*, Arch. Math. (Basel) **115** (2020), no. 6, 679–689. https://doi.org/10.1007/s00013-020-01524-6

[21] R. E. A. C. Paley, N. Wiener, and A. Zygmund, *Notes on random functions*, Math. Z. **37** (1933), no. 1, 647–668. https://doi.org/10.1007/BF01474606

[22] C. Park and D. Skoug, *A note on Paley-Wiener-Zygmund stochastic integrals*, Proc. Amer. Math. Soc. **103** (1988), no. 2, 591–601. https://doi.org/10.2307/2047184

[23] D. Skoug and D. Storvick, *A survey of results involving transforms and convolutions in function space*, Rocky Mountain J. Math. **34** (2004), no. 3, 1147–1175. https://doi.org/10.1216/rmjm/1181069848

[24] J. Yeh, *Inequalities and a limit theorem for certain Wiener integrals*, Proc. Amer. Math. Soc. **16** (1965), 915–918. https://doi.org/10.2307/2035582

Jae Gil Choi
Department of Mathematics
Basic Sciences and Mathematics Center
Dankook University
Cheonan 31116, Korea
Email address: jgchoi@dankook.ac.kr